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A

COURSE

OF

MATHEMATICS.

IN TWO VOLUMES.

COMPOSED FOR THE

USE OF THE ROYAL MILITARY ACADEMY.

BY CHARLES HUTTON, LL.D. F.R.S.

LATE PROFESSOR OF MATHEMATICS IN THAT INSTITUTION.

VOL. II.

THE ELEVENTH EDITION,

WITH MANY CORRECTIONS AND IMPROVEMENTS.

BY OLINTHUS GREGORY, LL.D. F.R.A.S.

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PREFACE

TO

THE SECOND VOLUME.

In this New Edition of the Woolwich Course, the substance of the second and third volumes of the former Edition is, by a new arrangement, incorporated into one. The matter, also, of several portions of the volume is entirely remodified; and my colleague, Mr. Davies, has been at the pains considerably to enlarge the part which relates to the Conic Sections; as well as to prepare a sketch of the Geometry of Co-ordinates. The doctrine of Fluxions is now introduced before the subject of Mechanics, a change which has enabled me to improve that department by introducing various propositions which could *only* be treated *adequately* by the fluxionary or an analogous calculus.

I had intended to attempt a concise sketch of the Elements of the *Differential Calculus*, according to my own view of the principle of limits, first defining the sense in which the term limit may be unobjectionably employed, and then, as occasion required, resorting to the familiar axiom, that "what is true *up to* the limit, is true *at* the limit;" but a very serious and continued indisposition, which commenced just at the time this should have been undertaken, compelled me to adopt another course. I got one of my own family to translate the *Lehrbuch des Höhern Kalkuls, für Lehrer und Selbstlernende*, of S. F. Lubbe of the University of Berlin: not because it was in *all* respects so satisfactory as I could have wished, for it sometimes falls into the paralogisms of various other authors in this department of science; but because several of its processes of investigation are both elegant and complete; and because it was

the only foreign elementary treatise which I could conveniently reduce to the space that could be assigned to the subject in this volume; simply subjoining a note or two for explication.

Altogether, I hope the Work will now be found considerably improved; although I have ventured upon very few more alterations than such as would have appeared necessary to the Author, had he been now alive, and acquainted with the present state of Mathematical instruction in this Institution, as well as in the more respectable private seminaries.

OLINTHUS GREGORY.

Royal Military Academy,
June, 1837.

CONTENTS

VOLUME II.

ERRATA IN THIS VOLUME.

PAGE
48, in the answer to Question 1, inclose $\alpha + \beta$, and $\beta + \gamma$ in vincula.
61, note, *for* $\frac{1}{2}$ *c read* $\frac{1}{2}$ C throughout.
78, Ex. 5, *for* wall *read* ball.
80, line 30, *for* some *read* same.
191, Ex. 10, *for* $\frac{\cos.}{a,}$ *read* $\frac{\cos.\ \theta}{a,}$; and in Ex. 12, *for* straight is, *read* straight line i
193, *for* Chapter II. *read* Chapter III.
195, *for* Chapter III. *read* Chapter IV.
197, *for* Chapter IV. *read* Chapter V.
213, line 12, *for* function *read* fluxion.
315, in the diagram, *for* A, B, C, *read* B, *b*, C.
331, line 8 from bottom, *for* g : $\frac{h}{l}$, *read* g. $\frac{h}{l}$.
400, line 5 from bottom, *for* $(2\,r\,x - x^2 \times$, *read* $(2\,r\,x - x^2) \times$.
405, line 9 from bottom, *for* 240 oz. 15 lb., *read* 240 oz. or 15 lb.
426, bottom line, *for* $d\,z$, *read* $\Delta\,z$.
431, line 9, *for* fractions, *read* functions
432, line 10, *for* $m\,x^{x\,-\,1}\,d\,x$, *read* $m\,x^{n\,-\,1}\,d\,x$.
447, line 1, *for* u, *read* n.

A

COURSE

OF

MATHEMATICS,

&c.

ELEMENTS OF ISOPERIMETRY.

Def. 1. When a variable quantity has its mutations regulated by a certain law, or confined within certain limits, it is called a *maximum* when it has reached the greatest magnitude it can possibly attain; and, on the contrary, when it has arrived at the least possible magnitude, it is called a *minimum.*

Def. 2. *Isoperimeters,* or *Isoperimetrical Figures,* are those which have equal perimeters.

Def. 3. The *Locus* of any point, or intersection, &c. is the right line or curve in which these are always situated.

The problem in which it is required to find, among figures of the same or of different kinds, those which, within equal perimeters, shall comprehend the greatest surfaces, has long engaged the attention of mathematicians. Since the admirable invention of the method of Fluxions, this problem has been elegantly treated by some of the writers on that branch of analysis; especially by Maclaurin and Simpson. A much more extensive problem was investigated at the time of "the war of problems," between the two brothers John and James Bernoulli: namely, "To find, among all the isoperimetrical curves between given limits, such a curve, that, constructing a second curve, the ordinates of which shall be functions of the ordinates or arcs of the former, the area of the second curve shall be a maximum or a minimum." While, however, the attention of mathematicians was drawn to the most abstruse inquiries connected with isoperimetry, the *elements* of the subject were lost sight of. Simpson was the first who called them back to this interesting branch of research, by giving in his neat little book of Geometry a chapter on the maxima and minima of geometrical quantities, and some of the simplest problems concerning isoperimeters. The next who treated this subject in an elementary manner was Simon Lhuillier, of Geneva, who, in 1782, published his treatise *De Relatione Mutua Capacitatis et Terminorum Figurarum,* &c. His principal object in the composition of that work was to supply the deficiency in this respect which he found in most of the Elementary Courses; and to determine, with regard to both the most usual surfaces and solids, those which possessed the minimum of contour with the

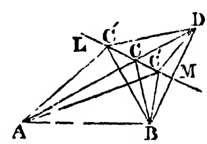
same capacity; and, reciprocally, the maximum of capacity with the same boundary. M. Legendre has also considered the same subject, in a manner somewhat different from either Simpson or Lhuillier, in his *Eléments de Géométrie.* An elegant geometrical tract, on the same subject, was also given, by Dr. Horsley, in the Philos. Trans. vol. lxxv. for 1775; contained also in the New Abridgment, vol. xiii. page 653*. The chief propositions deduced by these four geometers, together with a few additional propositions, are reduced into one system in the following theorems.

SECTION I.—SURFACES.

THEOREM I.

Of all triangles of the same base, and whose vertices fall in a right line given in position, the one whose perimeter is a minimum is that whose sides are equally inclined to that line.

Let AB be the common base of a series of triangles ABC′, ABC, &c., whose vertices C′, C, fall in the right line LM, given in position, then is the triangle of least perimeter that whose sides AC, BC, are inclined to the line LM in equal angles.



For, let BM be drawn from B, perpendicularly to LM, and produced till DM = BM: join AD, and from the point C where AD cuts LM draw BC: also from any other point C′, assumed in LM, draw C′A, C′B, C′D. Then the triangles DMC, BMC, having the angle DCM = angle ACL (theor. 1, Geom.) = MCB (by hyp.), DMC = BMC, and DM = BM, and MC common to both, have also DC = BC (theor. 1, Geom.)

So also, we have C′D = C′B. Hence AC + CB = AC + CD = AD, is less than AC′ + C′D (theor. 10, Geom.), or than its equal AC′ + C′B. And consequently, AB + BC + AC is less than AB + BC′ + AC′. Q. E. D.

Cor. 1. Of all triangles of the same base and the same altitude, or of all equal triangles of the same base, the isosceles triangle has the smallest perimeter.

For, the locus of the vertices of all triangles of the same altitude will be a right line LM *parallel* to the base; and when LM in the above figure becomes parallel to AB, since MCB = ACL, MCB = CBA (theor. 12, Geom), ACL = CAB; it follows that CAB = CBA, and consequently AC = CB (theor. 4, Geom.)

Cor. 2. Of all triangles of the same surface, that which has the minimum perimeter is equilateral.

For the triangle of the smallest perimeter, with the same surface, must be isosceles, whichever of the sides be considered as base: therefore, the triangle of smallest perimeter has each two or each pair of its sides equal, and consequently it is equilateral.

Cor. 3. Of all rectilinear figures, with a given magnitude and a given number of sides, that which has the smallest perimeter is equilateral.

For so long as any two adjacent sides are not equal, we may draw a diagonal

* Another work on the same general subject, containing many valuable theorems, has been published since the first edition of this brief treatise, by Dr. *Creswell*, of Trinity College, Cambridge.

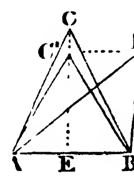
to become a base to those two sides, and then draw an isosceles triangle equal to the triangle so cut off, but of less perimeter: whence the corollary is manifest.

Scholium.

To illustrate the second corollary above, the student may proceed thus: assuming an isosceles triangle whose base is *not* equal to either of the two sides, and then, taking for a new base one of those sides of that triangle, he may construct another isosceles triangle equal to it, but of a smaller perimeter. Afterwards, if the base and sides of this second isosceles triangle are not respectively equal, he may construct a third isosceles triangle equal to it, but of a still smaller perimeter: and so on. In performing these successive operations, he will find that the new triangles will approach nearer and nearer to an equilateral triangle.

THEOREM II.

Of all triangles of the same base, and of equal perimeters, the isosceles triangle has the greatest surface.



Let ABC, ABD, be two triangles of the same base AB and with equal perimeters, of which the one ABC is isosceles, the other is not: then the triangle ABC has a surface (or an altitude) greater than the surface (or than the altitude) of the triangle ABD.

Draw C′D through D, parallel to AB, to cut CE (drawn perpendicular to AB) in C′: then it is to be demonstrated that CE is greater than C′E.

The triangles AC′B, ADB, are equal both in base and altitude; but the triangle AC′B is isosceles, while ADB is scalene: therefore the triangle AC′B has a smaller perimeter than the triangle ADB (theor. 1, cor. 1), or than ACB (by hyp.) Consequently AC′ $<$ AC; and in the right-angled triangles AEC′, AEC, having AE common, we have C′E $<$ CE*. Q. E. D.

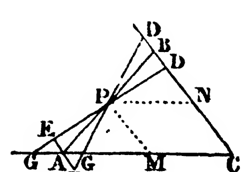
Cor. Of all isoperimetrical figures, of which the number of sides is given, that which is the greatest has all its sides equal. And in particular, of all isoperimetrical triangles, that whose surface is a maximum, is equilateral.

For, so long as any two adjacent sides are not equal, the surface may be augmented without increasing the perimeter.

Remark. Nearly as in this theorem may it be proved that, of all triangles of equal heights, and of which the sum of the two sides is equal, that which is isosceles has the greatest base. And, of all triangles standing on the same base and having equal vertical angles, the isosceles one is the greatest.

THEOREM III.

Of all right lines that can be drawn through a given point, between two right lines given in position, that which is bisected by the given point forms with the other two lines the least triangle.



Of all right lines GD, AB, GD, that can be drawn through a given point P, to cut the right lines CA, CD, given in position, that, AB, which is bisected by the given point P, forms with CA, CD, the least triangle, ABC.

For, let EE be drawn through A parallel to CD, meeting DG (produced if necessary) in E; then

* When two mathematical quantities are separated by the character $<$, it denotes that the preceding quantity is *less than* the succeeding one: when, on the contrary, the separating character is $>$, it denotes that the preceding quantity is *greater than* the succeeding one.

the triangles PED, PAE, are manifestly equiangular; and, since the corresponding sides PB, PA are equal, the triangles are equal also. Hence PBD will be less or greater than PAG, according as CG is greater or less than CA. In the former case, let PACD, which is common, be added to both; then will BAC be less than DGC (ax. 4, Geom.) In the latter case, if PGCB be added, DCG will be greater than BAC; and consequently in this case also BAC is less than DCG. Q. E. D.

Cor. If PM and PN be drawn parallel to CB and CA respectively, the two triangles PAM, PBN, will be equal, and these two taken together (since AM = PN = MC) will be equal to the parallelogram PMCN: and consequently the parallelogram PMCN is equal to half ABC, but less than half DGC. From which it follows (consistently with both the algebraical and geometrical solution of prob. 8, Application of Algebra to Geometry), that a parallelogram is always less than half a triangle in which it is inscribed, except when the base of the one is half the base of the other, or the height of the former half the height of the latter; in which case the parallelogram is just half the triangle: this being the maximum parallelogram inscribed in the triangle.

Scholium.

From the preceding corollary it might easily be shown, that the least triangle which can possibly be described about, and the greatest parallelogram which can be inscribed in, any curve concave to its axis, will be when the subtangent is equal to half the base of the triangle, or to the whole base of the parallelogram: and that the two figures will be in the ratio of 2 to 1. But this is foreign to the present inquiry.

THEOREM IV.

Of all triangles in which two sides are given in magnitude, the greatest is that in which the two given sides are perpendicular to each other.

For, assuming for base one of the given sides, the surface is proportional to the perpendicular let fall upon that side from the opposite extremity of the other given side: therefore, the surface is the greatest when that perpendicular is the greatest; that is to say, when the other side is not inclined to that perpendicular, but *coincides* with it: hence the surface is a maximum when the two given sides are perpendicular to each other.

Otherwise. Since the surface of a triangle, in which two sides are given, is proportional to the sine of the angle included between those two sides; it follows, that the triangle is the greatest when that sine is the greatest: but the greatest sine is the sine total, or the sine of a quadrant; therefore the two sides given make a quadrantal angle, or are perpendicular to each other. Q. E. D.

THEOREM V.

Of all rectilinear figures in which all the sides except one are known, the greatest is that which may be inscribed in a semicircle whose diameter is that unknown side.

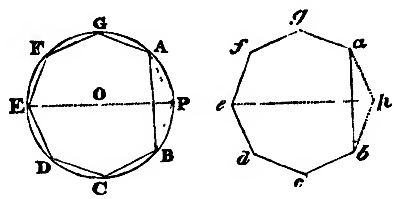
For, if you suppose the contrary to be the case, then whenever the figure made with the sides given, and the side unknown, is not inscribable in a semicircle of which this latter is the diameter, viz. whenever any one of the angles, formed by lines drawn from the extremities of the unknown side to one of the summits of the figure, is not a right angle; we may make a figure greater

than it, in which that angle shall be right, and which shall only differ from it in that respect: therefore, whenever all the angles, formed by right lines drawn from the several vertices of the figure to the extremities of the unknown line, are not right angles, or do not fall in the circumference of a semicircle, the figure is not in its maximum state. Q. E. D.

THEOREM VI.

Of all figures made with sides given in number and magnitude, that which may be inscribed in a circle is the greatest.

Let ABCDEFG be the polygon inscribed, and *abcdefg* a polygon with equal sides, but not inscribable in a circle; so that AB = *ab*, BC = *bc*, &c.; it is affirmed that the polygon ABCDEFG is greater than the polygon *abcdefg*.



Draw the diameter EP; join AP, PB; upon *ab* = AB make the triangle *abp*, equal in all respects to ABP; and join *ep*. Then, of the two figures *edcbp*, *pagfe*, one at least is not (by hyp.) inscribable in the semicircle of which *ep* is the diameter. Consequently, one at least of these two figures is smaller than the corresponding part of the figure APBCDEFG (theor. 5). Therefore the figure APBCDEFG is greater than the figure *apbcdefg*: and if from these there be taken away the respective triangles APB, *apb*, which are equal by construction, there will remain (ax. 5, Geom.) the polygon ABCDEFG greater than the polygon *abcdefg*. Q. E. D.

THEOREM VII.

The magnitude of the greatest polygon which can be contained under any number of unequal sides, does not at all depend on the order in which those lines are connected with each other.

For, since the polygon is a maximum under given sides, it is inscribable in a circle (theor. 6). And this inscribed polygon is constituted of as many isosceles triangles as it has sides, those sides forming the bases of the respective triangles, the other sides of all the triangles being radii of the circle, and their common summit the centre of the circle. Consequently, the magnitude of the polygon, that is, of the assemblage of these triangles, does not at all depend on their disposition, or arrangement around the common centre. Q. E. D.

THEOREM VIII.

If a polygon inscribed in a circle have all its sides equal, all its angles are likewise equal, or it is a regular polygon.

For, if lines be drawn from the several angles of the polygon, to the centre of the circumscribing circle, they will divide the polygon into as many isosceles triangles as it has sides; and each of these isosceles triangles will be equal to either of the others in all respects, and of course they will have the angles at their bases all equal: consequently, the angles of the polygon, which are each made up of two angles at the bases of two contiguous isosceles triangles, will be equal to one another. Q. E. D.

THEOREM IX.

Of all figures having the same number of sides and equal perimeters, the greatest is regular.

For, the greatest figure under the given conditions has all its sides equal (*theor.* 2, *cor.*) But since the sum of the sides and the number of them are given, each of them is given: therefore (theor. 6), the figure is inscribable in a circle: and consequently (theor. 8) all its angles are equal: that is, it is regular. Q. E. D.

Cor. Hence we see that regular polygons possess the property of a maximum of surface, when compared with any other figures of the same name, and with equal perimeters.

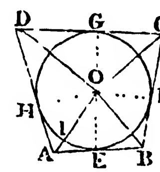
THEOREM X.

A REGULAR polygon has a smaller perimeter than an irregular one equal to it in surface, and having the same number of sides.

This is the converse of the preceding theorem, and may be demonstrated thus: Let R and I be two figures equal in surface and having the same number of sides, of which R is regular, I irregular: let also R′ be a regular figure similar to R, and having a perimeter equal to that of I. Then (theor. 9) $R' > I$; but $I = R$: therefore $R' > R$. But R′ and R are similar; consequently, perimeter of $R' >$ perimeter of R; while per. $R' =$ per. I (by hyp) Hence, per. $I >$ per. R. Q. E. D.

THEOREM XI.

THE surfaces of polygons, circumscribed about the same or equal circles, are respectively as their perimeters*.



Let the polygon ABCD be circumscribed about the circle EFGH; and let this polygon be divided into triangles, by lines drawn from its several angles to the centre O of the circle. Then, since each of the tangents AB, BC, &c. is perpendicular to its corresponding radius, OE, OF, &c., drawn to the point of contact (theor. 46, Geom); and since the area of a triangle is equal to the rectangle of the perpendicular and half the base (Mens. of Surfaces, pr. 2); it follows, that the area of each of the triangles ABO, BCO, &c. is equal to the rectangle of the radius of the circle and half the corresponding side AB, BC, &c.; and consequently, the area of the polygon ABCD, circumscribing the circle, will be equal to the rectangle of the radius of the circle and half the perimeter of the polygon. But, the surface of the circle is equal to the rectangle of the radius and half the circumference (th. 94, Geom.) Therefore, the surface of the circle, is to that of the polygon, as half the circumference of the former, to half the perimeter of the latter; or, as the circumference of the former, to the perimeter of the latter. Now, let P and P′ be any two polygons circumscribing a circle C: then, by the foregoing, we have

$$\text{surf. C : surf. P :: circum. C : perim. P.}$$
$$\text{surf. C : surf. P}' \text{ :: circum. C : perim. P}'.$$

But, since the antecedents of the ratios in both these proportions, are equal, the consequents are proportional: that is, surf. P : surf. P′ :: perim. P : perim P′.

Q. E. D.

* This theorem, together with the analogous ones respecting bodies circumscribing cylinders and spheres, were given by Emerson in his Geometry, and their use in the theory of Isoperimeters was just suggested: but the full application of them to that theory is due to Simon Lhuillier.

Cor. 1. Any one of the triangular portions ABO, of a polygon circumscribing a circle, is to the corresponding circular sector, as the side AB of the polygon, to the arc of the circle included between AO and BO.

Cor. 2. Every circular arc is greater than its chord, and less than the sum of the two tangents drawn from its extremities and produced till they meet.

The first part of this corollary is evident, because a right line is the shortest distance between two given points. The second part follows at once from this proposition: for EA + AH being to the arch EIH, as the quadrangle AEOH to the circular sector HIEO; and the quadrangle being greater than the sector, because it contains it; it follows that EA + AH is greater than the arch EIH *.

Cor. 3. Hence also, any single tangent EA, is greater than its corresponding arc EI.

THEOREM XII.

If a circle and a polygon, circumscribable about another circle, are isoperimeters, the surface of the circle is a geometrical mean proportional between that polygon and a similar polygon (regular or irregular) circumscribed about that circle.

Let C be a circle, P a polygon isoperimetrical to that circle, and circumscribable about some other circle, and P′ a polygon similar to P and circumscribable about the circle C: it is affirmed that $P : C :: C : P'$.

For, $P : P' ::$ perim.^{2} P : perim.^{2} $P' ::$ circum.^{2} C : perim.^{2} P' by theor. 89 Geom. and the hypothesis.

But (theor. 11) $P' : C ::$ per. P' : cir. $C ::$ per.^{2} P' : per $P' \times$ cir. C.

Therefore $P : C ::$ - - - - - cir.^{2} C : per $P' \times$ cir. C.

$::$ cir. C : per. $P' :: C : P'$. Q. E. D.

THEOREM XIII.

If a circle and a polygon, circumscribable about another circle, are equal in surface, the perimeter of that figure is a geometrical mean proportional between the circumference of the first circle and the perimeter of a similar polygon circumscribed about it.

Let $C = P$, and let P′ be circumscribed about C and similar to C: then it is affirmed that cir. C : per P : : per. P : per. P'.

For, cir. C : per. $P' :: C : P' :: P : P' ::$ per.^{2} P : per.^{2} P'.

Also, per. P' : per. P - - - - - - - $::$ per.^{2} P' : per. $P \times$ per. P'.

Therefore, cir. C : per. P - - - - - - - $::$ per.^{2} P : per. $P \times$ per. P'.

$::$ per. $P ::$ per. P'. Q. E. D.

THEOREM XIV.

The circle is greater than any rectilinear figure of the same perimeter; and it has a perimeter smaller than any rectilinear figure of the same surface.

For, in the proportion, $P : C :: C : P'$ (theor. 12), since $C < P'$, therefore $P < C$.

And, in the propor. cir. C : per. $P ::$ per. P : per. P' (theor. 13),

or, cir. C : per. $P' ::$ cir.^{2} C : per.^{2} P,

and cir. $C <$ per. P'; therefore, cir.^{2} $C <$ per.^{2} P, or cir. $C <$ per P. Q. E. D.

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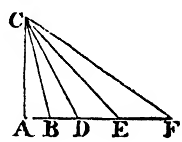
* This second corollary is introduced, not because of its immediate connexion with the subject under discussion, but because, notwithstanding its simplicity, some authors have employed whole pages in attempting its demonstration, and failed at last.

Cor. 1. It follows at once, from this and the two preceding theorems, that rectilinear figures which are isoperimeters, and each circumscribable about a circle, are respectively in the inverse ratio of the perimeters, or of the surfaces, of figures similar to them, and both circumscribed about one and the same circle. And that the perimeters of equal rectilineal figures, each circumscribable about a circle, are respectively in the subduplicate ratio of the perimeters, or of the surfaces, of figures similar to them, and both circumscribed about one and the same circle.

Cor. 2. Therefore, the comparison of the perimeters of equal regular figures, having different numbers of sides, and that of the surfaces of regular isoperimetrical figures, is reduced to the comparison of the perimeters, or of the surfaces of regular figures respectively similar to them, and circumscribable about one and the same circle.

Lemma 1.

If an acute angle of a right-angled triangle be divided into any number of equal parts, the side of the triangle opposite to that acute angle is divided into unequal parts, which are greater as they are more remote from the right angle.



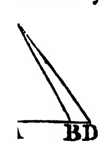
Let the acute angle C, of the right-angled triangle ACF, be divided into equal parts, by the lines BC, CD, CE, drawn from that angle to the opposite side; then shall the parts AB, BD, &c. intercepted by the lines drawn from C, be successively longer as they are more remote from the right angle A.

For, the angles ACD, BCE, &c. being bisected by CB, CD, &c. therefore by theor. 83 Geom. AC : CD :: AB : BD, and BC : CE :: BD : DE, and DC : CF :: DE : EF. And by theor. 21, Geom. CD > CA, CE > CB, CF > CC, and so on: whence it follows, that DB > AB, DE > DB, and so on. Q. E. D.

Cor. Hence it is obvious that, if the part the most remote from the right angle A, be repeated a number of times equal to that into which the acute angle is divided, there will result a quantity greater than the side opposite to the divided angle.

THEOREM XV.

If two regular figures, circumscribed about the same circle, differ in their number of sides by unity, that which has the greatest number of sides shall have the smallest perimeter.



Let CA be the radius of a circle, and AB, AD, the half sides of two regular polygons circumscribed about that circle, of which the number of sides differ by unity, being respectively $n+1$ and n. The angles ACB, ACD, therefore are respectively the $\frac{1}{n+1}$, and the $\frac{1}{n}$ th part of two right angles: consequently these angles are as n and $n+1$: and hence, the angle may be conceived divided into $n+1$ equal parts, of which BCD is one. Consequently (cor. to the lemma), $(n+1)$ BD > AD. Taking, then, unequal quantities from equal quantities, we shall have

$$(n+1)\,AD - (n+1)\,BD < (n+1)\,AD - AD,$$
$$\text{or } (n+1)\,AB < n \,.\, AD.$$

That is, the semiperimeter of the polygon whose half side is AB, is smaller than the semiperimeter of the polygon whose half side is AD: whence the proposition is manifest.

Cor. Hence, augmenting successively by unity the number of sides, it follows generally, that the perimeters of polygons circumscribed about any proposed circle, become smaller as the number of their sides becomes greater.

THEOREM XVI.

The surfaces of regular isoperimetrical figures are greater as the number of their sides is greater: and the perimeters of equal regular figures are smaller as the number of their sides is greater.

For, 1st. Regular isoperimetrical figures are (cor. 1, theor. 14) in the inverse ratio of figures similar to them circumscribed about the same circle. And (theor. 15) these latter are smaller when their number of sides is greater: therefore, on the contrary, the former become greater as they have more sides.

2dly. The perimeters of equal regular figures are (cor. 1, theor. 14) in the subduplicate ratio of the perimeters of similar figures circumscribed about the same circle: and (theor. 15) these latter are smaller as they have more sides: therefore the perimeters of the former also are smaller when the number of their sides is greater. Q. E. D.

SECTION II.—SOLIDS.

THEOREM XVII.

Of all prisms of the same altitude, whose base is given in magnitude and species, or figure, or shape, the right prism has the smallest surface.

For, the area of each face of the prism is proportional to its height; therefore the area of each face is the smallest when its height is the smallest, that is to say, when it is equal to the altitude of the prism itself: and in that case the prism is evidently a right prism. Q. E. D.

THEOREM XVIII.

Of all prisms whose base is given in magnitude and species, and whose lateral surface is the same, the right prism has the greatest altitude, or the greatest capacity.

This is the converse of the preceding theorem, and may readily be proved after the manner of theorem 2.

THEOREM XIX.

Of all right prisms of the same altitude, whose bases are given in magnitude and of a given number of sides, that whose base is a regular figure has the smallest surface.

For, the surface of a right prism of given altitude, and base given in magnitude, is evidently proportional to the perimeter of its base. But (theor. 10) the base being given in magnitude, and having a given number of sides, its perimeter is smallest when it is regular: whence, the truth of the proposition is manifest.

THEOREM XX.

Of two right prisms of the same altitude, and with irregular bases equal in surface, that whose base has the greatest number of sides has the smallest surface; and, in particular, the right cylinder has a smaller surface than any prism of the same altitude and the same capacity.

The demonstration is analogous to that of the preceding theorem, being at once deducible from theorems 16 and 14.

THEOREM XXI.

Of all right prisms whose altitudes and whose whole surfaces are equal, and whose bases have a given number of sides; that whose base is a regular figure is the greatest.

Let P, P′, be two right prisms of the same name, equal in altitude, and equal whole surface, the first of these having a regular, the second an irregular base; then is the base of the prism P′, less than the base of the prism P.

For, let P″ be a prism of equal altitude, and whose base is equal to that of the prism P′ and similar to that of the prism P. Then, the lateral surface of the prism P″ is smaller than the lateral surface of the prism P′ (theor. 19): hence, the total surface of P″ is smaller than the total surface of P′, and therefore (by hyp.) smaller than the whole surface of P. But the prisms P″ and P have equal altitudes and similar bases; therefore the dimensions of the base of P″ are smaller than the dimensions of the base of P. Consequently the base of P″, or that of P′, is less than the base of P; or the base of P greater than that of P′. Q. E. D.

THEOREM XXII.

Of two right prisms, having equal altitudes, equal total surfaces, and regular bases, that whose base has the greatest number of sides, has the greatest capacity. And, in particular, a right cylinder is greater than any right prism of equal altitude and equal total surface.

The demonstration of this is similar to that of the preceding theorem; and flows from theor. 20. [The student is required to go through it.]

THEOREM XXIII.

The greatest parallelopiped which can be contained under the three parts of a given line, any way taken, will be that constituted of equal length, breadth, and depth.

For, let AB be the given line, and, if possible, let two parts AE, ED, be unequal. Bisect AD and C, then will the rectangle under AE (= AC + CE) and ED (= AC − CE), be less

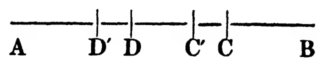
than AC², or than AC . CD, by the square of CE (theor. 33, Geom.) Consequently, the solid AE . ED . DB, will be less than the solid AC . CD . DB; which is repugnant to the hypothesis.

Cor. Hence, of all the rectangular parallelopipeds, having the sum of their three dimensions the same, the cube is the greatest.

THEOREM XXIV.

The greatest parallelopiped that can possibly be contained under the square of one part of a given line, and the other part, any way taken, will be when the former part is the double of the latter.

Let AB be a given line, and AC = 2CB, then is $AC^2 . CB$ the greatest possible.



For, let AC′ and C′B be any other parts into which the given line AB may be divided; and let AC, AC′ be bisected in DD′, respectively. Then shall $AC^2 . CB = 4AD . DC . CB$ (cor. to theor. 31, Geom.) $> 4AD' . D'C . CB$, or greater than its equal $C'A^2 . C'B$, by the preceding theorem.

THEOREM XXV.

Of all right parallelopipeds given in magnitude, that which has the smallest surface has all its faces squares, or is a cube. And reciprocally, of all parallelopipeds of equal surface, the greatest is a cube.

For, by theorems 19 and 21, the right parallelopiped having the smallest surface with the same capacity, or the greatest capacity with the same surface, has a square for its base. But any face whatever may be taken for base: therefore, in the parallelopiped whose surface is the smallest with the same capacity, or whose capacity is the greatest with the same surface, any two opposite faces whatever are squares: consequently, this parallelopiped is a cube.

THEOREM XXVI.

The capacities of prisms circumscribing the same right cylinder, are respectively as their surfaces, whether total or lateral.

For, the capacities are respectively as the bases of the prisms; that is to say (theor. 11), as the perimeters of their bases; and these are manifestly as the lateral surfaces: whence the proposition is evident.

Cor. The surface of a right prism circumscribing a cylinder, is to the surface of that cylinder, as the capacity of the former, to the capacity of the latter.

Def. The Archimedean cylinder is that which circumscribes a sphere, or whose altitude is equal to the diameter of its base.

THEOREM XXVII.

The Archimedean cylinder has a smaller surface than any other right cylinder of equal capacity; and it is greater than any other right cylinder of equal surface.

Let C and C′ denote two right cylinders, of which the first is Archimedean, the other not: then,

1st, If . . . $C = C'$, surf. $C <$ surf. C':

2dly, if surf. $C =$ surf. C', $C > C'$,

For, having circumscribed about the cylinders C, C′, the right prisms P, P′, with square bases, the former will be a cube, the second not: and the following series of equal ratios will obtain, viz. $C : P :: $ surf. C : surf. P :: base C : base P :: base C′ : base P′ :: $C' : P' ::$ surf. C′ : surf. P′.

Then, 1st: when $C = C'$. Since $C : P :: C' : P'$, it follows that $P = P'$; and therefore (theor. 25) surf. $P <$ surf. P'. But, surf. C : surf. P :: surf. C′ : surf. P′; consequently surf. $C <$ surf. C'. Q. E. 1D.

2dly. When surf. C = surf. C′. Then, since surf. C : surf. P : : surf. C′ : surf. P′, it follows that surf. P = surf. P′; and therefore (theor. 25) P > P′. But C : P : : C′ : P′ : consequently C > C′. Q. E. 2D.

THEOREM XXVIII.

Of all right prisms whose bases are circumscribable about circles, and given in species, that whose altitude is double the radius of the circle inscribed in the base, has the smallest surface with the same capacity, and the greatest capacity with the same surface.

This may be demonstrated exactly as the preceding theorem, by supposing cylinders inscribed in the prisms.

Scholium.

If the base cannot be circumscribed about a circle, the right prism which has the minimum surface, or the maximum capacity, is that whose lateral surface is quadruple of the surface of one end, or that whose lateral surface is two-thirds of the total surface. This is manifestly the case with the Archimedean cylinder; and the extension of the property depends solely on the mutual connection subsisting between the properties of the cylinder, and those of circumscribing prisms.

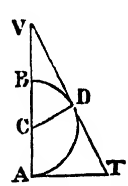
THEOREM XXIX.

The surfaces of right cones circumscribed about a sphere, are as their solidities.

For, it may be demonstrated, in a manner analogous to the demonstrations of theorems 11 and 26, that these cones are equal to right cones, whose altitude is equal to the radius of the inscribed sphere, and whose bases are equal to the total surfaces of the cones: therefore the surfaces and solidities are proportional.

THEOREM XXX.

The surface or the solidity of a right cone circumscribed about a sphere is directly as the square of the cone's altitude, and inversely as the excess of that altitude over the diameter of the sphere.



Let VAT be a right-angled triangle which, by its rotation upon VA as an axis, generates a right cone; and BDA the semicircle which by a like rotation upon VA forms the inscribed sphere: then, the surface or the solidity of the cone varies as $\frac{VA^2}{VB}$.

For, draw the radius CD to the point of contact of the semicircle and VT. Then, because the triangles VAT, VDC, are similar, it is AT : VT : : CD : VC.

And, by compos. AT : AT + VT : : CD : CD + CV VA;

Therefore AT² : (AT + VT) AT : : CD : VA, by multiplying the terms of the first ratio by AT.

But, because VB, VD, VA are continued proportions, it is VB : VA : : VD² : VA² :: CD² : AT² by sim. triangles. But CD : VA : : AT² : (AT + VT) AT by the last: and these mult. give CD . VB : VA² : : CD² : (AT + VT) AT,

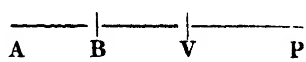
or VB : CD : : VA² : (AT + VT) AT = CD . $\frac{VA^2}{VB}$.

But the surface of the cone, which is denoted by $\pi \cdot AT^2 + \pi \cdot AT \cdot VT$*, is manifestly proportional to the first member of this equation, and is also proportional to the second member, or, since CD is constant, it is proportional to $\frac{AV^2}{VB}$, or to a third proportional to BV and AV. And, since the capacities of these circumscribing cones are as their surfaces (theor. 29), the truth of the whole proposition is evident.

Lemma 2.

The difference of two right lines being given, the third proportional to the less and the greater of them is a minimum when the greater of those lines is double the other.

Let AV and BV be two right lines, whose difference AB is given, and let AP be a third proportional to BV and AV; then is AP a minimum when $AV = 2BV$.



For, since AP : AV   AV   BV;
By division AP : AP—AV   AV   AV — BV;
That is, AP . VP   AV   AB.
Hence, $VP \cdot AV = AP \cdot AB$.

But $VP \cdot AV$ is either $=$ or $< \frac{1}{4} AP^2$ (cor. to theor. 31, Geom. and theor. 23 of this chapter).

Therefore $AP \cdot AB \leqq \frac{1}{4} AP^2$; whence $4AB \leqq AP$, or $AP \geqq 4AB$. Consequently the minimum value of AP is the quadruple of AB; and in that case $PV = VA = 2AB$. Q, E. D.

THEOREM XXXI.

Of all right cones circumscribed about the same sphere, the smallest is that whose altitude is double the diameter of the sphere.

For, by theor. 30, the solidity varies as $\frac{VA^2}{VB}$ (see the fig. to that theorem): and, by lemma 2, since $VA - VB$ is given, the third proportional $\frac{VA^2}{VB}$ is a minimum when $VA = 2AB$. Q. E. D.

Cor. 1. Hence, the distance from the centre of the sphere to the vertex of the least circumscribing cone, is triple the radius of the sphere.

Cor. 2. Hence also, the side of such cone is triple the radius of its base.

THEOREM XXXII.

The whole surface of a right cone being given, the inscribed sphere is the greatest when the slant side of the cone is triple the radius of its base.

For, let C and C′ be two right cones of equal whole surface, the radii of their respective inscribed spheres being denoted by R and R′; let the side of the cone C be triple the radius of its base, the same ratio not obtaining in C′; and let C″ be a cone similar to C, and circumscribed about the same sphere with C′. Then, (by theor. 31) surf. C″ < surf. C′: therefore surf. C″ < surf. C. But

* $\pi = 3{\cdot}141533$.

C″ and C are similar, therefore all the dimensions of C″ are less than the corresponding dimensions of C: and consequently the radius R′ of the sphere inscribed in C″ or in C′, is less than the radius R of the sphere inscribed in C, or R > R′. Q. E. D.

Cor. The capacity of a right cone being given, the inscribed sphere is the greatest when the side of the cone is triple the radius of its base.

For the capacities of such cones vary as their surfaces (theor. 29).

THEOREM XXXIII.

Of all right cones of equal whole surface, the greatest is that whose side is triple the radius of its base: and reciprocally, of all right cones of equal capacity, that whose side is triple the radius of its base has the least surface.

For, by theor. 29, the capacity of a right cone is in the compound ratio of its whole surface and the radius of its inscribed sphere. Therefore, the whole surface being given, the capacity is proportional to the radius of the inscribed sphere: and consequently is a maximum when the radius of the inscribed sphere is such; that is, (theor. 32) when the side of the cone is triple the radius of the base.

Again, reciprocally, the capacity being given, the surface is in the inverse ratio of the sphere inscribed: therefore, it is the smallest when that radius is the greatest; that is, (theor. 32) when the side of the cone is triple the radius of its base. Q. E. D.

THEOREM XXXIV.

The surfaces, whether total or lateral, of pyramids circumscribed about the same right cone, are respectively as their solidities. And, in particular, the surface of a pyramid circumscribed about a cone, is to the surface of that cone, as the solidity of the pyramid is to the solidity of the cone; and these ratios are equal to those of the surfaces or the perimeters of the bases.

For, the capacities of the several solids are respectively as their bases; and their surfaces are as the perimeters of those bases: so that the proposition may manifestly be demonstrated by a chain of reasoning exactly like that adopted in theorem 11

THEOREM XXXV.

The base of a right pyramid being given in species, the capacity of that pyramid is a maximum with the same surface, and, on the contrary, the surface is a minimum with the same capacity, when the height of one face is triple the radius of the circle inscribed in the base.

Let P and P′ be two right pyramids with similar bases, the height of one lateral face of P being triple the radius of the circle inscribed in the base, but this proportion not obtaining with regard to P′: then

1st. If surf. $P =$ surf. P', $P > P'$.

2dly. If . . $P =$ . . P', surf. $P <$ surf. P'.

For, let C and C′ be right cones inscribed within the pyramids P and P′: then, in the cone C, the slant side is triple the radius of its base, while this is not the case with respect to the cone C′. Therefore, if $C = C'$, surf. $C <$ surf. C': and, if surf. $C =$ surf. C', $C > C'$ (theor. 33).

But, 1st surf. P : surf, C :: surf. P′ : surf. C′;

whence, if surf. $P =$ surf. P', surf. $C =$ surf. C';

therefore $C > C'$. But $P : C :: P' : C'$. Therefore $P > P'$.

2dly. P : C :: P′ : C. Theref. if P = P′, C = C; consequently surf. C < surf. C′. But surf. P : surf. C :: surf. P′ : surf. C′. Whence, surf. P < surf. P′.

Cor. The regular tetrahedron possesses the property of the minimum surface with the same capacity, and of the maximum capacity with the same surface, relatively to all right pyramids with equilateral triangular bases, and, *à fortiori*, relatively to every other triangular pyramid.

THEOREM XXXVI.

A SPHERE is to any circumscribing solid, bounded by plane surfaces, as the surface of the sphere to that of the circumscribing solid.

For, since all the planes touch the sphere, the radius drawn to each point of contact will be perpendicular to each respective plane. So that, if planes be drawn through the centre of the sphere and through all the edges of the body, the body will be divided into pyramids whose bases are the respective planes, and their common altitude the radius of the sphere. Hence, the sum of all these pyramids, or the whole circumscribing solid, is equal to a pyramid or a cone whose base is equal to the whole surface of that solid, and altitude equal to the radius of the sphere. But the capacity of the sphere is equal to that of a cone whose base is equal to the surface of the sphere, and altitude equal to its radius. Consequently, the capacity of the sphere, is to that of the circumscribing solid, as the surface of the former to the surface of the latter: both having, in this mode of considering them, a common altitude. Q. E. D.

Cor. 1. All circumscribing cylinders, cones, &c. are to the sphere they circumscribe, as their respective surfaces.

For the same proportion will subsist between their indefinitely small corresponding segments, and therefore between their wholes.

Cor. 2. All bodies circumscribing the same sphere, are respectively as their surfaces.

THEOREM XXXVII.

THE sphere is greater than any polyhedron of equal surface.

For, first it may be demonstrated, by a process similar to that adopted in theorem 9, that a *regular* polyhedron has a greater capacity than any other polyhedron of equal surface to a sphere S. Then P must either circumscribe S, or fall partly within it and partly without it, or fall entirely within it. The first of these suppositions is contrary to the hypothesis of the proposition, because in that case the surface of P could not be *equal* to that of S. Either the 2d or 3d supposition therefore must obtain; and then each plane of the surface of P must fall either partly or wholly within the sphere S: whichever of these be the case, the perpendiculars demitted from the centre of S upon the planes, will be each less than the radius of that sphere: and consequently the polyhedron P must be less than the sphere S, because it has an equal base, but a less altitude.

Q. E. D.

Cor. If a prism, a cylinder, a pyramid, or a cone, be equal to a sphere either in capacity, or in surface; in the first case, the surface of the sphere is less than the surface of any of those solids; in the second, the capacity of the sphere is greater than that of either of those solids.

The theorems in this chapter will suggest a variety of practical examples to exercise the student in computation. A few such are given in the two next pages.

EXERCISES.

Ex. 1. Find the areas of an equilateral triangle, a square, a hexagon, a dodecagon, and a circle, the perimeter of each being 36.

Ex. 2. Find the difference between the area of a triangle whose sides are 3, 4, and 5, and of an equilateral triangle of equal perimeter.

Ex. 3. What is the area of the greatest triangle which can be constituted with two given sides 8 and 11: and what will be the length of its third side?

Ex. 4. The circumference of a circle is 12, and the perimeter of an irregular polygon which circumscribes it is 15: what are their respective areas?

Ex. 5. Required the surface and the solidity of the greatest parallelopiped, whose length, breadth, and depth, together make 18?

Ex. 6. The surface of a square prism is 546: what is its solidity when a maximum?

Ex. 7. The content of a cylinder is 169·645968: what is its surface when a minimum?

Ex. 8. The whole surface of a right cone is 201·061952: what is its solidity when a maximum?

Ex. 9. The surface of a triangular pyramid is 43·30127: what is its capacity when a maximum?

Ex. 10. The radius of a sphere is 10. Required the solidities of this sphere, of its circumscribed equilateral cone, and of its circumscribed cylinder

Ex. 11. The surface of a sphere is 28·274337, and of an irregular polyhedron circumscribed about it 35: what are their respective solidities?

Ex. 12. The solidity of a sphere, equilateral cone, and Archimedean cylinder, is each 500: what are the surfaces and respective dimensions of each?

Ex. 13. If the surface of a sphere be represented by the number 4, the circumscribed cylinder's convex surface and whole surface will be 4 and 6, and the circumscribed equilateral cone's convex and whole surface, 6 and 9 respectively. Show how these numbers are deduced.

Ex. 14. The solidity of a sphere, circumscribed cylinder, and circumscribed equilateral cone, are as the numbers, 4, 6, and 9. Required the proof.

Ex. 15. The area of the triangle is double the area of the inscribed circle: one of the angles is one-fifth of the sum of the other two: and the diameter of the circumscribing circle is 12. Find the sides, angles, and area of the triangle, and the radius of its inscribed circle.

Ex. 16. Two right-angled triangles are described about the same circle, having the following properties: the difference of their areas is 1·5 times the area of the circle; the sum of their areas is 6 times the area of the circle; the sum of the two legs of the one is triple that of the other; and the sum of their hypothenuses is 18. Find the angles and sides of the two triangles, and the ratio of the radii of the inscribed and circumscribed circles.

Ex. 17. The edge of the regular tetrahedron is 10, find the volumes of the inscribed and circumscribed spheres, and the ratio of the surface of the tetrahedron to that of the former sphere. Find also the vertical angle of the right cone circumscribing the former sphere, and the distance of its vertex from the centre of the sphere, also the ratio of its curve surface to the curve surface of the Archimedean cylinder.

PLANE TRIGONOMETRY CONSIDERED ANALYTICALLY.

Art. 1. There are two methods which are adopted by mathematicians in investigating the theory of Trigonometry; the one *Geometrical*, the other *Algebraical*. In the former, the various relations of the sines, cosines, tangents, &c. of single or multiple arcs or angles, and those of the sides and angles of triangles, are deduced immediately from the figures to which the several inquiries are referred; each individual case requiring its own particular method, and resting on evidence peculiar to itself. In the latter, the nature and properties of the linear-angular quantities (sines, tangents, &c.) being first defined, some general relation of these quantities, or of them in connection with a triangle, is expressed by one or more algebraical equations; and then every other theorem or precept, of use in this branch of science, is developed by the simple reduction and transformation of the primitive equation. Thus, the rules for the three fundamental cases in Plane Trigonometry, which are deduced by three independent geometrical investigations, in the first volume of this Course of Mathematics, are obtained algebraically, by forming, between the three data and the three unknown quantities, three equations, and obtaining, in expressions of known terms, the value of each of the unknown quantities, the others being exterminated by the usual processes. Each of these general methods has its peculiar advantages. The geometrical method carries conviction at every step; and by keeping the objects of inquiry constantly before the eye of the student, serves admirably to guard him against the admission of error: the algebraical method, on the contrary, requiring little aid from first principles, but merely at the commencement of its career is more properly mechanical than mental, and requires frequent checks to prevent any deviation from truth. The geometrical method is direct, and rapid, in producing the requisite conclusions at the outset of trigonometrical science: but slow and circuitous in arriving at those results which the modern state of the science requires: while the algebraical method, though sometimes circuitous in the developement of the mere elementary theorems, is very rapid and fertile in producing those curious and interesting formulæ, which are wanted in the higher branches of pure analysis, and in mixed mathematics, especially in Physical Astronomy. This mode of developing the theory of Trigonometry is, consequently, well suited for the use of the more advanced student: and is therefore introduced here with as much brevity as is consistent with its nature and utility.

2. To save the trouble of turning very frequently to the 1st volume, a few of the principal definitions, there given, are here repeated, as follows:

The sine of an arc, is the perpendicular let fall from one of its extremities upon the diameter of the circle, which passes through the other extremity.

The cosine of an arc, is the sine of the complement of that arc, and is equal to the part of the radius comprised between the centre of the circle and the foot of the sine.

The tangent of an arc, is a line which touches the circle in one extremity of that arc, and is continued from thence till it meets a line drawn from or through the centre and through the other extremity of the arc.

The secant of an arc, is the radius drawn through one of the extremities of that arc, and prolonged till it meets the tangent drawn from the other extremity.

The VERSED SINE of an arc, is that part of the diameter of the circle which lies between the beginning of the arc and the foot of the sine.

The COTANGENT, COSECANT, and COVERSED SINE of an arc, are the tangent, secant, and versed sine, of the complement of such arc.

3. Since arcs are proper and adequate measures of plane angles (the ratio of any two plane angles being constantly equal to the ratio of the two arcs of any circle whose centre is the angular point, and which are intercepted by the lines whose inclinations form the angle), it is usual, and it is perfectly safe, to apply the above names without circumlocution as though they referred to the angles themselves: thus, when we speak of the sine, tangent, or secant, of an angle, we mean the sine, tangent, or secant, of the arc which measures that angle; the radius of the circle employed being known.

4. It has been shown in the 1st vol. (p. 389), that the tangent is a fourth proportional to the cosine, sine, and radius; the secant, a third proportional to the cosine and radius; the cotangent, a fourth proportional to the sine, cosine, and radius; and the cosecant a third proportional to the sine and radius. Hence, making use of the obvious abbreviations, and converting the analogies into equations, we have

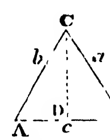
$$\text{tan.} = \frac{\text{rad.} \times \text{sine}}{\text{cos.}},\ \text{cot.} = \frac{\text{rad.} \times \text{cos.}}{\text{sine}},\ \text{sec.} = \frac{\text{rad.}^2}{\text{cos.}},\ \text{cosec.} = \frac{\text{rad.}^2}{\text{sine}}.$$

Or, assuming unity for the rad. of the circle, these will become

$$\text{tan.} = \frac{\text{sin.}}{\text{cos.}} \ \ldots\ \text{cot.} = \frac{\text{cos.}}{\text{sin.}} = \frac{1}{\text{tan.}} \ \ldots\ \text{sec.} = \frac{1}{\text{cos.}} \ \ldots.\ \text{cosec.} = \frac{1}{\text{sin.}}.$$

These preliminaries being borne in mind, the student may pursue his investigations.

5. Let ABC be any plane triangle, of which the side BC opposite the angle A is denoted by the small letter a, the side AC opposite the angle B by the small letter b, and the side AB opposite the angle C by the small letter c, and CD perpendicular to AB: then is $c = a$ cos. B $+ b$ cos. A.



For, since AC $= b$, AD is the cosine of A to that radius; consequently, supposing radius to be unity, we have AD $= b$ cos. A. In like manner it is BD $= a$ cos. B. Therefore, AD + BD $=$ AB $= c = a$ cos. B $+ b$ cos. A. By pursuing similar reasoning with respect to the other two sides of the triangle, exactly analogous results will be obtained. Placed together, they will be as below:

$$\left.\begin{aligned} a &= b \text{ cos. C} + c \text{ cos. B} \\ b &= a \text{ cos. C} + c \text{ cos. A} \\ c &= a \text{ cos. B} + b \text{ cos. A} \end{aligned}\right\} \quad \text{(I.)}$$

6. Now, if from these equations it were required to find expressions for the angles of a plane triangle, when the sides are given; we have only to multiply the first of these equations by a, the second by b, the third by c, and to subtract each of the equations thus obtained from the sum of the other two. For thus we shall have

$$\left.\begin{aligned} b^2 + c^2 - a^2 &= 2bc \text{ cos. A, whence cos. A} = \frac{b^2 + c^2 - a^2}{2bc} \\ a^2 + c^2 - b^2 &= 2ac \text{ cos. B,} \quad . \quad . \quad . \text{ cos. B} = \frac{a^2 + c^2 - b^2}{2ac} \\ a^2 + b \ - c &= 2ab \text{ cos. C,} \quad . \quad . \quad . \text{ cos. C} = \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\} \quad \text{(II.)}$$

7. More convenient expressions than these will be deduced hereafter: but

even these will often be found very convenient, when the sides of triangles are expressed in integers, and tables of sines and tangents, as well as a table of squares, (like that in our first vol.) are at hand.

Suppose, for example, the sides of the triangle are $a = 320$, $b = 562$, $c = 800$, being the numbers given in prop. 4, of the Introduction to the Mathematical Tables: then we have

$b^2 + c^2 - a^2 = 853444$ . . . . . log. $= 5\cdot9311751$

$2bc$ . . . $= 899200$ . . . . . log. $= 5\cdot9538080$

The remainder being log. cos. A, or of $18°\ 20' = 9\cdot9773671$

Again, $a^2 + c^2 - b^2 = 426556$ . . log. $= 5\cdot6299760$

$2ab$ . . . $= 512000$ . . log. $= 5\cdot7092700$

The remainder being log. cos. B, or of $33°\ 35' = 9\cdot9207060$

Then $180° - (18°\ 20' + 33°\ 35') = 128°\ 5' = C$; where all the three angles are determined in 7 lines.

8. If it were wished to get expressions for the sines, instead of the cosines, of the angles; it would merely be necessary to introduce into the preceding equations (marked II), instead of cos. A, cos. B, &c. their equivalents, cos. $A = \sqrt{(1 - \sin.^2 A)}$, cos. $B = \sqrt{(1 - \sin.^2 B)}$, &c. For then, after a little reduction, there would result,

$$\left.\begin{aligned} \sin. A &= \frac{1}{2bc}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)} \\ \sin\ B &= \frac{1}{2ac}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)} \\ \sin. C &= \frac{1}{2ab}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)} \end{aligned}\right\}$$

Or, resolving the expression under the radical into its four constituent factors, substituting $2s$ for $a + b + c$, and reducing, the equations will become

$$\left.\begin{aligned} \sin. A &= \frac{2}{bc}\sqrt{s\,(s-a)\,(s-b)\,(s-c)} \\ \sin. B &= \frac{2}{ac}\sqrt{s\,(s-a)\,(s-b)\,(s-c)} \\ \sin. C &= \frac{2}{ab}\sqrt{s\,(s-a)\,(s-b)\,(s-c)} \end{aligned}\right\} \quad \text{(III.)}$$

These equations are moderately well suited for computation in their latter form; they are also perfectly symmetrical: and as indeed the quantities under the radical are identical, and are constituted of known terms, they may be represented by the same character; suppose K: then shall we have

$$\sin. A = \frac{2K}{bc} \ldots \sin. B = \frac{2K}{ac} \ldots \sin. C = \frac{2K}{ab} \ldots (iii.)$$

Hence we may immediately deduce a very important theorem: for, the first of these equations, divided by the second, gives $\frac{\sin. A}{\sin. B} = \frac{a}{b}$, and the first divided by the third gives $\frac{\sin. A}{\sin. C} = \frac{a}{c}$: whence, since two equal fractions denote a proportion (vol. i. p. 83), we have

$$\sin. A : \sin. B : \sin. C :: a : b : c \ldots \text{(IV.)}$$

Or, in words, *the sides of plane triangles are proportional to the sines of their opposite angles*. (See theor. 1, Trig. vol i. p 390.)

9. Before the remainder of the theorems, necessary in the solution of plane triangles, are investigated, the fundamental proposition in the theory of sines,

etc. must be deduced, and the method explained by which Tables of these quantities, confined within the limits of the quadrant, are made to extend to the whole circle, or to any number of quadrants whatever. In order to this, expressions must be first obtained for the sines, cosines, &c. of the sums and differences of any two arcs or angles. Now, it has been found (I.) that $a = b \cos. C + c \cos. B$. And the equations (IV.) give $b = a \cdot \frac{\sin. B}{\sin. A}$ . . . and $c = a \frac{\sin. C}{\sin. A}$. Substituting these values of b and c for them in the preceding equation, and multiplying the whole by $\frac{\sin. A}{a}$, it will become

$$\sin. A = \sin. B \cos. C + \sin. C \cos. B.$$

But, in every plane triangle, the sum of the three angles is equal to two right angles; therefore, B and C are equal to the supplement of A: and, consequently, since an angle and its supplement have the same sine (cor. 1, p. 385, vol. i.), we have $\sin. (B + C) = \sin. B \cos. C + \sin. C \cos. B$.

10. If, in the last equation, C become subtractive, then would sin. C manifestly become subtractive also, while the cosine of C would not change its sign, since it would still continue to be estimated on the same radius in the same direction. Hence the preceding equation would become

$$\sin. (B - C) = \sin. B \cos. C - \sin. C \cos. B.$$

11. Let C′ be the complement of C, and $\frac{1}{4}\bigcirc$ be the quarter of the circumference: then will $C' = \frac{1}{4}\bigcirc - C$, $\sin. C' = \cos. C$, and $\cos. C' = \sin. C$. But (art. 10), $\sin. (B - C') = \sin. B \cos C' - \sin. C' \cos. B$. Therefore, substituting for sin. C′, cos. C′, their values, there will result $\sin. (B - C') = \sin. B, \sin. C - \cos. B \,.\, \cos. C$. But because $C' = \frac{1}{4}\bigcirc - C$, we have $\sin. (B - C') = \sin. (B + C - \frac{1}{4}\bigcirc) = \sin. [(B + C) - \frac{1}{4}\bigcirc] = - \sin. [\frac{1}{4}\bigcirc - (B + C)] = - \cos. (B + C)$ Substituting this value of sin. (B — C′) in the equation above, it becomes $\cos. (B + C) = \cos. B \,.\, \cos. C - \sin. B \,.\, \sin. C$.

12. In this latter equation, if C be made subtractive, sin. C will become — sin. C, while cos. C will not change: consequently the equation will be transformed to the following, viz. $\cos. (B - C) = \cos. B \,.\, \cos. C + \sin. B \,.\, \sin. C$.

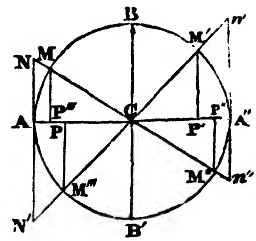
If, instead of the angles B and C, the angles had been A and B; or, if A and B represented the *arcs* which measure those angles, the results would evidently be similar: they may therefore be expressed generally by the two following equations, for the sines and cosines of the sums or differences of any two arcs or angles:

$$\left.\begin{array}{l}\sin. (A \pm B) = \sin. A \cos. B \pm \sin. B \cos. A. \\ \cos. (A \pm B) = \cos. A \cos. B \mp \sin. A \sin. B.^*\end{array}\right\} \quad (V.)$$

13. We are now in a state to trace completely the mutations of the sines, cosines, &c. as they relate to arcs in the various parts of a circle; and thence to perceive that tables which apparently are included within a quadrant, are, in fact, applicable to the whole circle.

Imagine that the radius MC of the circle, in the marginal figure, coinciding at first with AC, turns about the point C (in the same manner as a rod would turn on a pivot) and thus forming successively with AC all possible angles: the point M at its extremity passing over all the points of the circumference

* See, for a different mode of investigating these and some other useful formulæ, vol. i. pp. 401—5.



ABA′B′A, or describing the whole circle. Tracing this motion attentively, it will appear, that at the point A, where the arc is nothing, the sine is nothing also, while the cosine does not differ from the radius. As the radius MC recedes from AC, the sine PM keeps increasing, and the cosine CP decreasing, till the describing point M has passed over a quadrant, and arrived at B: in that case, PM becomes equal to CB the radius, and the cosine CP vanishes. The point M continuing its motion beyond B, the sine P′M′ will diminish, while the cosine C′P′, which now falls on the *contrary* side of the centre C, will increase. In the figure, P′M′ and CP′ are respectively the sine and cosine of the arc A′M′, or the sine and cosine of ABM′, which is the supplement of A′M′ to $\frac{1}{2}$○, half the circumference: whence it follows that an obtuse angle (measured by an arc greater than a quadrant) has *the same sine and cosine as its supplement;* the cosine, however, being reckoned subtractive or negative, because it is situated contrariwise with regard to the centre C.

When the describing point M has passed over $\frac{1}{2}$○, or half the circumference, and has arrived at A′, the sine P′M′ vanishes, or becomes nothing, as at the point A, and the cosine is again equal to the radius of the circle. Here the angle ACM has attained its maximum limit; but the radius CM may still be supposed to continue its motion, and pass *below* the diameter AA′. The sine. which will then be P″M″, will consequently fall below the diameter, and will augment as M moves along the third quadrant, while on the contrary CP″, the cosine, will diminish. In this quadrant, too, both sine and cosine must be considered as negative; the former being on a contrary side of the diameter, the latter a contrary side of the centre, to what each was respectively in the first quadrant. At the point B′, where the arc is three-fourths of the circumference, or $\frac{3}{4}$○, the sine P″M″ becomes equal to the radius CB, and the cosine CP″ vanishes. •Finally, in the fourth quadrant, from B′ to A, the sine P‴M‴, always *below* AA′, diminishes in its progress, while the cosine CP‴, which is then found on the same side of the centre as it was in the first quadrant, augments till it becomes equal to the radius CA. Hence the sine in this quadrant is to be considered as negative or subtractive, the cosine as positive. If the motion of M were continued through the circumference again, the circumstances would be exactly the same in the fifth quadrant as in the first, in the sixth as in the second, in the seventh as in the third, in the eighth as in the fourth: and the like would be the case in any subsequent revolutions.

14. If the mutations of the *tangent* be traced in like manner, it will be seen that its magnitude passes from nothing to infinity in the first quadrant; becomes negative, and decreases from infinity to nothing, in the second; becomes positive again, and increases from nothing to infinity, in the third quadrant; and lastly, becomes negative again, and decreases from infinity to nothing, in the fourth quadrant.

15. These conclusions admit of a ready confirmation, and others may be deduced, by means of the analytical expressions in arts. 4 and 12. Thus, if A be supposed equal to $\frac{1}{4}$○, in equa. V, it will become

$$\cos.\ (\tfrac{1}{4}\bigcirc \pm B) = \cos.\ \tfrac{1}{4}\bigcirc\ \cos.\ B \mp \sin.\ \tfrac{1}{4}\bigcirc\ \sin.\ B,$$
$$\sin.\ (\tfrac{1}{4}\bigcirc \pm B) = \sin.\ \tfrac{1}{4}\bigcirc\ \cos.\ B \pm \sin.\ B\ \cos.\ \tfrac{1}{4}\bigcirc.$$
$$\text{But } \sin.\ \tfrac{1}{4}\bigcirc = \text{rad.} = 1;\ \text{and } \cos.\ \tfrac{1}{4}\bigcirc = 0:$$

so that the above equations will become

$$\cos.\ (\tfrac{1}{4}\bigcirc \pm B) = \mp \sin.\ B.$$
$$\sin.\ (\tfrac{1}{4}\bigcirc \pm B) = \cos.\ B.$$

From which it is obvious, that if the sine and cosine of an arc, less than a quadrant, be regarded as positive, the cosine of an arc greater than $\frac{1}{4}\bigcirc$ and less than $\frac{1}{2}\bigcirc$ will be negative, but its sine positive. If B also be made $= \frac{1}{4}\bigcirc$; then shall we have $\cos.\ \frac{1}{2}\bigcirc = -1$; $\sin.\ \frac{1}{2}\bigcirc = 0$.

Suppose next, that in the equa. V, $A = \frac{1}{2}\bigcirc$; then shall we obtain

$$\cos.\ (\tfrac{1}{2}\bigcirc \pm B) = -\cos.\ B\ .$$
$$\sin.\ (\tfrac{1}{2}\bigcirc \pm B) = \mp \sin.\ B;$$

which indicates, that every arc comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{4}\bigcirc$, or that terminates in the third quadrant, will have its sine and its cosine both negative. In this case, too, when $B = \frac{1}{4}\bigcirc$, or the arc terminates at the *end* of the third quadrant, we shall have $\cos.\ \frac{3}{4}\bigcirc = 0$, $\sin.\ \frac{3}{4}\bigcirc = -1$.

Lastly, the case remains to be considered in which $A = \frac{3}{4}\bigcirc$, or in which the arc terminates in the fourth quadrant. Here the primitive equations (V.) give

$$\cos.\ (\tfrac{3}{4}\bigcirc \pm B) = \pm \sin.\ B\ .$$
$$\sin.\ (\tfrac{3}{4}\bigcirc \pm B) = -\cos.\ B;$$

so that in all arcs between $\frac{3}{4}\bigcirc$ and $\bigcirc$, the cosines are positive and the sines negative.

16. The changes of the tangents, with regard to positive and negative, may be traced by the application of the preceding results to the algebraic expression for tangent; viz. $\tan. = \frac{\sin.}{\cos.}$. For it is hence manifest, that when the sine and cosine are either both positive or both negative, the tangent will be positive; which will be the case in the first and third quadrants. But when the sine and cosine have different signs, the tangents will be negative, as in the second and fourth quadrants. The algebraic expression for the cotangent, viz. $\cot. = \frac{\cos.}{\sin.}$, will produce exactly the same results.

The expressions for the secants and cosecants, viz. $\sec. = \frac{1}{\cos.}$, $\operatorname{cosec.} = \frac{1}{\sin.}$, show, that the signs of the secants are the same as those of the cosines; and those of the cosecants the same as those of the sines.

The *magnitude* of the tangent at the end of the first and third quadrants will be infinite; because in those places the sign is equal to radius, the cosine equal to zero, and therefore $\frac{\sin.}{\cos.} = \frac{1}{0} = \infty$ (infinity). Of these, however, the former will be reckoned positive, the latter negative.

17. The magnitudes of the cotangents, secants, and cosecants, may be traced in like manner; and the results of the 13th, 14th, and 15th articles, recapitulated and tabulated as follows:

	0°	90°	180°	270°	360°	
Sin.	0	R	0	− R	0	
Tan.	0	∞	0	− ∞	0	
Sec.	R	∞	− R	− ∞	R	(VI.)
Cos.	R	0	− R	0	R	
Cot	∞	0	− ∞	0	∞	
Cosec.	∞	R	− ∞	− R	∞	

The changes of signs are these:

					sin.	cos.	tan.	cot.	sec.	cosec.	
1st.	5th.	9th.	13th.	quadrants.	+	+	+	+	+	+	(VII.)
2d.	6th.	10th.	14th.		+	−	−	−	−		
3d.	7th.	11th.	15th.		−	−	+	+	−	−	
4th.	8th.	12th.	16th.		−	+	−	−	+	−	

We have been thus particular in tracing the mutations, both with regard to value and algebraic signs, of the principal trigonometrical quantities, because a knowledge of them is absolutely necessary in the application of trigonometry to the solution of equations, and to various astronomical and physical problems.

18. We may now proceed to the investigation of other expressions relating to the sums, differences, multiples, &c. of arcs; and in order that these expressions may have the more generality, give to the radius any value R, instead of confining it to unity. This indeed may always be done in an expression, however complex, by merely rendering all the terms homogeneous; that is, *by multiplying each term by such a power of* R *as shall make it of the same dimension, as the term in the equation which has the highest dimension.* Thus, the expression for a triple arc

$$\sin. 3A = 3 \sin. A - 4 \sin.^3 A \text{ (radius } = 1)$$

becomes, when radius is assumed $= R$,

$$R^2 \sin. 3A = R^2\, 3\sin. A - 4\sin.^3 A,$$

$$\text{or } \sin. 3A = \frac{3R^2 \sin. A - 4\sin.^3 A}{R^2}.$$

Hence then, if, consistently with this precept, R be placed for a denominator of the second member of each equation V. (art. 12), and if A be supposed equal to B, we shall have

$$\sin. (A + A) = \frac{\sin. A \cos. A + \sin. A \cos. A}{R}.$$

$$\text{That is, } \sin. 2A = \frac{2\sin. A \cos. A}{R.}$$

And, in like manner, by supposing B to become successively equal to 2A, 3A, 4A, &c., there will arise

$$\left.\begin{aligned}\sin. 3A &= \frac{\sin. A \cos. 2A + \cos. A \sin. 2A}{R}\\ \sin. 4A &= \frac{\sin. A \cos. 3A + \cos. A \sin. 3A}{R}\\ \sin. 5A &= \frac{\sin. A \cos. 4A + \cos. A \sin. 4A}{R}\end{aligned}\right\} \text{(VIII.)}$$

And, by similar processes, the second of the equations just referred to namely, that for cos. (A + B), will give successively,

$$\left.\begin{aligned}\cos. 2A &= \frac{\cos.^2 A - \sin.^2 A}{R}\\ \cos. 3A &= \frac{\cos. A \cos. 2A - \sin. A \sin. 2A}{R}\\ \cos. 4A &= \frac{\cos. A \cos. 3A - \sin. A \sin. 3A}{R}\\ \cos. 5A &= \frac{\cos. A \cos. 4A - \sin. A \sin. 4A}{R}\end{aligned}\right\} \text{(IX.}$$

19. If in the expressions for the successive multiples of the sines, the values of the several cosines in terms of the sines were substituted for them, and a like process were adopted with regard to the multiples of the cosines, other expressions would be obtained, in which the multiple sines would be expressed in

terms of the radius and sine, and the multiple cosines in terms of the radius and cosine

$$\left.\begin{aligned}
\sin.\ A &= \sin.\ A \\
\sin.\ 2A &= 2 \sin.\ A \cos.\ A \\
\sin.\ 3A &= 3 \sin.\ A - 4 \sin.^3 A \\
\sin.\ 4A &= (4 \sin.\ A - 8 \sin.^3 A) \cos.\ A \\
\sin.\ 5A &= 5 \sin.\ A - 20 \sin.^3 A + 16 \sin.^5 A. \\
\sin.\ 6A &= (6 \sin.\ A - 32 \sin.^3 A + 32 \sin.^5 A) \cos.\ A. \\
&\&c. \qquad \&c. \qquad \&c.
\end{aligned}\right\} \text{(X.)}$$

$$\left.\begin{aligned}
\cos.\ A &= \cos.\ A \\
\cos.\ 2A &= 2 \cos.^2 A - 1 \\
\cos.\ 3A &= 4 \cos.^3 A - 3 \cos.\ A \\
\cos.\ 4A &= 8 \cos.^4 A - 8 \cos.^2 A + 1 \\
\cos.\ 5A &= 16 \cos.^5 A - 20 \cos.^3 A + 5 \cos.\ A \\
\cos.\ 6A &= 32 \cos.^6 A - 48 \cos.^4 A + 18 \cos.^2 A - 1 \\
&\&c. \qquad \&c. \qquad \&c.^*
\end{aligned}\right\} \text{(XI.)}$$

Other very convenient expressions for multiple arcs may be obtained thus.

Add together the expanded expressions for $\sin.\ (B + A)$, $\sin.\ (B - A)$, that is,

add - - - - $\sin.\ (B + A) = \sin.\ B \cos\ A + \cos.\ B \sin.\ A,$

to - - - - - $\sin.\ (B - A) = \sin.\ B \cos.\ A - \cos.\ B \sin.\ A;$

there results - $\sin.\ (B + A) + \sin.\ (B - A) = 2 \cos.\ A \sin.\ B:$

whence - - - $\sin.\ (B + A) = 2 \cos.\ A \sin.\ B - \sin.\ (B - A).$

Thus again, by adding together the expressions for $\cos.\ (B + A)$ and $\cos.\ (B - A)$, we have

$$\cos.\ (B + A) + \cos.\ (B - A) = 2 \cos.\ A \cos.\ B;$$

whence, $\cos.\ (B + A) = 2 \cos.\ A \cos.\ B - \cos.\ (B - A).$

Substituting in these expressions for the sine and cosine of $B + A$, the successive values A, $2A$, $3A$, &c. instead of B; the following series will be produced.

$$\left.\begin{aligned}
\sin.\ 2A &= 2 \cos.\ A \sin.\ A. \\
\sin.\ 3A &= 2 \cos.\ A \sin.\ 2A - \sin.\ A. \\
\sin.\ 4A &= 2 \cos.\ A \sin.\ 3A - \sin.\ 2A. \\
\sin.\ nA &= 2 \cos.\ A \sin.\ (n - 1) A - \sin.\ (n - 2) A.
\end{aligned}\right\} (x)$$

$$\left.\begin{aligned}
\cos.\ 2A &= 2 \cos.\ A \cos.\ A - \cos.\ 0\ (= 1). \\
\cos.\ 3A &= 2 \cos.\ A \cos.\ 2A - \cos.\ A. \\
\cos.\ 4A &= 2 \cos.\ A \cos.\ 3A - \cos\ 2A. \\
\cos.\ nA &= 2 \cos.\ A \cos.\ (n - 1) A - \cos.\ (n - 2) A.
\end{aligned}\right\} (xi.)$$

Several other expressions for the sines and cosines of multiple arcs might readily be found: but the above are the most useful and commodious.

20. From the equation $\sin.\ 2A = \dfrac{2 \sin.\ A \cos.\ A}{R}$, it will be easy, when the sine of an arc is known, to find that of its half. For, substituting for cos. A its value $\sqrt{(R^2 - \sin.^2 A)}$, there will arise $\sin.\ 2A = \dfrac{2 \sin.\ A \sqrt{(R^2 - \sin.^2 A)}}{R}$.

This squared gives $R^2 \sin.^2 2A = 4R^2 \sin.^2 A - 4 \sin.^4 A.$

Here taking sin. A for the unknown quantity, we have a quadratic equation, which solved after the usual manner, gives

$$\sin.\ A = \pm\sqrt{\tfrac{1}{2}R^2 \pm \tfrac{1}{2}R \sqrt{R^2 - \sin.^2 2A}}.$$

If we make $2A = A'$, then will $A = \frac{1}{2}A'$; and consequently the last equation becomes

* Here we have omitted the powers of R that were necessary to render all the terms homologous, merely that the expressions might be brought in upon the page; but they may easily be supplied, when needed, by the rule in art. 18.

$$\left.\begin{array}{l}\sin.\tfrac{1}{2}A' : \pm\sqrt{\tfrac{1}{2}R^2 \pm \tfrac{1}{2}R\sqrt{R^2 - \sin.^2 A'}} \\ \text{or } \sin.\tfrac{1}{2}A' = \pm\tfrac{1}{2}\sqrt{2R^2 \pm 2R\cos A'} :\end{array}\right\} \quad (\text{XII.})$$

by putting cos. A′ for its value $\pm\sqrt{R^2 - \sin.^2 A'}$, multiplying the quantities under the radical by 4, and dividing the whole second number by 2. Both these expressions for the sine of half an arch or angle will be of use to us as we proceed.

21. If the values of sin. (A + B) and sin. (A − B), given by equa. V, be added together, there will result

$$\sin.(A+B) + \sin.(A-B) = \frac{2\sin.A\cos.B}{R}; \text{ whence,}$$

$$\sin.A\cos.B = \tfrac{1}{2}R\sin.(A+B) + \tfrac{1}{2}R\sin.(A-B) \quad . \quad . \quad . \quad (\text{XIII.})$$

Also, taking sin. (A − B) from sin. (A + B), gives

$$\sin.(A+B) - \sin.(A-B) = \frac{2\sin B\cos.A}{R}; \text{ whence,}$$

$$\sin.B\cos.A = \tfrac{1}{2}R\sin.(A+B) + \tfrac{1}{2}R\sin.(A-B) \quad . \quad . \quad . \quad (\text{XIV.})$$

When A = B, both equa. XIII and XIV become

$$\cos.A\sin.A = \tfrac{1}{2}R\sin.2A. \quad . \quad . \quad . \quad (\text{XV.})$$

22. In like manner, by adding together the primitive expressions for cos. (A + B), cos. (A − B), there will arise

$$\cos.(A+B) + \cos.(A-B) = \frac{2\cos.A\cos.B}{R}; \text{ whence,}$$

$$\cos,A\cos.B = \tfrac{1}{2}R\cos.(A+B) + \tfrac{1}{2}R\cos(A-B) \quad . \quad . \quad . \quad (\text{XVI.})$$

And here, when A = B, recollecting that when the arc is nothing the cosine is equal to radius, we shall have

$$\cos.^2 A = \tfrac{1}{2}R\cos.2A + \tfrac{1}{2}R^2 \quad . \quad . \quad . \quad (\text{XVII.})$$

23. Deducting cos. (A + B) from cos. (A − B), there will remain

$$\cos.(A-B) - \cos.(A+B = \frac{2\sin.A\sin B}{R}; \text{ whence,}$$

$$\sin.A\sin.B = \tfrac{1}{2}R\cos.(A-B) - \tfrac{1}{2}R\cos.(A+B) \ . \ . \ . \ (\text{XVIII.})$$

When A = B, this formula becomes

$$\sin.^2 A = \tfrac{1}{2}R^2 - \tfrac{1}{2}R\cos.2A \quad . \quad . \quad . \quad (\text{XIX})$$

24. Multiplying together the expressions for sin. (A + B), and sin. (A − B), equation V, and reducing, there results

$$\sin.(A+B)\sin.(A-B) = \sin.^2 A - \sin.^2 B.$$

And, in like manner, multiplying together the values of cos. (A + B) and cos. (A − B), there is produced

$$\cos.(A+B)\cos.(A-B) = \cos^2 A - \sin.^2 B = \cos.^2 B - \sin.^2 A.$$

Here, since $\sin.^2 A - \sin.^2 B$, is equal to (sin. A + sin. B) × (sin. A − sin. B), that is, to the rectangle of the sum and difference of the sines; it follows, that the first of these equations converted into an analogy, becomes

$$\sin.(A-B) : \sin.A - \sin.B :: \sin.A + \sin B : \sin.(A+B) \ . \ . \ . \ (\text{XX.})$$

That is to say, *the sine of the difference of any two arcs or angles, is to the difference of their sines, as the sum of those sines is to the sine of their sum.*

If A and B be to each other as $n + 1$ to n, then the preceding proportion will be converted into $\sin.A : \sin.(n+1)A - \sin.nA :: \sin.(n+1)A + \sin.nA : \sin.(2n+1)A$ . . . (XXI.)

These two proportions are highly useful in computing a table of sines; as will be shown in the practical examples at the end of this chapter.

25. Let us suppose A + B = A′, and A − B = B′; then the half sum and the half difference of these equations will give respectively $A = \tfrac{1}{2}(A' + B')$, and $B = \tfrac{1}{2}(A' - B')$. Putting these values of A and B, in the expressions of sin.

cos. B, sin. B cos A, cos. A cos. B, sin. A sin. B, obtained in arts. 21, 22, 23, there would arise the following formulæ:

$$\sin.\ \tfrac{1}{2}(A' + B')\cos.\ \tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\sin.\ A' + \sin.\ B'),$$
$$\sin.\ \tfrac{1}{2}(A' - B')\cos.\ \tfrac{1}{2}(A' + B') = \tfrac{1}{2}R(\sin.\ A' - \sin.\ B'),$$
$$\cos.\ \tfrac{1}{2}(A' + B')\cos.\ \tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\cos.\ A' + \cos.\ B'),$$
$$\sin.\ \tfrac{1}{2}(A' + B')\sin.\ \tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\cos.\ B' - \cos.\ A').$$

Dividing the second of these formulæ by the first, there will be had

$$\frac{\sin.\ \frac{1}{2}(A'-B')\cos.\ \frac{1}{2}(A'+B')}{\sin.\ \frac{1}{2}(A'+B')\cos.\ \frac{1}{2}(A'-B')} = \frac{\sin.\ \frac{1}{2}(A'-B')}{\cos.\ \frac{1}{2}(A'-B')}\cdot\frac{\cos.\ \frac{1}{2}(A'+B')}{\sin.\ \frac{1}{2}(A'+B')} = \frac{\sin\ A'-\sin.\ B'}{\sin.\ A'+\sin.\ B'}.$$

But since $\frac{\sin.}{\cos.} = \frac{\tan.}{R}$, and $\frac{\cos.}{\sin.} = \frac{R}{\tan.}$, it follows that the two factors of the first member of this equation are

$\frac{\tan.\ \frac{1}{2}(A'-B')}{R}$, and $\frac{R}{\tan.\ \frac{1}{2}(A'+B')}$, respectively; so that the equation manifestly becomes $\frac{\tan.\ \frac{1}{2}(A' - B')}{\tan.\ \frac{1}{2}(A' + B')} = \frac{\sin.\ A' - \sin.\ B'}{\sin.\ A' + \sin.\ B'}$ . . . (XXII.)

This equation is readily converted into a very useful proportion, viz. *The sum of the sines of two arcs or angles, is to their difference, as the tangent of half the sum of those arcs or angles, is to the tangent of half their difference.*

26. Operating with the third and fourth formulæ of the preceding article, as we have already done with the first and second, we shall obtain

$$\frac{\tan.\ \frac{1}{2}(A' + B')\tan.\ \frac{1}{2}(A' - B')}{R^2} = \frac{\cos.\ B' - \cos.\ A'}{\cos.\ A' + \cos.\ B'}.$$

In like manner, we have by division,

$$\frac{\sin.\ A'+\sin.\ B'}{\cos\ A'+\cos.\ B} = \frac{\sin.\ \frac{1}{2}(A'+B')}{\cos.\ \frac{1}{2}(A'+B')} = \tan.\ \tfrac{1}{2}(A'+B');\ \frac{\sin.A'+\sin.B'}{\cos.B'-\cos.A'} = \cot.\tfrac{1}{2}(A'-B');$$
$$\frac{\sin.\ A'-\sin.B'}{\cos.A'+\cos.B'} = \tan.\ \tfrac{1}{2}(A'-B')\ \ldots\ \frac{\sin.\ A' - \sin.\ B'}{\cos.\ B' - \cos.\ A'} = \cot.\ \tfrac{1}{2}(A' + B'),$$
$$\frac{\cos.\ A' + \cos.\ B'}{\cos.\ B' - \cos.\ A'} = \frac{\cot.\ \frac{1}{2}(A' + B')}{\tan.\ \frac{1}{2}(A' - B')}.$$

Making B = 0, in one or other of these expressions, there results

$$\left.\begin{aligned}
&\frac{\sin.\ A'}{1 + \cos.\ A'} = \tan\ \tfrac{1}{2}A' = \frac{1}{\cot.\ \frac{1}{2}A'}.\\
&\frac{\sin.\ A'}{1 - \cos.\ A'} = \cot.\ \tfrac{1}{2}A' = \frac{1}{\tan.\ \frac{1}{2}A'}.\\
&\frac{1 + \cos.\ A'}{1 - \cos.\ A'} = \frac{\cot.\ \frac{1}{2}A'}{\tan.\ \frac{1}{2}A'} = \cot.^2\ \tfrac{1}{2}A = \frac{1}{\tan.^2\ \frac{1}{2}A}.
\end{aligned}\right\}\ (xxii.)$$

These theorems will find their application in some of the investigations of spherical trigonometry.

27. Once more, dividing the expression for sin. $(A \pm B)$ by that for cos. $(A \pm B)$, there results

$$\frac{\sin.\ (A \pm B)}{\cos.\ (A \pm B)} = \frac{\sin.\ A.\cos.\ B \pm \sin.\ B.\cos.\ A}{\cos.\ A.\cos.\ B \mp \sin.\ A.\sin.\ B}:$$

then dividing both numerator and denominator of the second fraction by cos. A cos. B, and recollecting that $\frac{\sin.}{\cos.} = \frac{\tan.}{R}$, we shall thus obtain

$$\frac{\tan.\ (A \pm B)}{R} = \frac{R\tan.\ (A \pm \tan.\ B)}{R^2 \mp \tan.\ A.\tan.\ B};$$

$$\text{or, lastly, } \tan.\ (A \pm B) = \frac{R^2(\tan.\ A \pm \tan.\ B)}{R^2 \mp \tan.\ A.\tan.\ B}\ \ldots\ (XXIII.)$$

Also, since cot. $= \frac{R^2}{\tan.}$ we shall have

$$\cot. (A. \pm B = \frac{R^2}{\tan. (A \pm B} = \frac{R^2 \mp \tan. A \tan. B}{\tan. A \pm \tan. B};$$

which, after a little reduction, becomes

$$\cot. (A \pm B) = \frac{\cot. A \cot. B \mp R^2}{\cot. B \pm \cot. A} \quad . \ . \ . \quad \text{(XXIV.)}$$

28. We might now, by making $A = B$, $A = 2B$, &c. proceed to deduce expressions for the tangents, cotangents, secants, &c. of multiple arcs; but we shall merely present a few for the tangents, as

$$\left.\begin{array}{l} \tan. 2A = \dfrac{2 \tan. A}{1 - \tan^2 A} \ . \ . \ . \ . \ \tan. 3A = \dfrac{3\tan. A - \tan.^3 A}{1 - 3\tan.^2 A}. \\ \tan. 4A = \dfrac{4\tan. A - 4\tan.^3 A}{1 - 6\tan.^2 A + \tan.^4 A}. \\ \tan. 5A = \dfrac{5\tan. A - 10\tan.^3 A + \tan.^5 A}{1 - 10\tan.^2 A + 5\tan.^4 A}. \end{array}\right\} \textit{(xxiii.)}$$

We might again from the obvious equation

$$\sec.^2 A - \tan.^2 A = \sec.^2 B - \tan.^2 B,$$

deduce the expression $\dfrac{\sec. A + \tan. A}{\sec. B + \tan. B} = \dfrac{\sec. B - \tan. B}{\sec. A - \tan. A}$;

and so, for many other analogies.

We might investigate also some of the usual formulæ of verification in the construction of tables, such as

$$\sin. (54^\circ + A) + \sin. (54^\circ - A) - \sin. (18^\circ + A) - \sin. (18^\circ - A) = \sin. (90^\circ - A)$$
$$\sin. A + \sin. (36^\circ - A) + \sin. (72^\circ + A) = \sin. (36^\circ + A) + \sin. (72^\circ - A),$$

&c. &c

But, as these inquiries would extend this chapter to too great a length, we shall pass them by; and merely investigate a few properties where *more* than two arcs or angles are concerned, and which may be of use in some subsequent parts of this volume.

29. Let A, B, C, be any three arcs or angles, and suppose radius to be unity; then

$$\sin (B + C) = \frac{\sin. A \sin. C + \sin. B \sin. (A + B + C)}{\sin. (A + B)}.$$

For, by equa. V, $\sin. (A + B + C) = \sin. A \cos. (B + C) + \cos. A \sin. (B + C)$, which [putting $\cos. B \cos. C - \sin. B \sin. C$ for $\cos. (B + C)$], is $= \sin. A \cos. B \cos. C - \sin. A \sin. B \sin. C + \cos. A \sin. (B + C)$; and, multiplying by sin. B, and adding sin. A sin. C, there results $\sin. A \sin. C + \sin. B \sin. (A + B + C) = \sin. A \cos. B \cos. C \sin. B + \sin. A \sin. C \cos.^2 B + \cos. A \sin. B \sin. (B + C) = \sin. A \cos. B (\sin. B \cos. C + \cos. B \sin. C) + \cos A \sin. B \sin. (B + C) = \sin. A \cos. B \sin. (B + C) + \cos. A \sin. A \sin. (B + C = (\sin. A \cos. B + \cos. A \sin. B) \times \sin. (B + C) = \sin. (A + B) \sin. (B + C)$. Consequently, by dividing by $\sin. (A + B)$, we obtain the expression above given.

In a similar manner it may be shown, that

$$\sin. (B - C) = \frac{\sin. A \sin. C - \sin. B \sin. (A - B + C)}{\sin. (A - B)}.$$

30. If A, B, C, D, represent four arcs or angles, then writing $C + D$ for C in the preceding investigation, there will result,

$$\sin. (B + C + D) = \frac{\sin. A \sin. (C + D) + \sin. B \sin. (A + B + C + D)}{\sin. (A + B)}.$$

A like process for five arcs or angles will give

$$\sin.(B+C+D+E) = \frac{\sin. A \sin.(C+D+E) + \sin. B \sin.(A+B+C+D+E)}{\sin.(A+B)}.$$

And for any number, A, B, C, &c. to L,

$$\sin.(B+C+\ldots L) = \frac{\sin. A \sin.(C+D+\ldots L) + \sin. B \sin.(A+B+C+\ldots L)}{\sin.(A+B)}.$$

31. Taking again the three, A, B, C, we have

$$\sin.(B-C) = \sin. B \cos. C - \sin. C \cos. B,$$
$$\sin.(C-A) = \sin. C \cos. A - \sin. A \cos. C,$$
$$\sin.(A-B) = \sin. A \cos. B - \sin. B \cos. A.$$

Multiplying the first of these equations by sin. A, the second by sin. B, the third by sin. C; then adding together the equations thus transformed, and reducing; there will result,

$$\sin. A \sin.(B-C) + \sin. B \sin.(C-A) + \sin. C \sin.(A-B) = 0,$$
$$\cos. A \sin.(B-C) + \cos. B \sin.(C-A) + \cos. C \sin.(A-B) = 0.$$

These two equations obtaining for any three angles whatever, apply evidently to the three angles of any triangle.

32. Let the series of arcs or angles A, B, C, D, ... L, be contemplated, then we have (art. 24),

$$\sin.(A+B) \sin.(A-B) = \sin.^2 A - \sin.^2 B,$$
$$\sin.(B+C) \sin.(B-C) = \sin.^2 B - \sin.^2 C,$$
$$\sin.(C+D) \sin.(C-D) = \sin.^2 C - \sin.^2 D,$$
$$\&c. \ \&c. \ \&c.$$
$$\sin.(L+A) \sin.(L-A) = \sin.^2 L - \sin.^2 A.$$

If all these equations be added together, the second member of the equation will vanish, and of consequence we shall have

$$\sin.(A+B) \sin.(A-B) + \sin.(B+C) \sin.(B-C) + \&c. \ldots$$
$$\ldots\ldots\ldots + \sin.(L+A) + \sin.(L-A) = 0.$$

Proceeding in a similar manner with sin. (A — B), cos. (A + B), sin. (B — C), cos. (B + C), &c. there will at length be obtained

$$\cos.(A+B) \sin.(A-B) + \cos.(B+C) \sin.(B-C) + \&c. \ldots$$
$$\ldots\ldots + \cos.(L+A) \sin.(L-A) = 0.$$

33. If the arcs A, B, C, &c. . . . L form an arithmetical progression, of which the first term is 0, the common difference D′, and the last term L any number n of circumferences; then will $B - A = D'$, $C - B = D'$, &c. $A + B = 2D'$, $B + C = 3D'$, &c.: and dividing the whole by sin D′, the preceding equations will become

$$\left.\begin{array}{l} \sin D' + \sin. 3D' + \sin. 5D' + \&c. = 0, \\ \cos. D' + \cos. 3D' + \cos. 5D' + \&c. = 0. \end{array}\right\} \ldots (XXV.)$$

If E′ were equal 2D′, these equations would become

$$\sin. D' + \sin.(D'+E') + \sin.(D'+2E') + \sin.(D'+3E') + \&c. = 0,$$
$$\cos. D' + \cos.(D'+E') + \cos.(D'+2E') + \cos.(D'+3E') + \&c. = 0,$$

34. The last equation, however, only shows the sums of sines and cosines of arcs or angles in arithmetical progression, when the common difference is to the first term in the ratio of 2 to 1. To investigate a *general* expression, let

$$s = \sin. A + \sin.(A+B) + \sin.(A+2B) + \sin.(A+3B) + \&c.$$

Then, since this series is a recurring series, whose scale of relation is $2 \cos. B - 1$, it will arise from the developement of a fraction whose denominator is $1 - 2z . \cos. B + z^2$, making $z = 1$.

Now this fraction will be $= \dfrac{\sin. A + z[\sin. A + B] - 2 \sin A \cos. B]}{1 - 2z \cos. B + z^2}$.

Therefore, when $z = 1$, we have

$s = \dfrac{\sin. A + \sin. (A + B) - 2 \sin. A \cos. B}{2 - 2 \cos. B}$; and this, because $2 \sin. A \cos. B = \sin. (A + B) + \sin. (A - B)$ (art. 21), is equal to $\dfrac{\sin. A - \sin. (A - B)}{2(1 - \cos. B)}$. But, since $\sin. A' - \sin. B' = 2 \cos. \frac{1}{2} (A' + B') \sin. \frac{1}{2} (A' - B')$, by art. 25, it follows, that $\sin. (A - B) = 2 \cos. (A - \frac{1}{2}B) \sin \frac{1}{2}B$; besides which, we have $1 - \cos. B = 2 \sin.^2 \frac{1}{2}B$. Consequently the preceding expression becomes $s = \sin. A + \sin. (A + B) + \sin. (A + 2B) + \sin. (A + 3B) +$ &c. *ad infinitum* $= \dfrac{\cos. (A - \frac{1}{2}B)}{2 \sin. \frac{1}{2}B}$ . . . . (XXVI.)

35. To find the sum of $n + 1$ terms of this series we have simply to consider that the sum of the terms past the $(n + 1)$th, that is, the sum of $\sin [A + (n + 1) B] + \sin. [A + (n + 2) B] + \sin [A + (n + 3) B] +$ &c. *ad infinitum*, is, by the preceding theorem, $= \dfrac{\cos. [A + (n + \frac{1}{2}) B]}{2 \sin. \frac{1}{2}B}$. Deducting this, therefore, from the former expression, there will remain, $\sin. A + \sin (A + B) + \sin (A + 2B) + \sin. (A + 3B) + \ldots . \sin. (A + nB) =$

$$\frac{\cos. (A - \frac{1}{2}B) - \cos. [A + (n + \frac{1}{2})B]}{2 \sin. \frac{1}{2}B} = \frac{\sin. (A + \frac{1}{2}nB) \sin. \frac{1}{2}(n + 1)B}{\sin. \frac{1}{2}B} . \text{(XXVII.)}$$

By like means it will be found, that the sums of the cosines of arcs or angles in arithmetical progression, will be $\cos. A + \cos. (A + B) + \cos. (A + 2B) + \cos. (A + 3b) +$ &c. *ad infinitum* $= - \dfrac{\sin. (A - \frac{1}{2}B)}{2 \sin. \frac{1}{2}B}$ . . . . (XXVIII.)

Also,

$$\cos. A + \cos. (A + B) + \cos. (A + 2B) + \cos. (A + 3B) + \ldots \ldots \ldots (\cos. A + nB) = \frac{\sin. (A + \frac{1}{2}nB) \sin. \frac{1}{2}(n + 1)B}{\sin. \frac{1}{2}B} . . \text{(XXIX.)}$$

36. With regard to the tangents of more than two arcs, the following property (the only one we shall here deduce) is a very curious one, which has seldom been inserted in works of Trigonometry, though it has been long known to mathematicians. Let the three arcs, A, B, C, together make up the whole circumference, $\bigcirc$: then, since $\tan (A + b) = \dfrac{R^2 (\tan A + \tan B)}{R^2 - \tan. A \tan. B}$ (by equation XXIII), we have $R^2 \times (\tan. A + \tan B + \tan. C) = R^2 \times [\tan. A + \tan. B - \tan (A + B)] = R^2 \times (\tan. A + \tan B - \dfrac{R^2 (\tan. A + \tan. B)}{R^2 - \tan A \tan. B}$ =, by actual multiplication and reduction, to $\tan. A \tan. B \tan. C$, since $\tan. C = \tan. [\bigcirc - (A + B)] = - \tan. (A + B) = - \dfrac{R^2 (\tan. A + \tan. B}{R^2 - \tan. A \tan. B}$, by what has preceded in this article. The result therefore is, that *the sum of the tangents of any three arcs which together constitute a circle, multiplied by the square of the radius, is equal to the product of those tangents*. . . .(XXX.)

Since both arcs in the second and fourth quadrants have their tangents considered negative, the above property will apply to arcs any way trisecting a semicircle; and it will therefore apply to the angles of a plane triangle, which are, together, measured by arcs constituting a semicircle. So that, if radius be considered as unity, we shall find, that *the sum of the tangents of the three angles of any plane triangle, is equal to the continued product of those tangents.*

(XXXI.)

37. Having thus given the chief properties of the sines, tangents, &c. of arcs, their sines, products, and powers, we shall merely subjoin investigations of theorems for the 2d and third cases in the solutions of plane triangles. Thus, with respect to the second case, where two sides and their included angle are given:

By equ. IV, $a : b :: \sin. A : \sin B$.

By compos. and division $\Big\}$ $a + b : a - b :: \sin A + \sin. B : \sin. A - \sin. B$;

but, eq. XXII., $\tan. \frac{1}{2}(A + B) : \tan. \frac{1}{2}(A - B) :: \sin. A + \sin. B : \sin. A - \sin. B$; whence, ex equal. $a + b : a - b :: \tan. \frac{1}{2}(A + B) : \tan. \frac{1}{2}(A - B)$. . . . (XXXII.)

Agreeing with the result of the geometrical investigation at p. 395, vol. i.

38. If, instead of having the two sides a, b, given, we know their *logarithms*, as frequently happens in geodesic operations, $\tan. \frac{1}{2}(A - B)$ may be readily determined without first finding the number corresponding to the logs of a and b. For if a and b were considered as the sides of a right-angled triangle, in which ϕ denotes the angle opposite the side a, then would $\tan. \phi = \frac{a}{b}$. Now, since a is supposed greater than b, this angle will be greater than half a right angle, or it will be measured by an arc greater than $\frac{1}{8}$ of the circumference, or than $\frac{1}{8}\bigcirc$. Then, because $\tan. (\phi - \frac{1}{8}\bigcirc) = \frac{\tan. \phi - \tan. \frac{1}{8}\bigcirc}{1 + \tan. \phi \tan. \frac{1}{8}\bigcirc}$, and because $\tan \frac{1}{8}\bigcirc = R = 1$,

we have $\tan (\phi - \frac{1}{8}\bigcirc) = (\frac{a}{b} - 1) \div (1 + \frac{a}{b}) = \frac{a - b}{a + b}$.

And, from the preceding article,

$\frac{a - b}{a + b} = \frac{\tan. \frac{1}{2}(A - B)}{\tan. \frac{1}{2}(A + B)} = \frac{\tan. \frac{1}{2}(A - B)}{\cot. \frac{1}{2}C}$: consequently,

$\tan. \frac{1}{2}(A - B) = \cot \frac{1}{2}C \tan. (\phi - \frac{1}{8}\bigcirc)$. . . . (XXXIII.)

From this equation we have the following practical rule: Subtract the less from the greater of the given logs, the remainder will be the log tan. of an angle: from this angle take 45 degrees, and to the log tan. of the remainder add the log cotan. of half the given angle; the sum will be the log tan. of half the *difference* of the other two angles of the plane triangle.

39. The remaining case is that in which the three sides of the triangle are known, and for which indeed we have already obtained expressions for the angles in arts. 6 and 8. But, as neither of these is best suited for logarithmic computation, (however well fitted they are for instruments of investigation), another may be deduced thus: In the equation for cos. A, (given equation II), viz. $\cos. A = \frac{b^2 + c^2 - a^2}{2bc}$, if we substitute, instead of cos. A, its value $1 - 2 \sin.^2 \frac{1}{2}A$, change the signs of all the terms, transpose the 1, and divide by 2, we shall have $\sin.^2 \frac{1}{2}A = \frac{a^2 - b^2 - c^2 + 2bc}{4bc} = \frac{a^2 - (b - c)^2}{4bc}$.

Here, the numerator of the second member being the product of the two factors $(a + b - c)$ and $(a - b + c)$, the equation will become $\sin.^2 \frac{1}{2}A = \frac{\frac{1}{2}(a + b - c)\,\frac{1}{2}(a - b + c)}{4bc}$. But, since $\frac{1}{2}(a + b - c) = \frac{1}{2}(a + b + c) - c$, and $\frac{1}{2}(a - b + c) = \frac{1}{2}(a + b + c) - b$; if we put $2s = a + b + c$, and extract the square root, there will result, (applying the method to the three angles in succession).

$$\left.\begin{aligned}\sin.\tfrac{1}{2}A &= \sqrt{\frac{(s-b(s-c)}{bc}}.\\ \sin.\tfrac{1}{2}B &= \sqrt{\frac{(s-a)(s-c)}{ac}},\\ \sin.\tfrac{1}{2}C &= \sqrt{\frac{(s-a)(s-b)}{ab}}.\end{aligned}\right\}\text{(XXXIV.)}$$

The following formulæ were investigated at p. 404, vol. i.:—

$$\left.\begin{aligned}\cos.\tfrac{1}{2}A &= \sqrt{\frac{s\,.\,s-a}{bc}}\\ \cos.\tfrac{1}{2}B &= \sqrt{\frac{s\,.\,s-b}{ac}}\\ \cos.\tfrac{1}{2}C &= \sqrt{\frac{s\,.\,s-c}{ab}}\end{aligned}\right\}\text{(XXXV.)}\qquad \left.\begin{aligned}\tan.\tfrac{1}{2}A &= \sqrt{\frac{s-b\,.\,s-c}{s\,.\,s-a}}\\ \tan.\tfrac{1}{2}B &= \sqrt{\frac{s-a\,.\,s-c}{s\,.\,s-b}}\\ \tan.\tfrac{1}{2}C &= \sqrt{\frac{s-a\,.\,s-b}{s\,.\,s-c}}\end{aligned}\right\}\text{(XXXVI)}$$

These expressions, besides their convenience for logarithmic computation, have the further advantage of being perfectly free from ambiguity, because the half of any angle of a plane triangle will always be *less* than a right angle.

40. The student will find it advantageous to collect into one place all those formulæ which relate to the computation of sines, tangents, &c.* ; and, in another place, those which are of use in the solution of plane triangles: the former of these are equations v, viii, ix, x, xi, *x*, *xi*, xii, xiii, xiv, xv, xvi, xvii, xviii, xix, xx, xxii, *xxii*, xxiii, xxiv, xxvii, xxxiv; the latter are equa. ii, iii, iv, vii, xxxii, xxxiii.

To exemplify the use of some of these formulæ, the following exercises are subjoined.

EXERCISES.

Ex. 1. Find the sines and tangents of 15°, 30°, 45°, 60°, and 75°: and show how from thence to find the sines and tangents of several of their submultiples.

First, with regard to the arc of 45°, the sine and cosine are manifestly equal; or they form the perpendicular and base of a right-angled triangle whose hypothenuse is equal to the assumed radius. Thus, if radius be R, the sine and cosine of 45° will each be $= \sqrt{\tfrac{1}{2}R^2} = R\sqrt{\tfrac{1}{2}} = R\sqrt{2}$. If R be equal to 1, as is the case with the tables in use, then

$$\sin.\,45^\circ = \cos.\,45^\circ = \tfrac{1}{2}\sqrt{2} = \cdot 7071068$$

$$\tan.\,45^\circ = \frac{\sin.}{\cos.} = 1\cdot = \frac{\cos.}{\sin.} = \text{cotangent } 45^\circ.$$

Secondly, for the sines of 60° and of 30°: since each angle in an equilateral triangle contains 60°, if a perpendicular be demitted from any one angle of such a triangle on the opposite side, considered as a base, that perpendicular will be the sine of 60°, and the half base the sine of 30°, the side of the triangle being the assumed radius. Thus, if it be R, we shall have $\tfrac{1}{2}$R for the sine of 30°, and $\sqrt{(R^2 - \tfrac{1}{4}R^2)} = \tfrac{1}{2}R\sqrt{3}$, for the sine of 60°. When R = 1, these become

$$\sin.\,30^\circ = \cdot 5 \;.\;.\;.\;.\;.\;.\; \sin.\,60^\circ = \cos 30 = \cdot 8660254.$$

Hence, $\tan.\,30^\circ = \dfrac{\cdot 5}{\frac{1}{2}\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = \cdot 5773503.$

$$\tan.\,60^\circ = \frac{\frac{1}{2}\sqrt{3}}{\cdot 5} = \sqrt{3} = \;.\;.\;.\;.\; 1\cdot 7320508.$$

* What is here given being only a brief sketch of an inexhaustible subject; the reader who wishes to pursue it further is referred to the copious Introduction to Hutton's Mathematical Tables, and the treatises on Trigonometry, by Emerson, Gregory, Bonnycastle, Woodhouse, Lardner, Hind, Young, Cagnoli, and many other modern writers on the same subject, where he will find his curiosity richly gratified.

Consequently, tan. $60^\circ = 3$ tan. 30°.

Thirdly, for the sines of 15° and 75°, the former arc is the half of 30°, and the latter is the complement of that half arc. Hence, substituting 1 for R and $\frac{1}{2}\sqrt{3}$ for cos. A, in the expression sin. $\frac{1}{2}A = \pm \frac{1}{2}\sqrt{(2R^2 + 2R \cos. A)}$ . . . (equa. XII), it becomes sin. $15^\circ = \frac{1}{2}\sqrt{(2 - \sqrt{3})} = \cdot 2588190$.

Hence, sin. $75^\circ =$ cos. $15^\circ = \sqrt{[1 - \frac{1}{4}(2 - \sqrt{3})]} = \frac{1}{2}\sqrt{(2 + \sqrt{3})} = \frac{\sqrt{6} + \sqrt{2}}{4} = \cdot 9659258$.

Consequently, tan. $15^\circ = \frac{\sin. 15^\circ}{\cos. 15^\circ} = \frac{\cdot 2588190}{\cdot 9659258} = \cdot 2679492$.

And, tan. $75^\circ = \frac{\cdot 9659258}{\cdot 2588190} = 3 \cdot 7320508$.

Now, from the sine of 30°, those of 6°, 2°, and 1°, may easily be found. For, if $5A = 30^\circ$, we shall have, from equation X, sin. $5A = 5$ sin. $A - 20$ sin.$^3 A + 16$ sin.$^5 A$: or, if sin $A = x$, this will become $16x^5 - 20x^3 + 5x = \cdot 5$. This equation solved by any of the approximating rules for such equations, will give $x = \cdot 1045285$, which is the sine of 6°.

Next, to find the sine of 2°, we have again, from equation X, sin. $3A = 3$ sin. $A - 4$ sin.$^3 A$: that is, if x be put for sin. 2°, $3x - 4x^3 = \cdot 1045285$. This cubic solved, gives $x = \cdot 0348995 =$ sin. 2°.

Then, if $s =$ sin. 1°, we shall, from the second of the equations marked X, have $2s\sqrt{1 - s^2} = \cdot 0348995$; whence s is found $= \cdot 0174524 =$ sin. 1°.

Had the expression for the sines of bisected arcs been applied successively from sin. 15°, to sin. $7^\circ 30'$, sin. $3^\circ 45'$, sin. $1^\circ 52\frac{1}{2}'$, sin. $56\frac{1}{4}'$, &c. a different series of values might have been obtained: or, if we had proceeded from the quinquisection of 45°, to the trisection of 9°, the bisection of 3°, and so on, a different series still would have been found. But what has been done above is sufficient to illustrate *this* method. The next example will exhibit a very simple and compendious way of ascending from the sines of smaller to those of larger arcs.

Ex. 2. Given the sine of 1°, to find the sine of 2°, and then the sines of 3°, 4°, 5°, 6°, 7°, 8°, 9°, and 10°, each by a single proportion.

Here, taking first the expression for the sine of a double arc, equa. X, we have sin. $2^\circ = 2$ sin. $1^\circ \sqrt{1 - \sin.^2 1^\circ} = \cdot 0348995$.

Then it follows from the rule in equa. XX. that

sin 1° : sin. 2° — sin. 1° :: sin. 2° + sin. 1° : sin. $3^\circ = \cdot 0523360$
sin 2° : sin. 3° — sin. 1° :: sin. 3° + sin. 1° : sin. $4^\circ = \cdot 0697565$
sin. 3° : sin. 4° — sin. 1° :: sin. 4° + sin. 1° : sin. $5^\circ = \cdot 0871557$
sin. 4° : sin. 5° — sin. 1° :: sin. 5° + sin. 1° : sin. $6^\circ = \cdot 1045285$
sin 5° : sin. 6° — sin. 1° :: sin. 6° + sin. 1° : sin. $7^\circ = \cdot 1218693$
sin. 6° : sin. 7° — sin. 1° :: sin 7° + sin. 1° : sin. $8^\circ = \cdot 1391731$
sin 7° : sin. 8° — sin. 1° :: sin. 8° + sin. 1° : sin. $9^\circ = \cdot 1564375$
sin. 8° : sin. 9° — sin. 1° :: sin. 9° + sin. 1° : sin. $10^\circ = \cdot 1736482$

To check and verify operations like these, the proportions should be changed at certain stages. Thus,

sin. 1° : sin. 3° — sin. 2° :: sin. 3° + sin. 2° : sin. 5°,
sin. 1° : sin. 4° — sin. 3° :: sin. 4° + sin. 3° : sin. 7°,
sin. 4° : sin. 7° — sin. 3° :: sin. 7° + sin. 3° : sin. 10°.

The coincidence of the results of these operations with the analogous results in the preceding, will manifestly establish the correctness of both.

Cor. By the same method, knowing the sines of 5°, 10°, and 15°, the sines of 20°, 25°, 35°, 55°, 65°, &c. may be found, each by a single proportion. And the

sines of 1°, 9°, and 10°, will lead to those of 19°, 29°, 39°, &c. So that the sines may be computed to any arc; and the tangents and other trigonometrical lines, by means of the expressions in art. 4, &c.

Ex. 3. Find the sum of all the natural sines to every minute in the quadrant, radius $= 1$.

In this problem the actual addition of all the terms would be a most tiresome labour: but the solution, by means of equation XXVII, is rendered very easy. Applying that theorem to the present case, we have sin. $(A + \frac{1}{2}nB) =$ sin. 45°, sin. $\frac{1}{2}(n + 1)$ B $=$ sin. 45°0′30″, and sin. $\frac{1}{2}$B $=$ sin. 30″. Therefore

$$\frac{\sin. 45^\circ \times \sin. 45^\circ\, 0'\, 30''}{\sin. 30''} = 3438{\cdot}2467465 \text{ the sum required.}$$

From another method, the investigation of which is omitted here, it appears that the same sum is equal to $\frac{1}{2}$ (cot. 30″ + 1).

Ex. 4. Explain the method of finding the *logarithmic* sines, cosines, tangents, secants, &c. the natural sines, cosines, &c. being known.

The *natural* sines and cosines being computed to the radius unity, are all proper fractions, or quantities less than unity, so that their logarithms would be negative. To avoid this, the tables of logarithmic sines, cosines, &c. are computed to a radius of 10000000000, or 10^{10}; in which case the logarithm of the radius is 10 times the log. of 10, that is, it is 10.

Hence, if sin a represent any sine to radius 1, then $10^{10} \times$ sin. $a =$ sine of the same arc or angle to rad. 10^{10}. And this, in logs., is, log 10^{10} sin. $a = 10$ log. 10 + log. sin. $a = 10 +$ log. sin. a.

The log. cosines are found by the same process, since the cosines are the sines of the complements.

The logarithmic expressions for the tangents, &c. are deduced thus:

tan $=$ rad. $\frac{\text{sin.}}{\text{cos.}}$. Theref. log. tan. $=$ log rad. + log. sin. — log. cos. $= 10$ + log. sin. — log cos.

Cot $= \frac{\text{rad.}^2}{\text{tan.}}$. Theref. log. cot. $=$ 2 log. rad. — log. tan. $= 20$ — log. tan.

Sec. $= \frac{\text{rad}^2}{\text{cos.}}$. Theref. log. sec. $=$ 2 log. rad. — log. cos. $= 20$ — log. cos.

Cosec. $= \frac{\text{rad.}^2}{\text{sin.}}$. Theref. log. cosec. $=$ 2 log. rad. — log sin. $= 20$ — log. sin.

$$\text{Versed sine} = \frac{\text{chord}^2}{\text{diam.}} = \frac{(2 \text{ sin. } \frac{1}{2} \text{ arc})^2}{2 \text{ rad.}} = \frac{2 \times \text{sin.}^2 \frac{1}{2} \text{ arc}}{\text{rad.}}.$$

Therefore, log. vers. sin. $=$ log. 2 + 2 log. sin. $\frac{1}{2}$ arc — 10.

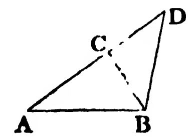
Ex. 5. Given the sum of the natural tangents of the angles A and B of a plane triangle $= 3{\cdot}1601988$, the sum of the tangents of the angles B and C $= 3{\cdot}8765577$, and the continued product, tan. A . tan. B . tan. C $= 5{\cdot}3047057$: to find the angles A, B, and C.

It has been demonstrated in art. 36, that when radius is unity, the sum of the natural tangents of the three angles of a plane triangle is equal to their continued product. Hence the process is this:

From tan. A + tan. B + tan. C	$= 5{\cdot}3047057$	
Take tan. A + tan. B	$= 3{\cdot}1601988$	
Remains tan. C	$= 2{\cdot}1445069$	$=$ tan. 65°.
From tan. A + tan. B + tan. C	$= 5{\cdot}3047057$	
Take tan. B + tan. C	$= 3{\cdot}8765577$	
Remains tan. A	$= 1{\cdot}4281480$	$=$ tan. 55°.

Consequently, the three angles are 55°, 60°, and 65°.

Ex. 6. There is a plane triangle, whose sides are three consecutive terms in the natural series of integer numbers, and whose largest angle is just double the smallest. Required the sides and angles of that triangle?



Let ACB be the given triangle, of which angle C $= 2$ angle A. Produce AC until its prolongation CD is $=$ CB; join BD. Then ACB $=$ CBD $+$ CDB $= 2$ CDB $= 2$ angle A, by quest. $\therefore$ A $=$ D, and the triangles ABD, BCD, being both isosceles, and D an angle common to both, are similar. Hence AB (or BD) : AD :: CB (or CD : BD. If AC $= x$, then CB $= x - 1$, AB $= x + 1$, and AD $=$ AC $+$ CB $= 2x - 1$; and therefore the above proportion becomes $x + 1 : 2x - 1 :: x - 1 : x + 1$. Consequently, equating the product of means and extremes, $2x^2 - 3x + 1 = x^2 + 2x + 1$. Hence $x^2 = 5x$, $x = 5$; and the sides are 4, 5, and 6.

Hence the cosines of the angles are, cos. A $= \frac{3}{4}$, cos. B $= \frac{9}{16}$, cos. C $= \frac{1}{8}$. The sines are, sin. A $= \frac{1}{4}\sqrt{7}$, sin. B $= \frac{5}{16}\sqrt{7}$, sin. C $= \frac{3}{8}\sqrt{7}$. And the angles are,

$$A = 41^\circ.409603 = 41^\circ\,24'\,34''\,34'''$$
$$B = 55^\circ.771191 = 55^\circ\,46'\,16'\,18'''$$
$$C = 82^\circ.819206 = 82^\circ\,49'\,9''\,8'''.$$

Ex. 7. Demonstrate that sin. $18^\circ =$ cos. 72° is $= \frac{1}{4} R(-1 + \sqrt{5})$, and sin. $54^\circ =$ cos. 36° is $= \frac{1}{4} R(1 + \sqrt{5})$.

Ex. 8. Demonstrate that the sum of the sines of two arcs which together make 60°, is equal to the sine of an arc which is greater than 60° by either of the two arcs: Ex, gr. sin. $3' +$ sin. $59^\circ\,57' =$ sin. $60^\circ\,3'$; and thus that the tables of sines may be continued beyond 60° by addition only.

Ex. 9. Show the truth of the following proportion: As the sine of half the difference of two arcs, which together make 60°, or 90°, respectively, is to the difference of their sines; so is 1 to $\sqrt{3}$, or $\sqrt{2}$, respectively.

Ex. 10. Demonstrate that the sum of the squares of the sine and versed sine of an arc, is equal to the square of double the sine of half the arc.*

Ex. 11. Demonstrate that the sine of an arc is a mean proportional between half the radius and the versed sine of double the arc.*

Ex. 12. Show that the secant of an arc is equal to the sum of its tangent and the tangent of half its complement.*

Ex. 13. Find the following continual products to six places of figures;— sin. $1^\circ\,5'\,10''$ sin. $91^\circ\,4'\,15''$ sin. $196^\circ\,10'\,18''$ sin. $250^\circ\,18'\,18''$ sin. $300^\circ\,10'\,15''$; tan. 18° tan. 108° tan. 196° tan. 271° tan. 305° tan. 375° tan. 400°,

and $\dfrac{\text{sin. }(-18^\circ)\text{ sin. }367^\circ\text{ cos. }95^\circ\text{ cos. }(-195^\circ)\text{ tan. }300^\circ}{\text{cos. }18^\circ\text{ cos. }(-367^\circ)\text{ sin. }(-95^\circ)\text{ cos. }195^\circ\text{ tan. }(-300^\circ)}$: and compute the values of the two following expressions:

$$\frac{\text{sin. }270^\circ\text{ sin. }175^\circ - \text{sin. }(-35^\circ)\text{ sin. }(-100^\circ)}{\text{cos. }17^\circ - \text{cos. }(-35^\circ)}\text{ and}$$

$$\frac{\text{sin. }100^\circ\text{ sin. }375^\circ\text{ cos. }92^\circ - \text{sin. }536^\circ\text{ sin. }(-182^\circ)\text{ cos. }276^\circ}{\text{cos. }100^\circ - \text{cos. }276^\circ}.$$

Ex. 14. How must three trees, A, B, C, be planted, so that the angle at A may be double the angle at B, the angle at B double that at C; and so that a line of 400 yards may just go round them?

* The examples marked with an asterisk in this series are required to be proved geometrically, as well as analytically.

Ex. 15. In a certain triangle, the sines of the three angles are as the numbers 17, 15, and 8, and the perimeter is 160. What are the sides, the angles, the perpendiculars, and the lines bisecting the sides?

Ex. 16. The logarithms of two sides of a triangle are 2·2407293 and 2·5378191, and the included angle is 37° 20′. It is required to determine the other angles, without first finding any of the sides.

Ex. 17. The sides of a triangle are to each other as the fractions, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$: what are the angles, and under what angles do the lines from the angles to the middles of the sides intersect?

Ex. 18. Show that the secant of 60° is double the tangent of 45°, and that the secant of 45° is a mean proportional between the tangent of 45° and the secant of 60 *.

Ex. 19. Demonstrate that 4 times the rectangle of the sines of two arcs, is equal to the difference of the squares of the chords of the sum and difference of those arcs *.

Ex. 20. Convert the equations marked XXXIV into their equivalent logarithmic expressions; and by means of them and equa. IV, find the angles of a triangle whose sides are 5, 6, and 7.

Ex. 21. Find the arc whose tangent and cotangent shall together be equal to 4 times the radius, and another n times the diameter †.

Ex. 22. Find the arc whose sine added to its cosine shall be equal to a; and show the limits of possibility †.

Ex. 23. Find the arc whose secant and cotangent shall be equal. †

Ex. 24. If one angle A of a right-angled plane triangle A, B, C, be given (B being the right angle), and the area or surface be given = S: demonstrate that

$$\text{AB} = \sqrt{(2\text{S cot. A})};\ \text{BC} = \sqrt{(2\text{S tan. A})};\ \text{and AC} = 2\sqrt{(\text{S sec. 2A})}.$$

Ex. 25. Demonstrate,

1. That
$$\begin{aligned}\sin.\text{A} &= \frac{1}{\sqrt{(1+\cot.^2\text{A})}} = \sqrt{(\tfrac{1}{2} - \tfrac{1}{2}\cos.\ 2\text{A})}\\ &= \frac{2}{\cot.\ \frac{1}{2}\text{A} + \tan.\ \frac{1}{2}\text{A}} = \frac{1}{\cot.\ \text{A} + \tan.\ \frac{1}{2}\text{A}}\\ &= 2\sin.^2(45° + \tfrac{1}{2}\text{A}) - 1 = 1 - 2\sin.^2(45° - \tfrac{1}{2}\text{A}).\end{aligned}$$

2. That
$$\begin{aligned}\tan.\ \text{A} &= \sqrt{\left(\frac{1}{\cos.^2\text{A}} - 1\right)} = \sqrt{\frac{1-\cos.\ 2\text{A}}{1+\cos.\ 2\text{A}}}\\ &= \frac{\sqrt{(1-\cos.^2\text{A})}}{\cos.\ \text{A}} = \frac{\sin.\ \text{A}}{\sqrt{(1-\sin.^2\text{A})}} =\\ &= \frac{1-\cos.\ 2\text{A}}{\sin.\ 2\text{A}} = \frac{\sin.\ 2\text{A}}{1+\cos.\ 2\text{A}}\\ &= \frac{2\cot.\ \frac{1}{2}\text{A}}{\cot.^2\frac{1}{2}\text{A} - 1} = \frac{2\tan.\ \frac{1}{2}\text{A}}{1-\tan.^2\frac{1}{2}\text{A}}\end{aligned}$$

$$= \frac{2}{\cot.\ \frac{1}{2}\text{A} - \tan.\ \frac{1}{2}\text{A}} = \cot.\ \text{A} - 2\cot\ 2\text{A} = \tfrac{1}{2}[\tan.(45°+\tfrac{1}{2}\text{A}) - \tan.(45° - \tfrac{1}{2}\text{A})].$$
$$= \tfrac{1}{2}[\tan.(45° + \tfrac{1}{2}\text{A}) - \tan.(45° - \tfrac{1}{2}\text{A})].$$

3. Let m and n be any two arcs, then $\cos.^4 m - \sin.^4 m = \cos.\ 2m$; and $\sin.^2 m \sin.^2 n + \cos.^2 m \cos.^2 n + \sin.^2 m \cos.^2 n + \cos.^2 m \sin.^2 n = 1$.

† Problems thus marked are to be also constructed geometrically.

ON DEMOIVRE'S THEOREM.

By means of the exponential theorem (vol. i. p. 243) we have

$$e^{x\sqrt{-1}} = 1 + \frac{x\sqrt{-1}}{1} - \frac{x}{1\cdot 2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \ldots$$

$$e^{x\sqrt{-1}} = 1 - \frac{x\sqrt{-1}}{1} - \frac{x^2}{1.2} + \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} - \frac{x^5\sqrt{-1}}{1.2.3.45} - \ldots$$

Whence by addition, subtraction, and division we obtain.

$$\tfrac{1}{2}(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}) = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{1}{2\sqrt{-1}}(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}) = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots (2)$$

But by reference to vol. i. p. 387, we find that the right side of these equations are respectively equal to cos. x and sin. x. Hence,

$$\text{cos. } x = \frac{1}{2}\left\{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right\} \ldots\ldots (3)$$

$$\text{sin. } x = \frac{1}{2.\sqrt{-1}}\left\{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right\} \ldots (4)$$

If we change the base to r, then since $x\sqrt{-1}$ and $-x\sqrt{-1}$ are the logarithms of the quantities which compose sin. x and cos. x to the base e, we have $m\,x\sqrt{-1}$ and $-m\,x\sqrt{-1}$ the component quantities to the new base: which, like the former, are also imaginary. We hence infer that all the forms under which the sines and cosines of x, of the *exponential form* can be expressed are essentially imaginary; and hence, like the value of the *one* real root of Cardan's formula (vol. i. p. 211) are to be considered only as *abridged algebraical expressions* of those quantities. Nevertheless, as by means of these abridged expressions, some curious and valuable properties of the sines, cosines, tangents, &c. of arcs are readily derived, they have been properly retained in a course of trigonometrical investigation.

Equations (3) and (4) are called "*the exponential expression of cos. x and sin. x.*"

Multiply the two equations,

cos. $x \pm \sqrt{-1}$ sin. x and cos. $x_, \pm \sqrt{-1}$, sin. x, which gives

(cos. x cos. $x_,$ — sin. x sin. $x_,$) $\pm$ (cos. x sin. $x_,$ + sin. x cos. $x_,$) $\sqrt{-1}$, or

(cos. $x \pm \sqrt{-1}$ sin. x) (cos. $x_, \pm \sqrt{-1}$ sin. x) = cos. $(x + x_,) \pm \sqrt{-1}$ sin. $(x + x_,)$

Let now $x = x_,$: then it becomes

(cos. $x \pm \sqrt{-1}$ sin. $x)^2$ = cos. $2x \pm \sqrt{-1}$ sin. $2x$.

Next, let this equation be multiplied by cos. $x \pm \sqrt{-1}$ sin. x, then we have in like manner

(cos. $x \pm \sqrt{-1}$ sin. $x)^3$ = cos. $3x \pm \sqrt{-1}$ sin. $3x$, and so on generally to

(cos. $x \sqrt{\pm -1}$ sin. $x)^n$ = cos. $nx \pm \sqrt{-1}$ sin. nx . . . (4)

Now this equation is true whatever value x may have: put then $x = \frac{z}{m}$; then

$$\left(\text{cos. } \frac{z}{m} \pm \sqrt{-1}\,\text{sin. } \frac{z}{m}\right)^m = \text{cos. } \frac{mz}{m} \pm \sqrt{-1}\,\text{sin. } \frac{mz}{m} = \text{cos. } z \pm \sqrt{-1}\,\text{sin. } z.$$

Hence, contracting the mth root,

$$(\cos. z \pm \sqrt{-1} \sin. z)^{\frac{1}{m}} = \cos. \frac{z}{m} \pm \sqrt{-1} \sin. \frac{z}{m}$$

and raising this to the nth power, we have,

$$(\cos. z \pm \sqrt{-1} \sin. z)^{\frac{n}{m}} = \left(\cos. \frac{z}{m} \pm \sqrt{-1} \sin. \frac{z}{m}\right)^n$$

$$= \cos. \frac{n}{m} z \pm \sqrt{-1} \sin. \frac{n}{m} z$$

in which since x was arbitrary, z is arbitrary also, and x may be written for it, and hence,

$$(\cos. x \pm \sqrt{-1} \sin. x)^{\frac{n}{m}} = \cos. \frac{n}{m} x \pm \sqrt{-1} \sin. \frac{n}{m} x \dots (5)$$

Again, dividing unity by each side, of this equation, we get

$$(\cos. x \pm \sqrt{-1} \sin. x)^{-\frac{n}{m}} = \frac{1}{\cos. \frac{n}{m} x \pm \sqrt{-1} \sin. \frac{n}{m}} \times \frac{\cos. \frac{n}{m} x \mp \sqrt{-1} \sin. \frac{n}{m} x}{\cos. \frac{n}{m} x \pm \sqrt{-1} \sin. \frac{n}{m} x}$$

But $(\cos. \frac{n}{m} x \pm \sqrt{-1} \sin. \frac{n}{m} x) (\cos. \frac{n}{m} x \mp \sqrt{-1} \sin. \frac{n}{m} x) \mp \cos.^2 \frac{n}{m} x + \sin.^2 \frac{n}{m} x = 1$, and hence,

$$(\cos. x \pm \sqrt{-1} \sin. x)^{-\frac{n}{m}} = \cos. \frac{n}{m} x \mp \sqrt{-1} \sin. \frac{n}{m} x$$

$$= \cos. \left(-\frac{n}{m} x\right) \pm \sqrt{-1} \sin. \left(-\frac{n}{m} x\right). \quad (6)$$

whence for all values of n and m, whether positive, negative, integer, or fractional, the same formula is true.

This formula, in honour of its discoverer, is called *Demoivre's theorem.*

Though this formula is of the utmost importance in the higher researches which occur in mathematical science, we can here only find room for two of the most elementary.

First, to express $(\sin. n)^n$ and $(\cos. x)^n$, or as they are generally written, $\sin.^n x$ and $\cos.^n x$, (meaning the nth powers of the sine and cosine of x) in terms of the sines and cosines of the successive multiples of x.

For brevity in writing, put

$$\cos. x + \sqrt{-1} \sin. x = u \text{ and } \cos. x - \sqrt{-1} \sin. x = v \dots\dots\dots (7)$$

then by Demoivre's theorem,

$$\cos. nx + \sqrt{-1} \sin. nx = u^n, \text{ and } \cos. nx - \sqrt{-1} \sin. x = v^n \dots (8)$$

By adding (1) which we get $\cos. x = \frac{1}{2}(u + v)$ . . . . (9)

By adding (2) we have $\cos. nx = \frac{1}{2}(u^n + v^n)$ . . . . (10)

By multiplying (2) we have $u^n v^n = 1$ . . . . (11)

From (9) we have

$$\cos.^n x = \frac{1}{2^n}(u + v)^n = \frac{1}{2^n}(v + u)^n \text{ as we arrange according to } u \text{ or } v.$$

$$= \frac{1}{2}\left\{u^n + \frac{n}{1} u^{n-1} v + \frac{n \,.\, n-1}{1 \,.\, 2} u^{n-} v^2 + \dots\right\}$$

$$= \frac{1}{2^n}\left\{v^n + \frac{n}{1} v^{n-1} u + \frac{n \,.\, n-1}{1 \,.\, 2} v^{n-3} u^2 + \dots\right\}$$

Add these equations together, and divide by 2, then we get

$$\cos.^n x = \frac{1}{2^{n+1}}\left\{(u^n + v^n) + \frac{n}{1}\cdot uv\,(u^{n-2} + v^{n-2}) + \frac{n\,.\,n-1}{1\,.\,2}\,u^2 v^2\,(u^{n-4} + u^{n-4}) +\right.$$

From the equations 9, 10, 11, substitute the values of these several quantities, and we have,

$$\cos.^n x = \frac{1}{2^{n-1}}\left\{\cos.\,nx + \frac{n}{1}\cos.\,(n-2)\,x + \frac{n\,.\,n-1}{1\,.\,2}\cos.\,(n-4)x + \,.\,.\right\}\ldots (12)$$

Again, subtracting one of the equations (7) from the other, we have $\sin.\,x = \frac{u-v}{\pm 2\sqrt{-1}}$, and hence $\sin.^n x = \frac{(u-v)^n}{(\pm 2\sqrt{-1})_n}$; and it will be convenient to consider the two cases where n is even and where n is odd, separately.

Let n be even: then $\frac{(u \mp v)^n}{(\pm 2\sqrt{-1})^n)} = \frac{(v-u)^n}{(\pm 2\sqrt{-1})^n}$, and hence by expansion and addition, as in the last case, we get

$$2\sin.^n x = \frac{1}{(\pm 2\sqrt{-1})^n}\left\{(u^n + v^n) - \frac{n}{1}\cdot uv\,(u^{n-2} + v^{n-2}) + \ldots\right\}$$

$$\text{or } \sin.^n x = \frac{\mp 1}{2^n}\left\{\sin.nx - \frac{n}{1}\cdot\cos.\,(n-2)\,x + \frac{n\,.\,\overline{n-1}}{1\,.\,2}\cos.\,(n-4)\,x + \ldots\right\}\ (13)$$

Next let n be odd: then, $(u-v)^n = -(v-n)^n$. Hence the expressions become $\sin.^n x = \frac{(u-v)^n}{(\pm 2\sqrt{-1})^n}$ and $\sin.^n x = \frac{-(v-u)^n}{(\pm 2\sqrt{-1})^n}$.

Expand these by the binomial theorem, as before, and add, then,

$$2\sin.^n x = \frac{1}{(\pm 2\sqrt{-1})^n}\left\{u^n - v^n - \frac{n}{1}\cdot uv\,(u^{n-2} - v^{n-2}) + \ldots\ldots\right\}$$

But by equations (8), $u^n v^n = 2\sqrt{-1}\sin.\,nx$, and $u^n v^n = 1$, hence this last becomes, sine $(\sqrt{-1})^{n-1} = \mp 1$,

$$\sin.^n x = \pm\frac{1}{2n}\left\{\sin.\,nx - \frac{n}{1}\sin.\,(n-2)\,x + \frac{n\,.\,n-1}{1\,.\,2}\sin.\,(n-4)\,x - \ldots\right\}$$

. . . . (15) where the upper or lower sign is to be taken according as n is of the form $4m + 2$ or $4m$, that is as n is 2, 6, 10, &c. or 4, 8, 12, &c.

Secondly, to express the sine and cosine of a multiple arc in terms of the powers of the sines and cosines of the simple arc.

By Demoivre's theorem, if we put $p = \cos.\,x$ and $q = \sqrt{-1}\sin.\,x$ then $\cos.\,n\,x$

$$\pm\sqrt{-1}\sin.\,nx = p^n \pm \frac{n}{1}\cdot p^{n-1}\,q + \frac{n\,.\,n-1}{1\,.\,2}\,p^{n-2}\,q^2 \pm \frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}\,p^{n-3}\,p^3$$

Now all the even powers of q are real, and all the odd ones imaginary; hence collecting those terms which are real into one class, and those which are imaginary into another, and putting the assemblage of all those of one kind equal to zero, and all those of the other kind equal to zero, we shall have two equations from which to determine $\cos.\,nx$ and $\sin.\,nx$ *.

* The principle employed here is one of extensive use in the higher departments of mathematics. It is this: that if there be one equation containing real quantities and imaginary quantities also, which is developed into a series of terms into which the imaginaries enter only as algebraic factors, then the assemblage of terms may be represented briefly by $A \pm B\sqrt{-1} = 0$, where A is the sum of all those which are free from imaginaries, and B the assemblage of all the coefficients of $\sqrt{-1}$: then we shall have separately $A = 0$ and $B = 0$. The reasoning by which it may be established is of this kind. If these two equations do not simultaneously hold good, then divide the equation by B; then $\frac{A}{B} = \pm\sqrt{-1}$. But A is free from imaginaries by the hypothesis; and B is also free from them: for if it were not, let it contain terms which

Hence we have, restoring the values of p and q, and reducing

$$\cos. nx = \cos.^n x - \frac{n.n-1}{1.2}\cos.^{n-2} x \sin.^2 x + \frac{n.n-1.n-2.n-3}{1.2.3.4.}\cos.^{n-4} x$$
$$\sin.^4 x \quad . \quad . \quad . \quad . \qquad (16)$$

$$\sin. nx = n \cos.^{n-1} x - \frac{n.n-1.n-2}{1.2.3}\sin.^{n-3} x \cos.^{3'} x + \quad . \quad . \quad . \quad . \qquad (17)$$

FARTHER EXAMPLES.

26. The ratio of the cosines of two arcs is $\frac{m+n}{m-n}$ and that of their sines is $\frac{m+n}{\sqrt{2mn}}$; find them, and the tangents of their sum and difference: and prove that if c be the hypothenuse of a right angled triangle; then

$$\log. \tan. \tfrac{1}{2}A = 10 - \tfrac{1}{2}\{\log. (c+b) - \log. (c-b).\}$$

Ex. 27. Two lines g and h are given, and a triangle, whose sides a, b, c are respectively the arithmetical, geometrical, and harmonical means between them is constructed: what are the angles A, B, C? Exemplify it when $g = 4h$.

Ex 28. The base of a triangle is a, and its sides b and c: and from its vertex a line is drawn to divide the base in the ratio of b^3 to c^3. What angle does it make with each of the three sides? And when it makes a right angle with the base what is the ratio of b to c?

Ex. 29. Given the radius of the inscribed circle and the angles at the base of a plane triangle, to find the sides, and the radius of the circumscribing circle.

Ex. 30. By trigonometry it is required to prove that if R and r be the radii of the circumscribing and inscribed circles, the distance of their centres is equal to the square root of $R^2 - 2Rr$ · and that if the four circles be described, which touch the sides of the triangle internally and externally, the sum of the squares of the distances of these four centres from the centre of the circumscribing circle, is equal to three times the square of the diameter of this last-named circle.

Ex. 31. If a plane cutting off one of the angles of a regular pyramid on a square base, at the distances from the vertex measured in order along the edges a, b, c, d, it is required to show that $\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}$.

Ex. 32. Given the perimeter of a triangle $= 2s$, the difference of the angles at the base, A — B, and the perpendicular from the vertical angle $= p$, to find the sides and angles of the triangle, and the rectangle of the radii of the circles described in and about it.

are collectively represented by $\pm C\sqrt{-1}$: then $B\sqrt{-1}$ would be of the form $(D \pm C\sqrt{-1})\sqrt{-1} = D\sqrt{-1} \mp C$, and hence the terms had not been collected as in the hypothesis. Hence neither A nor B contain imaginaries, and hence, moreover, the quotient $\frac{A}{B}$ contains none. But by the admitted contrary hypothesis $\frac{A}{B} = \mp\sqrt{-1}$, and the quotient does contain imaginaries. Hence the contra-hypothesis is false; or the one equation $A \pm B\sqrt{-1} = 0$ cannot be true, except also the two equations $A = 0$ and $B = 0$ be also true.

Ex. 33. Given the four sides of a quadrilateral figure inscriptible in a circle, a, b, c, d: to find the diagonals and the angle under which they intersect, together with the radius of the circle and the area of the quadrilateral: and exemplify it when the sides are 10, 12, 16, 18.

Ex. 34. Let A, B, C be the angles of a plane triangle; a, b, c, the sides; R the radius of the circumscribing circle; r the radius of the inscribed circle; $r_{,}$, $r_{,,}$, $r_{,,,}$, as at p. 405, vol. i., and $p_{,}, p_{,,}, p_{,,,}$ the perpendiculars from A, B, C, upon a, b, c respectively; then it is required to prove the following properties:

(1) $\tan. A + \tan. B + \tan. C = \tan. A \tan. B \tan. C.$

(2) $\cot. A \cot. B + \cot. B \cot. C + \cot. C \cot. A = 1$

(3) $\sin. (A - B) : \sin. C :: a - b : c$

(4) $\sin. 2A + \sin. 2B + \sin. 2C = 4 \sin. A \sin. B \sin. C.$

(5) $\tan. \frac{1}{2}A \tan. \frac{1}{2}B + \tan. \frac{1}{2}B \tan. \frac{1}{2}C + \tan. \frac{1}{2}C \tan. \frac{1}{2}A = 1.$

(6) $\cot. \frac{1}{2}A + \cot. \frac{1}{2}B + \cot. \frac{1}{2}C = \cot. \frac{1}{2}A \cot. \frac{1}{2}B \cot. \frac{1}{2}C.$

(7) $r = 4R \sin. \frac{1}{2}A \sin. \frac{1}{2}B \sin. \frac{1}{2}C.$

$$\left.\begin{array}{l} r_{,} = \\ r_{,,} = \\ r_{,,,} = \end{array}\right\} \text{in terms of R, A, B, C?}$$

(8) $p_{,} = \dfrac{b^2 \sin. c + c^2 \sin. B}{b + c}$, and the same form for $p_{,,}$ and $p_{,,,}$.

(9) $\dfrac{1}{r_{,}} + \dfrac{1}{r_{,,}} + \dfrac{1}{r_{,,,}} = \dfrac{1}{p_{,}} + \dfrac{1}{p_{,,}} + \dfrac{1}{p_{,,,}}.$

Ex. 35. Given the angles A, B, C, and radius R of the circumscribing circle to find the three lines from the angles bisecting the opposite sides, and those perpendicular to the sides.

Ex. 36. If $\tan.^3 x = \cos. b \sec. a$: then will $\dfrac{\cos. a}{\cos. x} + \dfrac{\cos. b}{\sin. x} = (\cos.^{\frac{2}{3}} a + \cos.^{\frac{2}{3}} b)^{\frac{3}{2}}$. Prove this.

Ex. 37. Investigate the radical expressions at p. xxxix. of Hutton's Tables, for the sines of the arcs specified there.

Ex. 38. Three of the angles of a quadrilateral figure, circumscribing a circle whose radius is 10, are 29° 15′ 10″, 87° 15′ 12″, and 105° 15′ 18″, what were its sides? And give the general solution.

Ex. 39. In the following several cases, the parts specified of a plane triangle are given, to find the sides and angles of the triangles:—

(1) $b, c, B \pm C$; (2) $B, C, b \pm c$: (3) $A, a, b \pm c$:

(4) $A, c, a \pm b$: (5) A, c, ab: (6) $A, B, a + b + c$.

Ex. 40. In addition to the notation of the last example, let $p_{,}, p_2, p_3$, be the perpendiculars from A, B, C on a, b, c, respectively:

Given (1) C, c, p_3 to find B and C

(2) $a + b + c$, C, and area, to find c.

(3) a, b, c to find the segments into which a line bisecting C divides c; and the segments into which a line bisecting c divides C. Also find p_1, p_2, p_3.

(4) p_1, p_2, p_3, to find A, B, C, and a, b, c.

(5) a, C, b, to find p_3.

(6) $a + b + c$, B, and $ac = b^2$ to find A and C.

(7) $a + b + c$, $\dfrac{a}{b}$, $\dfrac{a}{c}$ to find the angles and perpendiculars.

Ex. 41. Three given lines form the chords of a semicircle: what is the radius?

Ex. 42. Find the arc which is a third of the arc whose sine is s. Show when the inquiry admits of three answers; when of only two.

Ex. 43. Demonstrate that $\tan.^{-1}\frac{1}{2} + \tan.^{-1}\frac{1}{3} = 45°$ *, that $\tan.^{-1}\frac{1}{7} + 2\tan.^{-1}\frac{1}{3} = 45°$, and that $\tan.^{-1}a \pm \tan.^{-1}b = \frac{a \pm b}{1 \mp ab}$.

Ex. 44. Show that if $(\sin. \alpha + \sin. \beta\sqrt{-1}) \sin. \frac{\theta}{2} \pm (\sin. \alpha - \sin. \beta \sqrt{-1}) \cos. \frac{b}{2} = \sin. \alpha \pm \sin \beta$, the sines of α, 2θ and β are in harmonical progression, and find the value of θ: and find an arc whose tangent shall be half the harmonical mean between cot. α and cot. β.

45. In any triangle $\tan. C = \dfrac{1 + \sec. A \sqrt{\left(\frac{a}{b}\right)^2 - \sin.^2 A}}{1 - \operatorname{cosec.} A \sqrt{\left(\frac{a}{b}\right)^2 - \sin.^2 A}}$:

and if p be the perpendicular from c or c, show also that a and b have for values, respectively,

$\sqrt{c^2 + pc \cot. c} \pm \sqrt{c^2 + pc \tan. c}$; and $\cos. c = \dfrac{a - c \cos. B}{\sqrt{a^2 - 2ac \cos. B + c^2}} =$

$\frac{a}{b} \sin^2 B \mp \cos B \sqrt{1 - \left(\frac{a}{b}\right)^2 \sin.^2 B}$.

46. If α, β and $\alpha_{,}$, $\beta_{,}$ denote two angles of any two triangles, it is required to prove that

$$\frac{\sin.^2 \beta'}{\sin.^2 \overline{\alpha_{,} + \beta}} - \frac{2 \sin. \beta_{,} \sin, \beta \cos. \overline{\alpha - \alpha}}{\sin. \alpha_{,} + \beta_{,} \sin. \alpha + \beta} + \frac{\sin.^2 \beta}{\sin.^2 \overline{\alpha + \beta}} =$$
$$\frac{\sin.^2 \alpha_{,}}{\sin.^2 \overline{\alpha_{,} + \beta_{,}}} - \frac{2 \sin. \alpha_{,} \sin. \alpha \cos. \overline{\beta_{,} - \beta}}{\sin. \overline{\alpha + \beta_{,}} \sin. \overline{\alpha + \beta}} + \frac{\sin.^2 \alpha}{\sin^2 \overline{\alpha + \beta}}.$$

Ex. 47. Let a, b, c, be any three angles; it is required to prove that tan. a tan. b

$$\tan. c = \frac{-\sin. (a+b+c) + \sin. (-a+b+c) + \sin. (a-b+c) + \sin. (a+b-c)}{\cos. (a+b+c) + \cos. (-a+b+c) + \cos. (a-b+c) + \cos. (a+b-c)}:$$

and to find tan. a, tan. b, tan. c, separately, in terms of these quantities $a + b + c$, $-a + b + c$, $a - b + c$, and $a + b - c$.

Also to find tan. a tan. b, tan. a tan. c, and tan. b tan. c in terms of the same quantities.

Ex 48. Reduce sin. a sin. b sin. c sin. d . . . . . sin. m into expressions of the sums and differences in the same way as in the last, as well as the continued product of the cosines, or any product composed of the sines of some of the arcs and the cosines of the others.

Ex. 49. It is alleged that from a point within a triangle whose sides are in arithmetical progression, perpendiculars to the sides were observed to be also in

---

* This notation signifies the inverse of tan., or stands for the words "the arc whose tangent is $\frac{1}{2}$"," &c. Thus, also, $\sec.^{-1} 2$ denotes arc whose sec. is 2.

† Also, where tan. tan. . . . occurs, it signifies, not the nth power of the tangent, but a successive series of operations, the last written being the first performed. Thus tan. 4° being found, it is considered to be bent into an arc of the same radius, and its tangent found; then this being made an arc, its tangent is also found; and so on in succession. Till the notation is rendered familiar, it is better for the student to put the expression into words; in writing at first, then, when familiar with it in this form, to merely utter the words either orally or in his mind; and in a very little time, the whole will become sufficiently familiar to enable him merely to say, "tan. minus one power," &c.

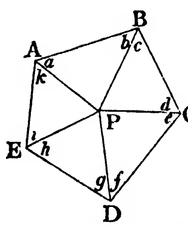
arithmetical progression, and farther that the angles which the sides subtended were also in the arithmetical progression of 1, 2, 3. Could this observation have been made?

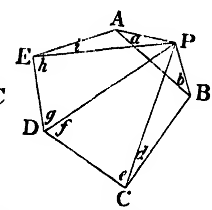
PROBLEMS IN PLANE TRIGONOMETRICAL SURVEYING.

LEMMA.

If straight lines be drawn from any point to all the angles of a polygon, the continual product of the sines of the alternate angles, made upon the angles of the polygon by the lines so drawn, will be equal.

Angles PAB $= a$
PAE $= k$
PBA $= b$
PBC $= c$





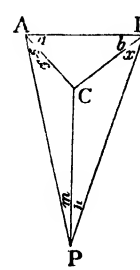
For, assuming a pentagon, and any point P, either within it, or out of it, we have, expressing the proportions fractionally,

$$\frac{PA}{PB} = \frac{\sin. b}{\sin. a}, \quad \frac{PB}{PC} = \frac{\sin. d}{\sin. c}, \quad \frac{PC}{PD} = \frac{\sin. f}{\sin. e}, \quad \frac{PD}{PE} = \frac{\sin. h}{\sin. g}, \quad \frac{PE}{PA} = \frac{\sin. k}{\sin. i}.$$

But PA . PB . PC . PD PE = PB . PC . PD . PE . PA;
or, prod. of numerators = prod. of denom^{rs}. in the first member,
$\therefore$ the same holds in the second members of the equation;
or, $\sin. b \,.\, \sin. d \,.\, \sin. f \,.\, \sin. h \,.\, \sin. k = \sin. a \,.\, \sin. c \,.\, \sin. e \,.\, \sin. g \,.\, \sin. i$.
And a similar demonstration will serve for any other polygon.

PROBLEM I.

Given AB, a, b; and the angles m, n, taken at some point P in the same horizontal plane (as ABC); to find x, and thence PA, PB, PC.



Put $S = 180° - (a + b + m + n)$
then $S - x = CAP$; and by the Lemma $\sin. b \,.\, \sin. n \,.\, \sin. (S - x) = \sin. m \,.\, \sin. a \,.\, \sin. x$;
or, $\sin. b \,.\, \sin. n \,.\, (\sin. S \,.\, \cos. x - \sin. x \,.\, \cos. S) = \sin. m \,.\, \sin. a \,.\, \sin. x$;
or, dividing by $\sin. x$,
$\sin. b \,.\, \sin. n\, (\sin. S \,.\, \cot. x - \cos. S) = \sin. m \,.\, \sin. a$;
or, $\sin. b \,.\, \sin. n \,.\, \sin. S \,.\, \cot. x - \sin. b \,.\, \sin. n \,.\, \cos. S = \sin. m \,.\, \sin. a$;
or, $\sin. b \,.\, \sin. n \,.\, \sin. S \,.\, \cot. x = \sin. b \,.\, \sin. n \,.\, \cos. S + \sin. m \,.\, \sin. a$.

Therefore, dividing by $\sin. b \,.\, \sin. n \,.\, \sin. S$,

$$\cot. x = \cot. S + \frac{\sin. m \,.\, \sin. a}{\sin. b \,.\, \sin. n \,.\, \sin. S}$$

$$= \cot. S + \sin. m \,.\, \sin. a \,.\, \operatorname{cosec}. b \,.\, \operatorname{cosec}. n \,.\, \operatorname{cosec}. S.$$

Then, sin. $(m+n)$: sin. $(b+x)$ :: AB : AP; whence

$$AP = \text{cosec.}\,(m+n)\,\text{sin.}\,(b+x)\,AB.$$

$$BP = \text{cosec.}\,(m+n)\,\text{sin.}\,(a+S-x)\,AB,$$

$$CP = \text{cosec.}\,(n+x)\,\text{sin.}\,x\,.\,BP.$$

EXAMPLE.

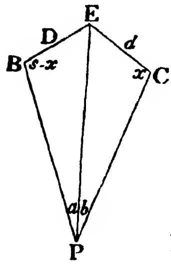
Given AB = 800 yards, $a = 44^\circ\ 24'$, $b = 52^\circ\ 36'$, $m = 22^\circ\ 18'$, $n = 19^\circ\ 24'$ } to find PB, AP, and CP.

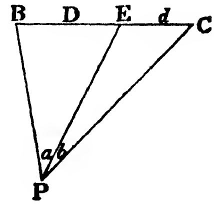
sin. m =	sin. 22° 18′ . .	9·5791616	
sin. a =	sin. 44 24 . .	9·8448891	
cosec. b =	cosec. 52 36 . .	10·0999528	
cosec. n =	cosec. 19 24 . .	10·4786512	
cosec. S =	cosec. 41 18 . .	10·1804550	nat. cot. S 1·1382761
		0·1831097 = log.	1·5244379
cot. 20° 35′ nearly		0989	2·6627140
		108	
$x = 20^\circ\ 35' =$ CPB		86	
S − x = 20 43 = CAP		22	

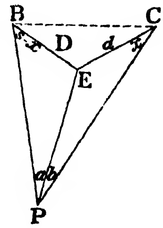
cosec. $(m+n)$ = cosec. 41° 42′	10·1770279
sin. $(a+S-x)$ = sin. 65 7	9·9576870
AB = 800 its log. =	2·9030900
BP = 1090·95	3·0378049
	347
	60
cosec. $(m+n)$ = cosec. (as above) . . .	10·1770279
sin. $(b+x)$ = sin. 73° 11′	9·9810187
AB = 800 (as above)	2·9030900
AP = 1151·138 its log. =	3·0611366
	131
	135
cosec. $(n+x)$ = 39° 59′	10·1920831
sin. x = sin. 20 35 . . .	9·5460110
BP = 1090·95 (as above) . . .	3 0378049
CP = 596·896 its log. =	2·7758990

PROBLEM II.

Given the distances D, d, the angle BEC, and the angles a, b, taken at P in the same horizontal plane; to find the angle x, and thence PB, PC, PE.







Put $S = 360^\circ - (BEC + a + b)$.

Then, $\sin. a : D :: \sin. (S - x) : PE = \dfrac{D \sin. (S - x)}{\sin. a}$.

$\sin. b : d :: \sin. x \quad : PE = \dfrac{d \sin. x}{\sin. b}$,

$$\therefore \frac{D \sin. (S - x)}{\sin. a} = \frac{d \sin. x}{\sin. b};$$

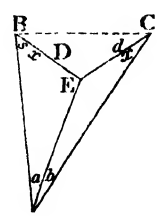
or, $D \sin. b . \sin. (S - x) = d \sin. a \sin. x$;
or, $D \sin. b (\sin. S . \cos. x - \sin. x \cos. S) = d \sin. a \sin. x$;
or, dividing by $\sin. x$, $D \sin. b (\sin. S . \cot. x - \cos. S) = d \sin. a$;
or, $D \sin. b . \sin. S . \cot. x - D \sin. b . \cos. S = d \sin. a$;
or, $D \sin. b . \sin. S . \cot. x = D \sin. b . \cos. S + d \sin. a$;
whence, dividing by $D \sin. b . \sin. S$, we have

$$\cot. x = \cot. S + \frac{d \sin. a}{D \sin. b \sin. S}$$

$$= \cot. S + \frac{d}{D} \sin. a . \operatorname{cosec.} b . \operatorname{cosec.} S.$$

Then $PE = d \operatorname{cosec.} b . \sin. x$.

EXAMPLE.



Given $BE = D = 840$, $EC = d = 760$, $BEC = 360^\circ\ 120^\circ$, $a = 26^\circ\ 14'$, $d = 18^\circ\ 18'$ Required PE and PC.

Arith. comp. $D = a.c. \log. 840$	=	7·0757207
$\log. d = \log. 760$	=	2·8808136
$\sin. a = \sin. 26^\circ\ 14'$	=	9·6454496
$\operatorname{cosec.} b = \operatorname{cosec} 18\ 18$	=	10·5030808
$\operatorname{cosec.} S = \operatorname{cosec.} 75\ 28$	=	10·0141238
		0·1191885
		1569
		316
		297
		19

nat. cot. $75^\circ\ 28'$ = ·2592384
No. to log. 0·1191885 is 1·3157959

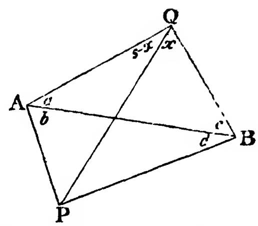
nat. cot. x $32^\circ\ 25'$ = 1·5750343
Hence $S - x = 43^\circ\ 3'$

$$\begin{array}{rcl}
\text{Log. } d = \text{(as above)} & = & 2{\cdot}8808136 \\
\text{cosec } b = \text{(as above)} & = & 10{\cdot}5030808 \\
\sin. x = \sin. 32^\circ\ 25' & = & 9{\cdot}7292234
\end{array}$$

$$\begin{array}{rl}
\text{PE} = 1297{\cdot}5 \text{ log.} = & 3{\cdot}1131178 \\
\text{cosec. } x = \text{cosec. } 32^\circ\ 25' & 10{\cdot}2707766 \\
\sin. (b + x) = \sin.\ \ 50\ \ 43 & 9.8887547
\end{array}$$

$$\text{PC} \ .\ \ 1873{\cdot}5 \text{ log.} = \ 3{\cdot}2726491$$

PROBLEM III.



Given AB and the angles a, b, c, d, to find x; and thence PQ.

Here $S = 180^\circ - (a + c)$, and P is taken as a point out of the polygon: then

$\sin. b \ .\ \sin. (c + d) \sin. (S - x) = \sin. d$
$\sin. (a + b) \sin. x$,

$\sin. b \ .\ \sin. (c + d) [\sin. S \cos. x - \sin. x \cos. S] = \sin. d \ .\ \sin. (a + b) \sin. x$;
and, dividing by $\sin. x$,

$\sin. b \ .\ \sin. (c + d) [\sin. S \cot. x - \cos. S] = \sin. d \ .\ \sin. (a + b)$;

or, $\sin. b \ .\ \sin. (c + d) \sin. S \ .\ \cot. x - \sin. b \ .\ \sin. (c + d) \cos. S = \sin. d \ .$
$\sin. (a + b)$;

or, $\sin. b \ .\ \sin. (c + d) \sin. S \ .\ \cot. x = \sin. b \ .\ \sin. (c + d) \cos. S + \sin. d \ .$
$\sin. (a + b)$;

whence, dividing by $\sin b \ .\ \sin. (c + d) \sin S$, we have

$\cot. x = \cot. S + \sin. d \ .\ \sin. (a + b) \text{ cosec. } b \ .\ \text{cosec. } (c + d) \text{ cosec. } S$;
whence x becomes known.

Then $\sin. S : AB :: \sin. a : BQ = \text{cosec. } S \ .\ \sin\ a \ .\ AB$,
and $\sin. (x + d + c) : BQ :: \sin. (c + d) : PQ =$
$\text{cosec. } (x + d + c) \text{ cosec. } S \ .\ \sin. a \ .\ \sin. (c + d)\ AB$.

Cor. If PQ were given, and the same angles, x would be found as above, and AB easily from the same triangles: viz. $AB = PQ \sin. (x + d + c) \sin. S \ .$ $\text{cosec. } a \ .\ \text{cosec. } (c + d)$.

EXAMPLE.

Given PQ, from a correct map, = 5218 yards; and by a theodolite

$$a = 52^\circ\ 16' \qquad b = 63^\circ\ 27'$$
$$c = 47\ \ 25 \qquad d = 45\ \ 18.$$

Required AB and AQ.

$$\text{Here } S = 180^\circ - (a + c) = 180^\circ - 99^\circ\ 41' = 80^\circ\ 19'.$$

$$\begin{array}{rcl}
\text{Log. sin. } d = \sin.\ \ 45^\circ\ 18' & = & 9{\cdot}8517471 \\
\sin. (a + b) = \sin.\ \ 115\ \ 43 & = & 9{\cdot}9547011 \\
\text{cosec. } b = \text{cosec. } 63\ \ 27 & = & 10{\cdot}0483980 \\
\text{cosec. } (c + d) = \text{cosec. } 92\ \ 43 \text{ or } 87\ \ 17 & = & 10{\cdot}0004884 \\
\text{cosec. } S = \text{cosec. } 80\ \ 19 & = & 10{\cdot}0062321
\end{array}$$

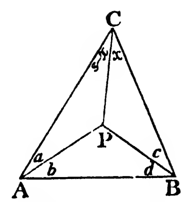
$$9{\cdot}8615667$$

$$\begin{array}{rl}
\text{nat. cot. } S = & {\cdot}1706338 \\
\log.^{-1} 9{\cdot}8615667 = & {\cdot}7270540 \\
\hline
\cot. x = \cot. 48^\circ\ 5' = & {\cdot}8976878
\end{array}$$

Log. PQ = log. 5218 =	3·7175041
log. sin. $(x + d + c)$ = sin. 140° 48′ =	9·8007372
sin. S = sin. 80 19 =	9·9937679
cosec. a = cosec. 52 16 =	10·1018962
cosec. $(c + d)$ = as above =	10·0004884
AB = 4115·2 its log. =	3·6143938

Then cosec. S = cosec. (as above)	10·0062321
sin. c = sin. 47 25′	9·8679512
AB = (as above)	3·6143938
AQ = 3073·8 its log. =	3·4876771

PROBLEM IV.



Supposing P to fall within the triangle ABC. Then, given AB, and the angles a, b, c, d; it is required to find x and CP.

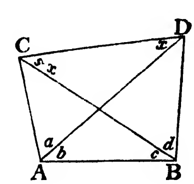
Put $S = 180° - (a + b + c + d)$.

Then, by the lemma,
$\sin. b \,.\, \sin. c \,.\, \sin. (S - x) = \sin. a \,.\, \sin. d \,.\, \sin. x$;
or, $\sin. b \,.\, \sin. c \,.\, (\sin. S \cos. x - \sin. x - \cos. S) = \sin. a \,.\, \sin. d \,.\, \sin. x$;
or, dividing by sin. x,
$\sin. b \,.\, \sin. c \,.\, (\sin. S \,.\, \cot\, x - \cos. S) = \sin. a \,.\, \sin. d$,
$\sin. b \,.\, \sin. c \,.\, \sin. S \,.\, \cot. x - \sin. b \,.\, \sin. c \,.\, \cos. S = \sin. a \,.\, \sin. d$,
whence, dividing by sin. b . sin. c . sin. S, we have
$\cot. x = \cot. S + \sin. a \,.\, \sin. d \,.\, \text{cosec}. b \,.\, \text{cosec}. c \,.\, \text{cosec}. S$.
Thus x becomes known.

Then, $\sin. (b + d) : AB :: \sin. d : BP = \text{cosec}. (b + d) \sin. b \,.\, AB$, and $\sin. x : BP :: \sin. c : CP =$
$\text{cosec}. x \,.\, \text{cosec}. (b + d) \sin. b \,.\, \sin. c \,.\, AB$.

Cor. If CP, and the angles a, b, c, d, were given, AB might be found by reversing the two last proportions: viz.
$AB = \sin. x \,.\, \sin. (b + d) \,\text{cosec}. b \,.\, \text{cosec}. c \,.\, CP$.

PROBLEM V.



Given AB, and the angles a, b, c, d, to find x, and thence CD.

Put $S = b + c$; then, if $CDA = x$, $BCD = S - x$.

Also, $\sin. ADB = \sin. (b + c + d)$,
$\sin. ACB = \sin. (a + b + c)$.

And, by the lemma,
$\sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) \sin. (S - x) = \sin. b \,.\, \sin. d \,.\, \sin. (a + b + c) \sin. x$;
or, $\sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) [\sin. S \cos. x - \sin. x \cos. S] = \sin. b \,.\, \sin d \,.\, \sin. (a + b + c) \sin. x$.

Then, dividing by sin. x,
$\sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) [\sin. S \cot x - \cos. S] = \sin. b \,.\, \sin. d \,.\, \sin. (a + b + c)$;

or, $\sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) \,.\, \sin. S \,.\, \cot. x - \sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) \cos. S = \sin. b \,.\, \sin. d \,.\, \sin. (a + b + c)$,

$\sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) \,.\, \sin. S \cot. x =$
$\sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) \,.\, \cos. S + \sin. b \,.\, \sin. d \,.\, \sin. (a + b + c)$;
whence, by dividing by $\sin. a \,.\, \sin. c \,.\, \sin. (b + c + d) \sin. S$,
$\cot. x = \cot. S + \sin. b \,.\, \sin. d \,.\, \sin. (a + b + c) \,.\, \text{cosec.}\, a \,.\, \text{cosec.}\, c \,.\, \text{cosec.}\, (b + c + d) \text{ cosec.}\, S$;
whence x and $S - x$ are known.

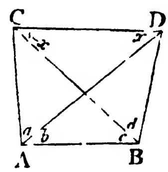
Then, $\sin. (a + b + c) : \sin. c :: AB : AC$,

$$\text{and } \sin. x : \sin. a :: AC : CD = \frac{AB \,.\, \sin. c}{\sin. (a + b + c)} \,.\, \frac{\sin. a}{\sin. x} =$$

$\text{cosec.}\, (a + b + c) \text{ cosec.}\, x \,.\, \sin. a \,.\, \sin. c \,.\, AB$.

Cor. If the same angles, and CD, be given to find AB; it is $AB = \sin. (a + b + c) \sin. x \,.\, \text{cosec.}\, a \,.\, \text{cosec.}\, c \,.\, CD$.

EXAMPLE I.



Ex. 24, p. 413, vol. i. Course.
Where $AB = 600$, $a = 37°$, $b = 58° 20'$,
$C = 53° 30'$, $d = 45° 15'$,
$S = b + c = 111° 50'$.

Log. sin. b = sin.	58° 20′	.	9·9299891
sin. d = sin.	45 15	.	9·8513717
sin. $(a + b + c)$ = sin.	148 50 or 31 10	.	9·7139349
cosec. a = cosec.	37 0	.	10·2205370
cosec. c = cosec.	53 30	.	10·0948213
cosec. $(b + c + d)$ = cosec.	157 5 or 22 25	.	10·4096131
cosec. S = cosec.	111 50 or 68 10	.	10·0323259
			0·2525930

nat. cot. S = — ·4006465
nat. numb. to ·2525930 is 1·7889250

cot. x = cot. 35° 46′ 1·3882795

Log. AB	2·7781513
sin. a = 20 — cosec. a, as above . . .	9·7794630
sin. c = 20 — cosec. c, ditto	9·9051787
cosec. $(a+b+c)$ = 20 — sin. $(a+b+c)$.	10·2860651
cosec. x	10·2332261
CD = 959·52 log. = . . .	2·9820842

EXAMPLE II.

Suppose a, b, c, d, to be as in the former example, and CD = 863·64 : required AB.

$$\begin{array}{lll} \text{Log. CD} & = & 2\cdot9363327 \\ \sin\ (a+b+c) & = & 9\cdot7139349, \text{ as above} \\ \text{cosec. } a & = & 10\cdot2205370, \text{ ditto} \\ \text{cosec. } c & = & 10\cdot0948213, \text{ ditto} \\ \sin.\ x & = & 9\cdot7667739, = 20 - \text{cosec. } x \end{array}$$

$$AB = 540\cdot0 / \log. = 2\cdot7323098$$

EXAMPLES FOR EXERCISE.

1. An obstacle prevented the measurement of the part BC of a line AD; but a point E was selected from which the angles subtended by each segment were measured: viz. AB $= a$, CD $= c$, AEB $= \alpha$, BEC $= \beta$ and CED $= \gamma$. Which is the length of the time? And give the calculation when $\alpha = 23° 10'$ $\beta = 24° 15'$ and $\gamma = 33° 18'$, the segments a and b being 151 and 176 yards respectively.

Ans. x (the segment BC) is found from $(a + x)\ (c + x) \dfrac{ac \sin.\ \alpha + \beta \sin.\ \beta + \gamma}{\sin.\ \alpha \sin.\ \gamma}$

2. Being on the opposite side of a river from two steeples O and W, whose distance I knew by a previous survey to be 6954 yards, to save the trouble of measuring the distance between two other objects, A and B, on account of the irregularity of the intervening ground, I took the angles subtended at its extremities by each steeple and the other extremity, viz. OAW $= 85° 46'$, WAB $= 23° 56'$, OBW $= 31° 48$, OBA $= 68° 2'$. What was the distance AB? And give a general formula for the solution of this and its converse problem, viz. Ex. 24, p. 413, vol. i. Ans. to the case proposed AB $= 4694$ yards.

3. From a point within a regular hexagonal field, whose area was known to be 6 acres, 2 roods, 15·18 perches, the two opposite sides, whose bearing was E. by N., subtended angles of 48° 10′ and 76° 18′: what were the bearing and distance of the centre of the field? the angles subtended by the other sides?

4. A person walking from C to D on a straight horizontal road can see plainly an object on the summit of the hill A at every point but E, where he can just distinguish the *top* of the object over the hill B. He then measures a base EC of 150 yards, and at C observes the elevations of B and A to be 56° 10′ 15″ and 59° 18′ 15″, and he also finds the angles ACB and ACE to be 10° 12′ 20″, and 69° 18′ 30″, and the angle ACE $= 108° 12' 15''$. Can he from these data ascertain the heights of the hills and their *horizontal* distances from each other, and from the places of observation C and E?

5. From three positions in the same horizontal plane ABC whose distances from each other taken in order were 150·25, 179·69, and 205·36 yards, the elevations of the top of a tower on a hill were found to be 6° 10′ 55″, 7° 18′ 3″, and 6° 58′ 58″, whilst the elevation of its base from the first station was 6° 2′ 58″. Find the altitude of the tower.

6. Four points A, B, C, D are known from the following data, AB $= 815$ yards, BC $= 670$, CD $= 660$; ABC $= 49° 54'$, BCD $= 73° 57'$; and from these data the three points M, N, P, which are inaccessible, are to be found: to accomplish which, the following angles are measured, viz. AMB $= 80° 8'$, BMN $= 24° 55'$ CNM $= 124° 16'$, CNP $= 98° 14'$, CPN $= 29° 13'$, and CPD $= 51° 19$.

SPHERICAL TRIGONOMETRY.

SECTION I.

General Properties of Spherical Triangles.

ART. 1. *Def.* 1.—Any portion of a spherical surface bounded by three arcs of great circles, is called *Spherical Triangle.*

Def. 2. Spherical Trigonometry is the art of computing the measures of the sides and angles of spherical triangles.

Def. 3. A *right-angled* spherical triangle has one right angle: the sides about the right angle are called *legs;* the side opposite to the right angle is called the *hypothenuse.*

Def. 4. A *quadrantal* spherical triangle has one side equal to 90° or a quarter of a great circle.

Def. 5. The two arcs or angles, when compared together, are said to be *alike,* or of the *same affection,* when both are less than 90°, or both are greater than 90°. But when one is greater and the other less than 90°, they are said to be *unlike,* or of *different affections.*

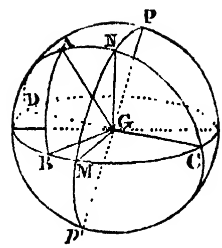
ART 2. The small circles of the sphere do not fall under consideration in Spherical Trigonometry; but such only as have the same centre with the sphere itself. And hence it is that spherical trigonometry is of so much use in Practical Astronomy, the apparent heavens assuming the shape of a concave sphere, whose centre is the same as the centre of the earth.

3. Every spherical triangle has three sides and three angles: and if any three of these six parts be given, the remaining three may be found, by some of the rules which will be investigated in this chapter.

4. In *plane* trigonometry, the knowledge of the three angles is not sufficient for ascertaining the sides: for in that case the *relations* only of the three sides can be obtained, and not their absolute values: whereas, in *spherical* trigonometry, where the sides are circular arcs, whose values depend on their proportion to the whole circle, that is, on the number of degrees they contain, the sides may always be determined when the three angles are known. Other remarkable differences between the plane and spherical triangles are, 1st. That in the former, two angles always determine the third; while in the latter they never do 2ndly. The surface of a plane triangle cannot be determined from a knowledge of the angles alone; while that of a spherical triangle always can.

5. The *sides* of a spherical triangle are all arcs of great circles, which, by their intersection on the surface of the sphere, constitute that triangle.

6. The *angle* which is contained between the arcs of two great circles, intersecting each other on the surface of the sphere, is called a spherical angle; and its measure is the same as the measure of the plane angle which is formed by two lines issuing from the same point of, and perpendicular to, the common section of the planes which determine the containing sides; that is to say, it is the same as the angle made by those planes. Or, it is equal to the plane angle formed by the tangents to those arcs at their point of intersection.



7. Hence it follows, that the surface of a spherical triangle BAC, and the three planes which determine it, form a kind of triangular pyramid, BCGA, of which the vertex G is at the centre of the sphere, the base ABC a portion of the spherical surface, and the faces AGC, AGB, BGC, sectors of the great circles whose intersections determine the side of the triangle.

Def. 6. A line perpendicular to the plane of a great circle, passing through the centre of the sphere, and terminated by two points, diametrically opposite, at its surface, is called the *axis* of such circle; and the extremities of the axis, or the points where it meets the surface, are called the *poles*, of that circle. Thus, PGP′ is the axis, and P, P′, are the poles, of the great circle CND.

7. If we conceive any number of less circles, each parallel to the said great circle, this axis will be perpendicular to them likewise; and the points P, P′, will be their poles also.

8. Hence, each pole of a great circle is 90° distant from every point in its circumference; and all the arcs drawn from either pole of a less circle to its circumference, are equal to each other.

9. It likewise follows, that all the arcs of great circles drawn through the poles of another great circle, are perpendicular to it: for, since they are great circles by the supposition, they all pass through the centre of the sphere, and consequently through the axis of the said circle. The same thing may be affirmed with regard to small circles.

10. Hence, in order to find the *poles* of any circle, it is merely necessary to describe, upon the surface of the sphere, two great circles perpendicular to the plane of the former; the points where these circles intersect each other will be the poles required.

11. It may be inferred also, from the preceding, that if it were proposed to draw, from any point assumed on the surface of the sphere, an arc of a circle which may measure the shortest distance from that point, to the circumference of any given circle; this arc must be so described, that its prolongation may pass through the poles of the given circle. And conversely, if an arc pass through the poles of a given circle, it will measure the shortest distance from any assumed point to the circumference of that circle.

12. Hence again, if upon the sides, AC and BC, (produced if necessary) of a spherical triangle BCA, we take the arcs CN, CM, each equal 90°, and through the radii GN, GM (figure to art. 7) draw the plane NGM, it is manifest that the point C will be the pole of the circle coinciding with the plane NGM: so that, as the lines GM, GN, are both perpendicular to the common section GC, of the planes AGC, BGC, they measure, by their inclination, the angle of these planes; or the arc NM measures that angle, and consequently the spherical angle BCA.

13. It is also evident that every arc of a less circle, described from the pole C as centre, and containing the same number of degrees as the arc MN, is equally proper for measuring the angle BCA; though it is customary to use only arcs of great circles for this purpose.

14. Lastly, we infer, that if a spherical angle be a right angle, the arcs of the great circles which form it, will pass mutually through the poles of each other, and that, if the planes of two great circles contain each the axis of the other, or

pass through the poles of each other, the angle which they include is a right angle.

These obvious truths being premised and comprehended, the student may pass to the consideration of the following theorems.

THEOREM I.

Any two sides of a spherical triangle are together greater than the third.

This proposition is a necessary consequence of the truth, that the shortest distance between any two points, measured on the surface of the sphere, is the arc of a great circle passing through these points.

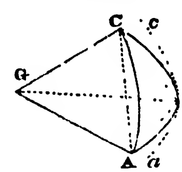
THEOREM II.

The sum of the three sides of any spherical triangle is less than a complete great circle, or 360 degrees.

For, let the sides AC, BC, (fig. to art. 7) containing any angle A, be produced till they meet again in D : then will the arcs DAC, DBC, be each 180°, because all great circles cut each other into two equal parts : consequently DAC + DBC = 360°. But (theorem 1) DA and DB are together greater than the third side AB of the triangle DAB; and therefore, since CA + CB + DA + DB = 360°, the sum CA + CB + AB is less than 360°. Q. E. D.

THEOREM III.

The sum of the three angles of any spherical triangle is always greater than two right angles, but less than six.



For, let ABC be a spherical triangle, G the centre of the sphere, and let the chords of the arcs AB, BC, AC, be drawn : these chords constitute a rectilinear triangle, the sum of whose three angles is equal to two right angles. But the angle at B made by the chords, AB, BC, is *less* than the angle aBc, formed by the two tangents Ba, Bc, or less than the angle of inclination of the two planes GBC, GBA, which (art. 6) is the spherical angle at B; consequently the spherical angle at B is *greater* than the angle at B made by the chords AB, CB. In like manner, the spherical angles at A and C are greater than the respective angles made by the chords meeting at those points. Consequently the sum of the three angles of the spherical triangle ABC, is greater than the sum of the three angles of the rectilinear triangle made by the chords AB, BC, AC, that is, greater than two right angles. Q. E. 1° D.

2. The angle of inclination of no two of the planes can be so great as two right angles; because, in that case, the two planes would become but one continued plane, and the arcs, instead of being arcs of distinct circles, would be joint arcs of one and the same circle. Therefore, each of the three spherical angles must be less than two right angles; and consequently their sum less than six right angles. Q. E. 2° D.

Cor. 1. Hence it follows, that a spherical triangle may have all its angles either right or obtuse; and therefore the knowledge of any two angles is not sufficient for the determination of the third.

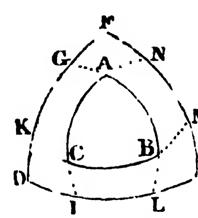
Cor. 2. If the three angles of a spherical triangle be right or obtuse, the three

sides are likewise each equal to or greater than 90°: and, if each of the angles be acute, each of the sides is also less than 90°; and conversely.

Scholium. From the preceding theorem the student may clearly perceive what is the essential difference between plane and spherical triangles, and how absurd it would be to apply the rules of plane trigonometry to the solution of cases of spherical trigonometry. Yet, though the difference between the two kinds of triangles be really so great, still there are various properties which are common to both, and which may be demonstrated exactly in the same manner. Thus, for example, it might be demonstrated here (as well as with regard to plane triangles in the elements of Geometry, vol. i.), that two spherical triangles are equal to each other, 1st. When the three sides of the one are respectively equal to the three sides of the other: 2ndly. When each of them has an equal angle contained between equal sides: and, 3rdly. When they have each two equal angles at the extremities of equal bases. It might also be shown, that a spherical triangle is equilateral, isosceles, or scalene, according as it hath three equal, two equal, or three unequal angles: and again, that the greatest side is always opposite to the greatest angle, and the least side to the least angle. But the brevity that our plan requires compels us merely to *mention* these particulars. It may be added, however, that a spherical triangle may be at once *right-angled* and *equilateral;* which can never be the case with a plane triangle *.

THEOREM IV.

If from the angles of a spherical triangle, as poles, there be described, on the surface of the sphere, three arcs of great circles, which by their intersections form another spherical triangle; each side of this new triangle will be the supplement to the measure of the angle which is at its pole, and the measure of each of its angles the supplement to that side of the primitive triangle to which it is opposite.



From B, A, and C, as poles, let the arcs DF, DE, FE, be described, and by their intersections form another spherical triangle DEF; either side, as DE, of this triangle, is the supplement of the measure of the angle A at its pole; and either angle, as D, has for its measure the supplement of the side AB.

Let the sides AB, AC, BC, of the primitive triangle, be produced till they meet those of the triangle DEF, in the points I, L, M, N, G, K: then, since the point A is the pole of the arc DILE, the distance of the points A and E (measured on an arc of a great circle) will be 90°; also, since C is the pole of the arc EF, the points C and E will be 90° distant: consequently (art. 8) the point E is the pole of the arc AC. In like manner it may be shown, that F is the pole of BC, and D that of AB.

This being premised, we shall have DL = 90°, and IE = 90°; whence DL + IE = DL + EL + IL = DE + IL = 180°. Therefore DE = 180° − IL: that is, since IL is the measure of the angle BAC, the arc DE is = the supplement of that measure. Thus also may it be demonstrated that EF is equal the supplement to MN, the measure of the angle BCA, and that DF is equal the

* These properties are readily derived from the expressions given in the following pages; as they may also from a consideration of the relations between the planes whose intersections with the sphere produce the several triangles.

supplement to GK, the measure of the angle ABC: which constitutes the first part of the proposition.

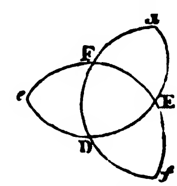
2ndly. The respective measures of the angles of the triangle DEF are supplemental to the opposite sides of the triangle ABC. For, since the arcs AL and BG are each 90°, therefore is $AL + BG = GL + AB = 180°$; when $GL = 180° - AB$; that is, the measure of the angle D is equal to the supplement to AB. So likewise may it be shown that AC, BC, are equal to the supplements to the measures of the respectively opposite angles E and F. Consequently, the measures of the angles of the triangle DEF are supplemental to the several opposite sides of the triangle ABC. Q. E. D.

Cor. 1. Hence these two triangles are called *supplemental* or *polar* triangles *.

Cor. 2. Since the three sides DE, EF, DF, are supplements to the measures of the three angles A, B, C; it results that $DE + EF + DF + A + B + C = 3 \times 180° = 540°$. But (theor. 2), $DE + EF + DE < 360°$; consequently $A + B + C > 180°$. Thus the first part of theorem 3 is very compendiously demonstrated.

Cor. 3. This theorem suggests mutations that are sometimes of use in computation. Thus, if three angles of a spherical triangle are given, to find the sides: the student may subtract each of the angles from 180°, and the three remainders will be the three sides of a new triangle; the angles of this new triangle being found, if their measures be each taken from 180°, the three remainders will be the respective sides of the primitive triangle, whose angles were given.

Scholium. The invention of the preceding theorem is due to *Philip Langsberg.* Vide Simon Stevin, liv. 3, de la Cosmographie, prop. 31, and Alb. Girard in loc. It is often, however, treated very loosely by authors on trigonometry: some of them speaking of sides as the supplements of angles, and scarcely any of them remarking which of the several triangles formed by the intersection of the arcs DE, EF, DF, is the one in question. Besides the triangle DEF, three others may be formed by the intersection of the semicircles, and if the *whole* circles be considered, there will be *seven* other triangles formed.



The Editor avails himself of the complete investigation of these triangles first given by his colleague, Mr. Davies, in his Supplement to Young's Trigonometry. They are not only curious in themselves, but lead to some important simplifications in the investigations of Spherical Trigonometry.

It is necessary to premise the following definitions:—The three spherical triangles formed by producing the sides till they meet in A′, B′, C′, lay them with the *fundamental triangle* ABC (next figures) constitute an *associated system of triangles.*

THEOREM V.

If the associated systems of the two polar triangles ABC, *abc*, be completed as in figure, they will be separable into four pairs (one of each system forming a pair) of mutually polar triangles. Thus,

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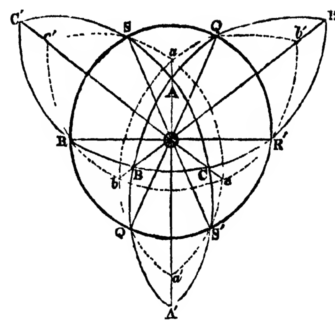
* The term *supplemental* has been generally used by English, and *polar* by the continental writers. The former expresses a *property*, the latter the *genesis* of the triangles; and is the preferable, not on that account only, but also on account of the completeness of its definition.



ABC, *abc*, are the first pair,
AB′C, *ab′c*, are the second,
A′BC, *a′bc*, are the third,
and ABC′, *abc′*, are the fourth.

"Now the first pair ABC, *abc* are by hypothesis *polars* *. Hence B is the pole of *ca*, that is of *ac′*; A is the pole of *bc*, that is of *bc′*; and since C is the pole of *ab*, therefore C′ is the other pole of *ab*. The points A, B, C′ are the poles of the sides, therefore, of the triangle *abc′*. Whence, also, it follows by the reciprocity of the polar system, that *a*, *b*, *c′* are the poles of ABC′.

"In the same way it is shown that ACB′ and *acb′* form a polar system, and that BA′C, *ba′c*, form another. Hence it follows that if one triangle of an associated system be polar to one of another associated system, then each of the triangles of one system is polar to one triangle of the other associated system.

"*Cor.* 1. Draw the great circle *a*A (next figure), and produce it to meet BC, *bc* in G and H. Then, because *a* is the pole of BC, and that A is the pole of *bc*, the arcs *a*G, AH, are quadrants. Hence,

$$aG + AH = aH + AG = \pi$$

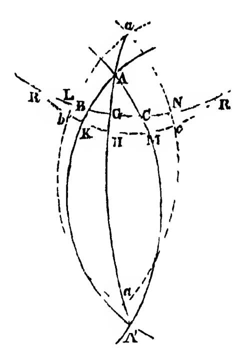
and the angles at G and H are right angles.

"Let R, R′, be the intersections of BC, *bc*; then, because the angles at G and H are right, R, R′, are the poles of *a*A.

"Let AB meet *bc* in K, BC meet *ab* in L, AC meet *bc* in M, and *ac* meet Bc in N.

"Then

$$\left\{\begin{array}{l} bH \backsim Hc = GAC \backsim GAB;\ \text{and} \\ bH + GAB = \dfrac{\pi}{2} = Hc + GAC. \end{array}\right\}$$



For

$$bM = bH + HM = bH + HAM = bH + GAC$$

In like manner,

$$cK = cH + HK = cH + HAK = cH + GAB.$$

"Also $bM = \dfrac{\pi}{2} = cK$, since b and c are the poles AM and CK.

"Hence the two parts of the proposition as stated are true.

"* When we speak of parallel lines, without specifying which is taken as the line of reference, and knowing that the second is related to the first in the same way that the first is to the second, we simply denominate them parallels. The same practice also holds in speaking of two mutually supplemental angles. But when we previously fix upon one line or one angle, we say the parallel, or the supplement in the singular number to express the other line or angle. Just so in respect to the two triangles which constitute a polar system, when we speak of both without assigning the reference, we call them polars, as a common epithet; but when we have fixed upon one already, as that to which the other is referred, we call it the *primary*, and the other the *polar triangle*."

"*Cor.* 2. Returning now to our original figure, let the three arcs Aa, Bb, Cc, be drawn; these will pass through the same point O, because, as has been just shown, they are perpendicular to the three sides BC, AC, AB, (vide p. 54). Also, because aA is perpendicular to bc, it will pass through the opposite pole a'. In like manner it will pass through A$'$. Or the points aAOa' A$'$ are in the same great circle, whose poles are R and R$'$, the intersections of BC, bc.

"In like manner bBOB$'$ B$'$ are in one great circle, whose poles are S, S$'$, the intersections of AC, ac; cCOc' C$'$ are in one great circle, whose poles are Q, Q$'$, the intersections of AB, ab.

"Again, because R, R$'$, are the poles of aa', the arcs RO, OR$'$, are quadrants. In like manner SO, OS$'$, and QO, OQ$'$, are respectively quadrants; and as the quadrants are drawn from the poles of aa', bb', cc', respectively, R, O, R$'$; S, O, S$'$, and Q, O, Q$'$, are respectively in the same great circles.

"*Cor.* 3. Also since $\mathrm{OR} = \mathrm{OR}' = \mathrm{OS} = \mathrm{OS}' = \mathrm{OQ} = \mathrm{OQ}' = \frac{\pi}{2}$, the points R, R$'$, S, S$'$, Q, Q$'$, are in the same great circle, and O is its pole.

"The complexity of the figure which would result from a further detail of these interesting researches, compels us to leave them to the industry and ingenuity of the student. They are exceedingly easy, and furnish an excellent exercise in spherical investigations; and we therefore hope he will give it a proper degree of consideration."

SECTION II.

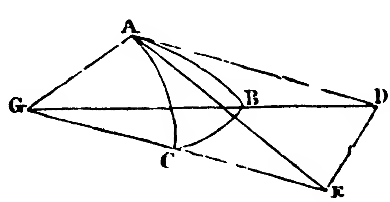
Resolution of Spherical Triangles.

THE different cases of spherical trigonometry, like those in plane trigonometry, may be solved either geometrically or algebraically. We shall here adopt the analytical method, as well on account of its being more compatible with brevity, as because of its correspondence and connexion with the substance of the preceding chapters *. The whole doctrine may be comprehended in the subsequent problems and theorems.

PROBLEM I.

To find equations, from which may be deduced the solution of all the cases of spherical triangles.

Let ABC be a spherical triangle; AD the tangent, and GD the secant, of the arc AB; AE the tangent, and GE the secant, of the arc AC; let the capital letters A, B, C, denote the angles of the triangle, and the small letters a, b, c, the opposite sides BC, AC, AB. Then the first equations in art. 6 Pl. Trig. applied to the two triangles ADE, GDE, give, for the



* For the geometrical method, the reader may consult Simson's or Playfair's *Euclid*, or Bishop Horsley's *Elementary Treatises on Practical Mathematics*.

former, $DE^2 = \tan.^2 b + \tan.^2 c - 2 \tan. b \tan. c \cos. A$; for the latter, $DE^2 = \sec.^2 b + \sec.^2 c - 2 \sec. b \sec. c \cos. a$. Subtracting the first of these equations from the second, and observing that $\sec.^2 b - \tan.^2 b = R^2 = 1$, we shall have after a little reduction, $1 + \frac{\sin. b \sin. c}{\cos. b \cos. c} \cos. A. - \frac{\cos. a}{\cos. b \cos. c} = 0$.

Whence the three following symmetrical equations are obtained:

$$\left.\begin{array}{l} \cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A \\ \cos. b = \cos. a \cos. c + \sin. a \sin. c \cos. B \\ \cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C \end{array}\right\} \quad . \quad . \quad (\text{I.})$$

THEOREM VI.

[The *numerical* exercises having been originally placed after the investigations, are so retained in this edition. The teacher will, of course, require those belonging to each case to be worked after the formula for its solution has been studied, in the same manner as in plane trigonometry.]

In every spherical triangle, the sines of the angles are proportional to the sines of their opposite sides.

If, from the first of the equations marked I. the value of cos. A be drawn, and substituted for it in the equation $\sin.^2 A = 1 - \cos.^2 A$, we shall have

$$\sin.^2 A = 1 - \frac{\cos.^2 a + \cos.^2 b \cos.^2 c - 2 \cos. a \cos. b \cos. c}{\sin.^2 b \sin.^2 c}.$$

Reducing the terms of the second side of this equation to a common denominator, extracting the square root, and multiplying both numerator and denominator by sin. a, there will result

$$\sin. A = \sin. a \,.\, \frac{\sqrt{(1 - \cos.^2 a - \cos.^2 b - \cos.^2 c + 2 \cos. a \cos. b \cos c)}}{\sin. a \sin. b \sin. c}.^*$$

Here, if the whole fraction which multiplies sin. a, be denoted by K (see art. 8, page 19), we may write $\sin. A = K . \sin. a$. And, since the fractional factor, in the above equation, contains terms in which the sides a, b, c, are alike affected, we have similar equations for sin. B, and sin. c. That is to say, we have

$\sin. A = K \sin. a$ . . . $\sin. B = K \sin. b$ . . . $\sin. C = K \sin. c$.

Consequently, $\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c}$ . . . . . . (II.)

which is the algebraical expression of the theorem.

PROBLEM II.

Given the three sides of a spherical triangle: it is required to find expressions for the determination of the angles.

Retaining the notation of prob. 1, in all its generality, we soon deduce from the equations marked 1 in that problem, the following, viz.

* The quantity under the radical is reducible to factors, thus;—

$$\left\{\sin. a \sin. c + \cos. b - \cos. a \cos. c\right\} \left\{\sin. a \sin. c - \cos. b + \cos. a \cos. c\right\}$$
$$= \left\{\cos. b - \cos. (a + c)\right\} \left\{-\cos. b + \cos. (a - c)\right\} = 2 \sin. \frac{a+b+c}{2} \sin. \frac{a+b-c}{2} \times$$
$$2 \sin. \frac{a-b+c}{2} \sin. \frac{-a+b+c}{2}. \text{ Or if } s = \frac{a+b+c}{2}, \text{ it becomes}$$

$4 \sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)$; a form of great elegance and of frequent use in Spherical researches.

$$\left.\begin{aligned}\cos. A &= \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}\\ \cos. B &= \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c}\\ \cos. C &= \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}\end{aligned}\right\}.$$

As these equations, however, are not well suited for logarithmic computation; they must be so transformed, that their second members will resolve into factors. In order to this, substitute in the known equation $1 - \cos. A = 2\sin^2 \frac{1}{2}A$, the preceding value of cos. A, and there will result

$$2\sin^2 \tfrac{1}{2}A = \frac{\cos. (b-c) - \cos. a}{\sin. b \sin. c}.$$

But, because $\cos. B' - \cos. A' = 2\sin. \frac{1}{2}(A' + B') \sin. \frac{1}{2}(A' - B')$ (art. 25, page 25), and consequently

$$\cos. (b - c) - \cos. a = 2 \sin. \frac{a+b-c}{2} . \sin. \frac{a+c-b}{2}$$

we have obviously,

$$\sin.^2 \tfrac{1}{2}A = \frac{\sin. \frac{1}{2}(a+b-c) \sin. \frac{1}{2}(a+c-b)}{\sin. b \sin. c}.$$

Whence, making $2s = a + b + c$, there results

$$\left.\begin{aligned}\sin. \tfrac{1}{2}A &= \sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}}\\ \text{So, also, } \sin. \tfrac{1}{2}B &= \sqrt{\frac{\sin. (s-a) \sin. (s-c)}{\sin. a \sin. c}}\\ \text{And, } \sin. \tfrac{1}{2}C &= \sqrt{\frac{\sin. (s-a) \sin. (s-b)}{\sin. a \sin. b}}\end{aligned}\right\}\text{(III.)}$$

The expressions for the cosines and tangents of the half angles might have been deduced with equal facility; and we should have obtained,

$$\left.\begin{aligned}\cos. \tfrac{1}{2}A &= \sqrt{\frac{\sin. s \sin. (s-a)}{\sin. b \sin. c}} & \tan. \tfrac{1}{2}A &= \sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. s \sin. (s-a)}}\\ \cos. \tfrac{1}{2}B &= \sqrt{\frac{\sin. s \sin. (s-b)}{\sin. c \sin. a}} & \tan. \tfrac{1}{2}B &= \sqrt{\frac{\sin. (s-c) \sin. (s-a)}{\sin. s \sin. (s-b)}}\\ \cos. \tfrac{1}{2}C &= \sqrt{\frac{\sin. s \sin. (s-c)}{\sin a \sin. b}} & \tan. \tfrac{1}{2}C &= \sqrt{\frac{\sin. (s-a) \sin. (s-b)}{\sin. s \sin. (s-c)}}\end{aligned}\right\}(iii)$$

The analogy between these expressions, and the expressions for the same quantities in a plane triangle are well worthy of the student's remark.

Cor. 1. When two of the sides, as b and c, become equal, then the expression for $\sin. \frac{1}{2}A$ becomes $\sin. A = \frac{\sin. (s-b)}{\sin. b} = \frac{\sin. \frac{1}{2}A}{\sin. c}$.

Cor. 2. When all the three sides are equal, or $a = b = c$, then $\sin. \frac{1}{2}A = \frac{\sin. \frac{1}{2}a}{\sin. a}$.

Cor. 3. In this case, if $a = b = c = 90°$; then $\sin. \frac{1}{2}A = \frac{\frac{1}{2}\sqrt{2}}{1} = \frac{1}{2}\sqrt{2} = \sin. 45°$: and $A = B = C = 90°$.

Cor. 4. If $a = b = c = 60°$: then $\sin. \frac{1}{2}A = \frac{\frac{1}{2}}{\sqrt{3}/2} = \frac{1}{3}\sqrt{3} = \sin. 35° 15' 51''$: and $A = B = C = 70° 31' 42''$: the same as the angle between two contiguous planes of a tetraëdron.

Cor. 5. If $a = b = c$ were assumed $= 120°$: then $\sin. \frac{1}{2}A = \frac{\sin. 60°}{\sin. 120°} =$

$\frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}\sqrt{3}} = 1$; and $A = B = C = 180°$; which shows that no such triangle can be constructed (conformably to theor. 2); but that the three sides would, in such case, form three continued arcs completing a great circle of the sphere.

PROBLEM III.

Given the three angles of a spherical triangle, to find expressions for the sides.

If from the first and third of the equations marked I (prob. 1), cos. c be exterminated, there will result

$$\cos. A \sin. c + \cos. C \sin. a \cos. b = \cos. a \sin. b.$$

But, it follows from theor. 6, that $\sin. c = \frac{\sin. a . \sin. C}{\sin. A}$. Substituting for sin. c this value of it, and for $\frac{\cos. A}{\sin. A}, \frac{\cos. a}{\sin. a}$, their equivalents cot. A, cot. a, we shall have, $\cot. A \sin. C + \cos. C \cos. b = \cot. a \sin. b$.

Now, $\cot. a \sin. B = \frac{\cos\ a}{\sin. a} \sin. b = \cos. a \frac{\sin. b}{\sin. a} = \cos. a . \frac{\sin. B}{\sin. A}$, (theor. 6).

So that the preceding equation at length becomes

$$\cos. A \sin. C = \cos. a \sin. B - \sin. A \cos. C \cos. b.$$

In like manner we have,

$$\cos. B \sin. C = \cos. b \sin. A - \sin. B \cos. C \cos. a.$$

Exterminating cos. b from these, and applying the same process to the other combinations, there results,

$$\left.\begin{array}{l} \cos. A = \cos. a \sin. B \sin. C - \cos. B \cos. C. \\ \cos. B = \cos. b \sin. A \sin. C - \cos. A \cos. C. \\ \cos. C = \cos. c \sin. A \sin. B - \cos. A \cos. B. \end{array}\right\} \text{(IV.)}$$

This system of equations is manifestly analogous to equation I; and if they be reduced in the manner adopted in the last problem, they will give, (making $2S = A + B + C$),

$$\left.\begin{array}{l} \sin. \frac{1}{2}a = \sqrt{-\frac{\cos. \frac{1}{2}(A+B+C) \cos. \frac{1}{2}(B+C-A)}{\sin. B \sin. C}} = \sqrt{-\frac{\cos. S \cos. (S-A)}{\sin. B \sin. C}} \\ \sin. \frac{1}{2}b = \sqrt{-\frac{\cos. \frac{1}{2}(A+B+C) \cos\ \frac{1}{2}(A+C-B)}{\sin. A \sin. C}} = \sqrt{-\frac{\cos. S \cos. (S-B)}{\sin. C \sin. A}} \\ \sin. \frac{1}{2}c = \sqrt{-\frac{\cos. \frac{1}{2}(A+B+C) \cos. \frac{1}{2}(A+B-C)}{\sin. A \sin. B}} = \sqrt{-\frac{\cos. S \cos. (S-C)}{\sin. A \sin. B}} \end{array}\right\} \text{(V.)}$$

The expression for the cosine and the cotangent of half a side is, in like manner

$$\left.\begin{array}{l} \cos. \frac{1}{2}a = \sqrt{\frac{\cos. (S-B) \cos. (S-C)}{\sin. B \sin. C}}; \cot. \frac{1}{2}a = \sqrt{-\frac{\cos. (S-B) \cos. (S-C)}{\cos. S \cos. (S-A)}} \\ \cos. \frac{1}{2}b = \sqrt{\frac{\cos. (S-C) \cos. (S-A)}{\sin. C \sin. A}}; \cot. \frac{1}{2}b = \sqrt{-\frac{\cos. (S-C) \cos. (S-A)}{\cos. S \cos. (S-B)}} \\ \cos. \frac{1}{2}c = \sqrt{\frac{\cos. (S-A) \cos. (S-B)}{\sin. A \sin. B}}; \cot. \frac{1}{2}c = \sqrt{-\frac{\cos. (S-A) \cos. (S-B)}{\cos. S \cos. (S-C)}} \end{array}\right\} (v.)$$

Cor. 1. When two of the angles, as B and C, become equal, then the value of cos. $\frac{1}{2}a$ becomes $\cos. \frac{1}{2}a = \frac{\cos. \frac{1}{2}A}{\sin. B}$.

Cor. 2. When $A = B = C$; then $\cos. \frac{1}{2}a = \frac{\cos. \frac{1}{2}A}{\sin. A}$.

Cor. 3. When $A = B = C = 90°$, then $a = b = c = 90°$.

Cor. 4. If $A = B = C = 60°$; then $\cos. \frac{1}{2}a = \frac{\sin. 60°}{\sin. 60} = 1$.

So that $a = b = c = 0$. Consequently no such triangle can be constructed: conformably to theor. 3.

Cor. 5. If $A = B = C = 120°$: then $\cos. \frac{1}{2}a = \frac{\cos. 60°}{\sin. 120°} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}\sqrt{3} = \cos. 54°44'9''$. Hence $a = b = c = 109°28'18''$.

Schol. If, in the preceding values of $\sin. \frac{1}{2}a$, $\sin. \frac{1}{2}b$, &c. the quantities under the radical were negative in reality, as they are in appearance, it would obviously be impossible to determine the value of $\sin. \frac{1}{2}a$, &c. But this value is in fact always real. For, in general, $\sin. (x - \frac{1}{4}\bigcirc) = -\cos. x$: therefore, $\sin. \left(\frac{A+B+C}{2} - \frac{1}{4}\bigcirc\right) = -\cos. \frac{1}{2}(A + B + C)$, a quantity which is always positive; because, as $A + B + C$ is necessarily comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{2}\bigcirc$, we have $\frac{1}{2}(A + B + C) - \frac{1}{4}\bigcirc$ greater than nothing, and less than $\frac{1}{2}\bigcirc$. Further, any one side of a spherical triangle being smaller than the sum of the other two, we have, by the property of the polar triangle (theorem 4), $\frac{1}{2}\bigcirc - A$ less than $\frac{1}{2}\bigcirc - A$ less than $\frac{1}{2}\bigcirc - B + \frac{1}{2}\bigcirc - C$; whence $\frac{1}{2}(B + C - A)$ less than $\frac{1}{4}\bigcirc$; and of course its cosine is positive *.

PROBLEM IV.

Given two sides of a spherical triangle, and the included angle; to obtain expressions for the other angles.

1. In the investigation of the last problem, we had

$$\cos. A \sin. c = \cos. a \sin. b - \cos. C \sin. a \cos. b$$

and by a simple permutation of letters, we have

$$\cos. B \sin. c = \cos. b \sin. a - \cos. C \sin. b \cos. a.$$

adding together these two equations, and reducing, we have

$$\sin. c (\cos. A + \cos. B) = (1 - \cos. C) \sin. (a + b).$$

Now, we have from theor. 6,

$$\frac{\sin. a}{\sin. A} = \frac{\sin. c}{\sin. C}, \text{ and } \frac{\sin. b}{\sin. B} = \frac{\sin. c}{\sin. C}.$$

Freeing these equations from their denominators, and respectively adding and subtracting them, there results

$$\sin. c (\sin. A + \sin. B) = \sin. C (\sin. a + \sin. b),$$
$$\text{and } \sin. c (\sin. A - \sin. B) = \sin. C (\sin. a - \sin. b).$$

Dividing each of these two equations by the preceding, there will be obtained

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. C}{1 - \cos. C} \cdot \frac{\sin. a + \sin. b}{\sin. (a + b)}.$$
$$\frac{\sin. A - \cos. B}{\cos. A + \cos. B} = \frac{\sin. C}{1 - \cos. C} \cdot \frac{\sin. a - \sin. b}{\sin. (a + b)}.$$

Comparing these with the equations in arts. 25, 26, 27, page 26, there will at length result

$$\left.\begin{aligned} \tan. \tfrac{1}{2}(A + B) &= \cot. \tfrac{1}{2}C \cdot \frac{\cos. \frac{1}{2}(a - b)}{\cos. \frac{1}{2}(a + b)} \\ \tan. \tfrac{1}{2}(A - B) &= \cot. \tfrac{1}{2}C \cdot \frac{\sin. \frac{1}{2}(a - b)}{\sin. \frac{1}{2}(a + b)} \end{aligned}\right\} \quad . \qquad \text{(VI.)}$$

Cor. When $a = b$, the first of the above equations becomes $\tan. A = \tan. B = \cot. \frac{1}{2}C \sec. a$.

* The notation here employed, and at p. 20, &c. for the circle, has many advantages. Writers on trigonometry, however, often employ instead of it 2π, meaning by it $2 \times 3{\cdot}141593$, the circumference to radius 1.

And in this case it will be, as rad. : sin. $\frac{1}{2}$C : : sin. a or sin. b : sin. $\frac{1}{2}c$

And, as rad. : cos. A or cos. B : : tan. a or tan. b : tan. $\frac{1}{2}c$.

2. The preceding values of tan. $\frac{1}{2}$(A + B) and tan. $\frac{1}{2}$(A — B) are very well fitted for logarithmic computation : it may, notwithstanding, be proper to investigate a theorem which will at once lead to one of the angles, by means of a subsidiary angle. In order to this, we deduce immediately from the second equation in the investigation of prob. 3,

$$\text{cot. A} = \frac{\text{cot. } a \text{ sin. } b}{\text{sin. C}} - \text{cot. C cos. } b.$$

Then, choosing the subsidiary angle, ϕ, so that

$$\text{tan. } \phi = \text{tan. } a \text{ cos. C}\,^*,$$

that is, finding the angle ϕ, whose tangent is equal to the product tan. a cos. C, (which is equivalent to dividing the original triangle into two right-angled triangles,)the preceding equation will become

$$\text{cot. A} = \text{cot. C (cot. } \phi \text{ sin. } b - \text{cos. } b) = \frac{\text{cot. C}}{\text{sin. } \phi} (\text{cos. } \phi \text{ sin. } b - \text{sin. } \phi \,.\, \text{cos. } b).$$

And this, since sin. $(b - \phi) =$ cos. ϕ sin. b — sin. ϕ . cos. b, becomes

$$\text{cot. A} = \frac{\text{cot. C}}{\text{sin. } \phi} \,.\, \text{sin } (b - \phi).$$

Which is a very simple and convenient expression.

PROBLEM V.

Given two angles of a spherical triangle, and the side comprehended between them; to find expressions for the other two sides.

1 Here, a similar analysis to that employed in the preceding problem, being pursued with respect to the equations IV, in prob. 3, will produce the following formulæ:

$$\frac{\text{sin. } a + \text{sin. } b}{\text{cos. } a + \text{cos. } b} = \frac{\text{sin. } c}{1 + \text{cos. } c} \,.\, \frac{\text{sin. A} + \text{sin. B}}{\text{sin. (A + B)}},$$

$$\frac{\text{sin. } a - \text{sin. } b}{\text{cos. } a + \text{cos. } b} = \frac{\text{sin. } c}{1 + \text{cos. } c} \,.\, \frac{\text{sin. A} - \text{sin. B}}{\text{sin. (A + B)}}.$$

Whence, as in prob. 4, we obtain

$$\left.\begin{aligned} \text{tan. } \tfrac{1}{2}(a + b) &= \text{tan. } \tfrac{1}{2}c \,.\, \frac{\text{cos. } \frac{1}{2}(\text{A} - \text{B})}{\text{cos. } \frac{1}{2}(\text{A} + \text{B})}. \\ \text{tan. } \tfrac{1}{2}(a - b) &= \text{tan. } \tfrac{1}{2}c \,.\, \frac{\text{sin. } \frac{1}{2}(\text{A} - \text{B})}{\text{sin. } \frac{1}{2}(\text{A} + \text{B})}. \end{aligned}\right\} \text{(VII.)}\,\dagger$$

* In order to comprehend the principle of the simplifying transformations that spring from the introduction of a subsidiary angle, let it be recollected, that in all such cases the object is to change into a product a binomial of the general form

$$\text{M sin. } \theta + \text{N cos. } \theta.$$

Here the contrivance consists in putting as a common factor one of the quantities M or N; whence, for example, we have

$$\text{M} \left(\text{sin. } \theta + \text{cos. } \theta \,\frac{\text{N}}{\text{M}}\right).$$

Thus, putting $\frac{\text{N}}{\text{M}}$ equal to a tangent, or to a cotangent, lines which are susceptible of receiving all possible values, we shall have

$$\text{M} \left(\text{sin. } \theta + \text{cos. } \theta \,\frac{\text{sin. } \phi}{\text{cos. } \phi}\right) = \frac{\text{M sin. } (\theta + \phi)}{\text{cos. } \phi}$$

$$\text{or, M} \left(\text{sin. } \theta + \text{cos. } \theta \,\frac{\text{cos. } \phi}{\text{sin. } \phi}\right) = \frac{\text{M cos. } (\theta - \phi)}{\text{sin. } \phi}.$$

The two latter are evidently fitted for logarithmic computation.

† The formulæ marked VI and VII, converted into analogies, by making the denominator of

2. If it be wished to obtain a side at once, by means of a subsidiary angle, find ϕ so that $\frac{\text{cot. A}}{\text{cot. } c} = \text{tan. } \phi$; then will $\text{cot. } a = \text{cos. } (\text{B} - \phi) \frac{\text{cot. } b}{\text{cos. } \phi}$.

THEOREM VI.

Given two sides of a spherical triangle, and an angle opposite to one of them; to find the other opposite angle.

Suppose the sides given are a, b, and the given angle B: then from theor. 7, we have $\text{sin. A} = \frac{\text{sin. } a \text{ sin. B}}{\text{sin. } b}$; or, sin, A, a fourth proportional to sin. b, sin. B, and sin. a.

PROBLEM VII.

Given two angles of a spherical triangle, and a side opposite to one of them; to find the side opposite to the other.

Suppose the given angles are A, and B, and b the given side; then theor. 7

the second member the first term, the other two factors the second and third terms, and the first member of the equation the fourth term of the proportion, as

$$\text{cos. } \tfrac{1}{2}(a + b) : \text{cos. } \tfrac{1}{2}(a - b) :: \text{cot. } \tfrac{1}{2}\text{C} : \text{tan. } \tfrac{1}{2}(\text{A} + \text{B}),$$
$$\text{sin. } \tfrac{1}{2}(a + b) : \text{sin. } \tfrac{1}{2}(a - b) :: \text{cot. } \tfrac{1}{2}\text{C} : \text{tan. } \tfrac{1}{2}(\text{A} - \text{B}), \text{ \&c. \&c.}$$

are called *the Analogies of Napier*, being invented by that celebrated geometer. He likewise invented other rules for spherical trigonometry, known by the name of *Napier's Rules for the circular parts*, which will be found under the head *Mnemonics in Spherical Trigonometry*. Four other analogies, corresponding to these, have been given by the celebrated astronomer, Gauss, of Göttingen, which, in subsequent investigations in Spherical Trigonometry, are of great value, and even for the solution of the cases of spherical triangles to which they are adapted, are often more convenient than those of Napier. Those two which apply to the solution of Problem IV. bear the same relation to Thacker's theorems on plane triangles (Course, vol. i. p. 395) that Napier's do to the corresponding plane problem solved in the usual manner. Their peculiar advantage is felt when the base alone is sought, without requiring the other two angles, as they save three openings of the tables and two other steps of the process. We give them in this note, with an indication of the method employed in their investigation annexed.

From the formulæ in Problem II. we have, by multiplication,

$$\text{sin. } \tfrac{1}{2}\text{A cos. } \tfrac{1}{2}\text{B} = \frac{\text{sin. }(s - b)}{\text{sin. } c} \sqrt{\frac{\text{sin. } s \text{ sin. } (s - c)}{\text{sin. } b \text{ sin. } c}}$$

$$\text{sin. } \tfrac{1}{2}\text{B cos. } \tfrac{1}{2}\text{A} = \frac{\text{sin. }(s - a)}{\text{sin. } c} \sqrt{\frac{\text{sin. } s \text{ sin. } (s - c)}{\text{sin. } b \text{ sin. } c}}$$

And by addition and subtraction these give

$$\text{sin. } \tfrac{1}{2}(\text{A} + \text{B}) = \frac{\text{sin. }(s - b) + \text{sin. }(s - a)}{2 \text{ sin. } \tfrac{1}{2}c \text{ cos. } \tfrac{1}{2}c} \text{cos. } \tfrac{1}{2}c$$

$$\text{sin. } \tfrac{1}{2}(\text{A} - \text{B}) = \frac{\text{sin. }(s - b) + \text{sin. }(s - a)}{2 \text{ sin. } \tfrac{1}{2}c \text{ cos. } \tfrac{1}{2}c} \text{cos. } \tfrac{1}{2}c.$$

Or these are ultimately changed into

$$\text{sin. } \tfrac{1}{2}(\text{A} + \text{B}) = \frac{\text{cos. } \tfrac{1}{2}(a - b)}{\text{cos. } \tfrac{1}{2}c} \text{cos. } \tfrac{1}{2}c \quad \ldots (1)$$

$$\text{sin. } \tfrac{1}{2}(\text{A} - \text{B}) = \frac{\text{sin. }(a - b)}{\text{sin. } \tfrac{1}{2}c} \text{cos } \tfrac{1}{2}c \quad \ldots (2)$$

which are the first and second analogies of Gauss.

Also, taking the values of sin. $\tfrac{1}{2}$A sin. $\tfrac{1}{2}$B and cos. $\tfrac{1}{2}$A cos. $\tfrac{1}{2}$B from problem III., we shall find in precisely the same manner

$$\text{cos. } \tfrac{1}{2}(\text{A} - \text{B}) = \frac{\text{sin. } \tfrac{1}{2}(a + b)}{\text{sin. } \tfrac{1}{2}c} \text{sin. } \tfrac{1}{2}c \quad \ldots\ldots (3)$$

$$\text{cos. } \tfrac{1}{2}(\text{A} + \text{B}) = \frac{\text{cos. } \tfrac{1}{2}(a + b)}{\text{cos. } \tfrac{1}{2}c} \text{sin. } \tfrac{1}{2}c \quad \ldots\ldots (4)$$

which are the third and fourth analogies of Gauss: and from these four, the analogies of Napier might be at once deduced, by division.

gives $\sin. a = \frac{\sin. b \sin. A}{\sin. B}$; or, sin. a, a fourth proportional to sin. B, sin. b, and sin. A.

THEOREM VIII.

In every right-angled spherical triangle, the cosine of the hypothenuse is equal to the product of the cosines of the sides including the right angle.

For if A be measured by $\frac{1}{4}\bigcirc$, its cosine becomes nothing, and the first of the equations I becomes $\cos. a = \cos. b \,.\, \cos. c$. Q E. D.

THEOREM IX.

In every right-angled spherical triangle, the cosine of either oblique angle, is equal to the quotient of the tangent of the adjacent side divided by the tangent of the hypothenuse.

If, in the second of the equations I, the preceding value of cos. a be substituted for it, and for sin. a its value $\tan. a \cos. a = \tan. a \cos. b \cos. c$; then, recollecting that $1 - \cos.^2 c = \sin.^2 c$, there will result, $\tan. a \cos. c \cos. B = \sin. c$. whence it follows that,

$$\tan. a \cos. B = \tan. c, \text{ or } \cos. B = \frac{\tan. c}{\tan. a}.$$

Thus also it is found that $\cos. C = \frac{\tan b}{\tan. a}$.

THEOREM X.

In any right-angled spherical triangle, the cosine of one of the sides about the right angle, is equal to the quotient of the cosine of the opposite angle divided by the sine of the adjacent angle.

From theor. 6, we have $\frac{\sin. B}{\sin A} = \frac{\sin. b}{\sin. a}$: which, when A is a right angle, becomes simply $\sin. B = \frac{\sin. b}{\sin. a}$. Again, from theor. 9, we have $\cos. C = \frac{\tan. b}{\tan a}$.
Hence, by division,

$$\frac{\cos. C}{\sin. B} = \frac{\tan. b}{\sin. b} \cdot \frac{\sin. a}{\tan. a} = \frac{\cos. a}{\cos. b}.$$

Now theor. 8 gives $\frac{\cos. a}{\cos. b} = \cos. c$. Therefore $\frac{\cos. C}{\sin B} = \cos. c$; and in like manner, $\frac{\cos. B}{\sin. C} = \cos. b$. Q. E. D.

THEOREM XI.

In every right-angled spherical triangle, the tangent of either of the oblique angles, is equal to the quotient of the tangent of the opposite side, divided by the sine of the other side about the right angle.

$$\text{For, since } \sin. B = \frac{\sin. b}{\sin. a}, \text{ and } \cos. B = \frac{\tan. c}{\tan. a},$$

$$\text{we have } \frac{\sin. B}{\cos. B} = \frac{\sin. b}{\sin. a} \cdot \frac{\tan. a}{\tan. c}.$$

Whence, because (theor. 8) $\cos a = \cos. b \,.\, \cos. c$, and since $\sin. a = \cos. a \tan. a$, we have

$$\tan. B = \frac{\sin b}{\cos. a \tan. c} = \frac{\sin. b}{\cos. b \cos. c \tan. c} = \frac{\sin. b}{\cos. b} \cdot \frac{1}{\cos. c \tan. c} = \frac{\tan b}{\sin. c}.$$

In like manner, $\tan. C = \frac{\tan c}{\sin. b}$. Q. E. D.

THEOREM XII.

In every right-angled spherical triangle, the cosine of the hypothenuse, is equal to the quotient of the cotangent of one of the oblique angles, divided by the tangent of the other angle.

For, multiplying together the resulting equations of the preceding theorem, we have

$$\tan. B \cdot \tan. C = \frac{\tan. b}{\sin. b} \cdot \frac{\tan. c}{\sin. c} = \frac{1}{\cos. b \cdot \cos. c}.$$

But, by theor. 8, $\cos. b \cdot \cos. c = \cos. a$.

Therefore $\tan. B \cdot \tan. C = \frac{1}{\cos. a}$, or $\cos. a = \frac{\cot. C}{\tan. B}$. Q. E. D

THEOREM XIII.

In every right-angled spherical triangle, the sine of the difference between the hypothenuse and base, is equal to the continued product of the sine of the perpendicular, cosine of the base, and tangent of half the angle opposite to the perpendicular; or equal to the continued product of the tangent of the perpendicular, cosine of the hypothenuse, and tangent of half the angle opposite to the perpendicular *.

Here retaining the same notation, since we have $\sin. a = \frac{\sin. b}{\sin. B}$, and $\cos. B = \frac{\tan. c}{\tan. a}$; and if for the tangents there be substituted their values in sines and cosines there will arise $\sin. c \cos. a = \cos. B \cos. c \sin. a = \cos. B \cos. c \cdot \frac{\sin b}{\sin. B}$.

Then substituting for $\sin. a$, and $\sin. c \cdot \cos. a$, their values in the known formula (equ. v. page 20,) viz.

$$\text{in } \sin. (a - c) = \sin. a \cdot \cos. c - \cos. a \cdot \sin. c,$$

and recollecting that $\frac{1 - \cos. B}{\sin. B} = \tan. \frac{1}{2}B$,

it will become, $\sin. (a - c) = \sin. b \cos c \tan. \frac{1}{2}B$;

which is the first part of the theorem: and, if in this result we introduce, instead of $\cos c$, its value $\frac{\cos. a}{\cos. b}$ (theor. 8), it will be transformed into $\sin. (a - c) = \tan. b \cos. a \tan. \frac{1}{2}B$; which is the second part of the theorem †. Q. E. D.

Scholium.

In problems 2 and 3, if the circumstances of the question leave any doubt, whether the arcs or the angles sought are greater or less than a quadrant, or than a right angle, the difficulty will be entirely removed by means of the table of mutations of signs of trigonometrical quantities, in different quadrants, marked VII in chap. 3. In the 6th and 7th problems, the question proposed will often

* This theorem is due to M. Prony, who published it without demonstration in the *Connaissance des Tems* for the year 1808, and made use of it in the construction of a chart of the course of the Po.

† This theorem leads manifestly to an analogous one with regard to rectilinear triangles, which, if h, b, and p denote the hypothenuse, base, and perpendicular, and B, P, the angles respectively opposite to b, p; may be expressed thus:

$$h - b = p \tan. \tfrac{1}{2}P \; \ldots\ldots \; h - p = b \tan. \tfrac{1}{2}B.$$

These theorems may be found useful in reducing inclined lines to the plane of the horizon.

be susceptible of two solutions: by means of the subjoined table the student may always tell when this will or will not be the case.

1. When the data a, b, and B, there can be only one solution
when $B = \frac{1}{4}\bigcirc$ (a right angle),

or, when $B < \frac{1}{4}\bigcirc$	$a < \frac{1}{4}\bigcirc$	$b \geq a$
$B < \frac{1}{4}\bigcirc$	$a \geq \frac{1}{4}\bigcirc$	$b \geq \frac{1}{2}\bigcirc - a$,
$B \geq \frac{1}{4}\bigcirc$ . . .	$a < \frac{1}{4}\bigcirc$	$b < \frac{1}{2}\bigcirc - a$,
$B > \frac{1}{4}\bigcirc$	$a > \frac{1}{4}\bigcirc$	$b < a$.

The triangle is susceptible of two forms and solutions

when $B < \frac{1}{4}\bigcirc$ . . .	$a < \frac{1}{4}\bigcirc$	$b < a$
$B < \frac{1}{4}\bigcirc$ . . .	$a > \frac{1}{4}\bigcirc$ . . .	$b < \frac{1}{2}\bigcirc - a$,
$B > \frac{1}{4}\bigcirc$	$a < \frac{1}{4}\bigcirc$	$b > \frac{1}{2}\bigcirc - a$,
$B > \frac{1}{4}\bigcirc$	$a > \frac{1}{4}\bigcirc$	$b > a$,
$B <$ or $> \frac{1}{4}\bigcirc$	$a = \frac{1}{4}\bigcirc$.	

2. With the data A, B, and b, the triangle can exist but in one form,
when $b = \bigcirc$ (one quadrant),

$b \geq \frac{1}{4}\bigcirc$ . . .	$A \geq \frac{1}{4}\bigcirc$	$B < A$,
$b \geq \frac{1}{4}\bigcirc$ . . .	$A < \frac{1}{4}\bigcirc$	$B < \frac{1}{2}\bigcirc - A$,
$b < \frac{1}{4}\bigcirc$ . . .	$A \geq \frac{1}{4}\bigcirc$	$B \geq \frac{1}{2}\bigcirc - A$,
$b < \frac{1}{4}\bigcirc$ . .	$A < \frac{1}{4}\bigcirc$	$A \geq A$.

It is susceptible of two forms,

when $b > \frac{1}{4}\bigcirc$ . . .	$A > \frac{1}{4}\bigcirc$ . .	$B > A$,
$b > \frac{1}{4}\bigcirc$ . .	$A > \frac{1}{4}\bigcirc$ . .	$B > \frac{1}{2}\bigcirc - A$.
$b < \frac{1}{4}\bigcirc$	$A > \frac{1}{4}\bigcirc$ . . .	$B < \frac{1}{2}\bigcirc - A$,
$b < \frac{1}{4}\bigcirc$ . . .	$A < \frac{1}{4}\bigcirc$ . .	$B < A$,
$b >$ or $\geq \frac{1}{4}\bigcirc$ . . .	$A = \frac{1}{4}\bigcirc$.	

Those formulæ are always to be preferred in calculation in which the functions, as sine, tangent, &c. are given by means of the half arcs or angles, to those where they are given by means of the whole arcs or angles: as in this case no ambiguity can possibly occur All such expressions, if adapted to logarithmic calculation, are real contributions to practical science; and to the discovery of such, the student should assiduously direct his attention.

Notes.—It may here be observed, that all the analogies and formulæ, of spherical trigonometry, in which *cosines* or *cotangents* are not concerned, may be applied to *plane* trigonometry; taking care to use only a *side* instead of the *sine* or the *tangent of a side;* or the *sum* or *difference* of the sides instead of the *sine* or *tangent* of such sum or difference. The reason of this is obvious: for analogies or theorems raised, not only from the consideration of a triangular figure, but the curvature of the sides also, are of consequence more general; and therefore, though the curvature should be deemed evanescent, by reason of a diminution of the surface, yet what depends on the *triangle* alone will remain, notwithstanding.

We have now deduced all the rules that are essential in the operations of spherical trigonometry; and explained under what limitations ambiguities may exist. That the student, however, may want nothing further to direct his practice in this branch of science, we shall add three tables, in which the several formulæ, already given, are respectively applied to the solution of all the cases of right and oblique-angled spherical triangles, that can possibly occur.

TABLE I.

For the Solution of all the Cases of Right-Angled Spherical Triangles.

Given.	Required.	Values of the terms required.	Cases in which the terms required are less than 90°.
I. Hypothenuse, and one leg.	Angle opposite to the given leg.	Its sin. $= \frac{\text{sin. given leg}}{\text{sin. hypoth.}}$	If the given leg be less than 90°.
	Angle adjacent to the given leg.	Its cos. $= \frac{\text{tan. given leg}}{\text{tan. hypoth}}$.	If the things given be of the same affection.
	Other leg.	Its cos. $= \frac{\text{cos. hypoth}}{\text{cos. given leg}}$.	Idem.
II. One leg, and its opposite angle.	Hypothenuse.	Its sin. $= \frac{\text{sin. given leg}}{\text{sin. given ang}}$.	Ambiguous.
	Other leg.	Its sin. $= \frac{\text{tan. given leg}}{\text{tan. given ang}}$.	Idem.
	Other angle.	Its sin. $= \frac{\text{cos given ang}}{\text{cos. given leg}}$	Idem.
III. One leg, and the adjacent angle.	Hypothenuse.	Its tan $= \frac{\text{tan. given leg}}{\text{cos. given ang}}$.	If the things given be of like affection.
	Other angle.	Its cos. $=$ cos. giv leg $\times$ sin giv ang.	If the given leg be less than 90°.
	Other leg.	Its tan $=$ sin. giv leg $\times$ tan. giv. ang.	If the given angle be less than 90°.
IV. Hypothenuse, and one angle.	Adjacent leg	Its tan. $=$ tan. hyp. $\times$ cos. giv. ang.	If the things given be of like affection.
	Leg opp. to the given angle.	Its sin. $=$ sin. hyp. $\times$ sin giv. ang	If the given angle be acute.
	Other angle.	Its tan. $= \frac{\text{cot giv. angle}}{\text{cos. hypothen}}$.	If the things given be of like affection.
V. The two legs.	Hypothenuse.	Its cos. $=$ rectan. cos. giv. legs.	If the given legs be of like affection.
	Either of the legs	Its tan. $= \frac{\text{tan oppos. leg}}{\text{sin. adjac. leg}}$.	If the opposite leg be less than 90°.
VI. The two angles.	Hypothenuse	Its cos. $=$ rect. cot. giv. angles.	If the angles be of like affection.
	Either of the legs.	Its cos $= \frac{\text{cos. opposite angle}}{\text{sin. adjacent angle}}$.	If the opposite angle be acute.

In working by the logarithms, the student must observe that when the resulting logarithm is the log. of a quotient, 10 must be *added* to the index; when it is the log. of a product, 10 must be *subtracted* from the index. Thus when the two angles are given,

Log. cos. hypothen. $=$ log. cos. one angle $+$ log. cos. other angle $-$ 10 :

Log. cos. either leg. $=$ log. cos. opp. angle $-$ log. sin. adjac. angle $+$ 10.

In a quadrantal triangle, if the quadrantal side be called radius, the supplement of the angle opposite to that side be called hypothenuse, the other sides be called angles, and their opposite angles be called legs: then the solutions of all the cases will be as in this table; merely changing *like* for *unlike* in their determinations.

TABLE II.

For the Solution of Oblique-Angled Spherical Triangles.

An angle or a side being divided by a perpendicular, the first and second segments are denoted by 1 seg. and 2 seg.

Given.	Required.	Values of the Quantities required.	
I. Two angles and a side opposite to one of them.	The side opp. to other angle.	By the common analogy.	Sines of angles are as sines of oppos. sides.
	Third side.	Let fall a per. on the side contained between the given angles.	Tan. 1 seg. of this side = cos. adj. angle $\times$ tan. given side. Sin. 2 seg. $= \dfrac{\text{sin. 1 seg.} \times \text{tan. ang. adj. given side}}{\text{tan. ang. opp. given side}}$.
	Third angle.	Let fall a per. as before.	Cot. 1 seg. of this ang. = cos. giv. side $\times$ tan. adj. angle. Sin. 2 seg. $= \dfrac{\text{sin. 1 seg.} \times \text{cos. ang. opp. given side}}{\text{cos. ang. adj. given side}}$.
II. Two sides and an angle opposite to one of them.	The angle opp. to the other side.	By the common analogy.	Sines of sides are as sines of their opposite angles.
	Angle included between the given sides.	Let fall a perpendicular from the included angle.	Cot. 1 seg. ang. req. = tan. given ang. $\times$ cos. adj. side. Cos. 2 seg. $= \dfrac{\text{cos. 1 seg.} \times \text{tan. giv. side adj. giv. angle}}{\text{tan. side opp. given angle}}$.
	Third side.	Let fall a perpendicular as before.	Tan. 1 seg. side req. = cos. given ang. $\times$ tan. adj. side. Cos. 2 seg. $= \dfrac{\text{cos. 1 seg.} \times \text{cos. side opp. given angle}}{\text{cos. side adj. given angle}}$.
III. Two sides and the included angle.	An angle oppos. to one of the given sides.	Let fall a perpen. from the third angle.	Tan. 1 seg. of div. side = cos. giv. ang. $\times$ tan. side opp. ang sought. Tan. ang. sought $= \dfrac{\text{tan. giv. ang.} \times \text{sin. 1 seg.}}{\text{sin. 2 seg. of div. side}}$.
	Third side.	Let fall a perpen. on one of the given sides	Tan. 1 seg of div. side = cos. giv. ang. $\times$ tan. other given side. Cos. side sought $= \dfrac{\text{cos. side not div.} \times \text{cos 2 seg.}}{\text{cos. 1 seg. of side divided}}$
IV. A side and the two adjacent angles.	A side opposite to one of the given angles.	Let fall a perpendicular on the third side.	Cot. 1 seg. of div. ang. = cos. giv. side $\times$ tan. ang. opp. side sought. Tan. side sought $= \dfrac{\text{tan. giv. side} \times \text{cos. 1 seg. div. ang.}}{\text{cos. 2 seg. of divided angle}}$.
	Third angle.	Let fall a perpen. from one of the given angles.	Cot. 1 seg. div. ang. = cos. giv. side $\times$ tan. other give angle. Cos. angle sought $= \dfrac{\text{cos. ang. not div.} \times \text{sin. 2 seg.}}{\text{sin. 1 seg. div. angle}}$.
V. The three sides.	An angle by the sine or cosine of its half.	Let a, b, c, be the sides; A, B, C, the angles, b and c including the angle sought, and $2s = a + b + c$. Then, $\sin.\tfrac{1}{2}A = \sqrt{\dfrac{\sin.(s-b)\sin.(s-c)}{\sin. b \sin. c}}$ .... $\cos.\tfrac{1}{2}A = \sqrt{\dfrac{\sin. s \sin.(s-a)}{\sin. b \sin. c}}$.	
VI. The three angles.	A side by the sine or cosine of its half.	Let 2S be the sum of the angles A, B, and C; and let B and C be adjacent to a the side required. Then, $\sin.\tfrac{1}{2}a = \sqrt{-\dfrac{\cos. S \cos.(S-A)}{\sin. B \sin. C}}$ .. $\cos.\tfrac{1}{2}a = \sqrt{-\dfrac{\sin.(S-B)\sin.(S-C)}{\sin. B \sin. C}}$.	

TABLE III.

For the Solution of all the Cases of Oblique-Angled Spherical Triangles, by the Analogies of Napier.

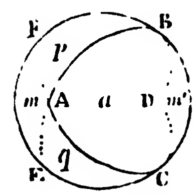
Given.	Required.	Values of the terms required.
I. Two angles and one of their opposite sides.	Side opp. to the other given angle.	By the common analogy, sines of angles are as sines of opp. sides.
	Third side.	Tan. of its half $= \dfrac{\text{tan. } \frac{1}{2} \text{ diff. giv. sides} \times \text{sin. } \frac{1}{2} \text{ sum opp. angles}}{\text{sin. } \frac{1}{2} \text{ diff. of those angles}}$. $= \dfrac{\text{tan. } \frac{1}{2} \text{ sum giv. sides} \times \text{cos. } \frac{1}{2} \text{ sum opp. angles}}{\text{cos. } \frac{1}{2} \text{ diff. of those angles}}$.
	Third angle.	By the common analogy.
II. Two sides and an opposite angle.	Angle opposite to the other known side.	By the common analogy.
	Third angle.	Cot. of its half $= \dfrac{\text{tan. } \frac{1}{2} \text{ diff other two ang.} \times \text{sin. } \frac{1}{2} \text{ sum giv. sides}}{\text{sin. } \frac{1}{2} \text{ diff. of those sides}}$. $= \dfrac{\text{tan } \frac{1}{2} \text{ sum of other two ang} \times \text{cos. } \frac{1}{2} \text{ sum giv. sides}}{\text{cos. } \frac{1}{2} \text{ diff. of those sides}}$.
	Third side.	By the common analogy.
III. Two sides and the included angle.	The other two angles.	Tan. $\frac{1}{2}$ their diff. $= \dfrac{\text{cot. } \frac{1}{2} \text{ giv. ang.} \times \text{sin } \frac{1}{2} \text{ diff giv. sides}}{\text{sin. } \frac{1}{2} \text{ sum of those sides}}$. Tan $\frac{1}{2}$ their sum $= \dfrac{\text{cot } \frac{1}{2} \text{ giv. ang.} \times \text{cos. } \frac{1}{2} \text{ diff. giv. sides}}{\text{cos. } \frac{1}{2} \text{ sum of those sides}}$.
	Third side	By the common analogy.
IV. Two angles and the side between them.	The other two sides.	Tan. $\frac{1}{2}$ their diff. $= \dfrac{\text{tan. } \frac{1}{2} \text{ giv. side} \times \text{sin. } \frac{1}{2} \text{ diff. giv angle}}{\text{sin. } \frac{1}{2} \text{ sum of those angles}}$. Tan $\frac{1}{2}$ their sum. $= \dfrac{\text{tan. } \frac{1}{2} \text{ giv. side} \times \text{cos. } \frac{1}{2} \text{ diff. give angles}}{\text{cos. } \frac{1}{2} \text{ sum of those angles}}$.
	Third angle	By the common analogy.
V. The three sides.	Either of the angles.	Let fall a perpen. on the side adjacent to the angle sought. Tan. $\frac{1}{2}$ sum or $\frac{1}{2}$ diff. of the seg. of the base $= \dfrac{\text{tan. } \frac{1}{2} \text{ sum} \times \text{tan. } \frac{1}{2} \text{ diff. of the sides.}}{\text{tan. } \frac{1}{2} \text{ base}}$ Cos. angle sought = tan. adj. seg. $\times$ cot. adja. side.
VI. The three angles.	Either of the sides.	Will be obtained by finding its correspondent angle, in a triangle which has all its parts supplemental to these of the triangle whose three angles are given.

SECTION III.

On the Area of a Spherical Triangle. *

THEOREM XIV.

In every spherical triangle the following proportion obtains, viz. as four right angles (or 360°) to the surface of a hemisphere; or, as two right angles (or 180°) to a great circle of the sphere; so is the excess of the three angles of the triangle above two right angles, to the area of the triangle.



Let ABC be the spherical triangle. Complete one of its sides as BC into the circle BCEF, which may be supposed to bound the upper hemisphere. Prolong also, at both ends, the two sides AB, AC, until they form semicircles estimated from each angle, that is, until BAE = ABD = CAF = ACD = 180°. Then will CBF = 180° = BFE; and consequently the triangle AEF, on the anterior hemisphere, will be equal to the triangle BCD on the opposite hemisphere. Put m, m', to represent the surface of these triangles, p for that of the triangle BAF, q for that of CAE, and a for that of the proposed triangle ABC. Then a and m' together (or their equal a and m together) make up the surface of a spheric lune comprehended between the two semicircles ACD, ABD, inclined in the angle A : a and p together make up the lune included between the semicircles CAF, CBF, making the angle C : a and q together make up the spheric lune included between the semicircles BCE, BAE, making the angle B. And the surface of each of these lunes, is to that of the hemisphere, as the angle made by the comprehending semicircles, to two right angles. Therefore, putting $\frac{1}{2}$S for the surface of the hemisphere, we have

$$180^\circ : A :: \tfrac{1}{2}S : a + m.$$
$$180^\circ : B :: \tfrac{1}{2}S : a + q.$$
$$180^\circ : C :: \tfrac{1}{2}S : a + p.$$

Whence, $180^\circ : A + B + C :: \frac{1}{2}S : 3a + m + p + q = 2a + \frac{1}{2}S$; and consequently, by division of proportion,

as $180^\circ : A + B + C - 180^\circ :: \frac{1}{2}S : 2a + \frac{1}{2}S - \frac{1}{2}S = 2a$;

or $180^\circ : A + B + C - 180^\circ :: \frac{1}{4}S : a = \frac{1}{4}S \; \dfrac{A + B + C - 180^\circ}{360^\circ}$.

Q. E. D †.

Cor. 1. Hence the excess of the three angles of any spherical triangle above two right angles, termed technically the ***spherical excess***, furnishes a correct measure of the surface of that triangle.

Cor. 2. If $\pi = 3{\cdot}141593$, and d the diameter of the sphere, then is . .
$\pi d^2 \dfrac{A + B + C - 180^\circ}{720^\circ}$ = the area of the spherical triangle.

* This quantity is technically called the "Spherical Excess," for a reason that will appear in cor. 1.

† This determination of the area of a spherical triangle is due to *Albert Girard* (who died about 1633). But the demonstration now commonly given of the rule was first published by Dr Wallis. It was considered as a mere speculative truth, until Mr. Dalby, in 1787, employed it very judiciously in the great Trigonometrical Survey, to correct the errors of spherical angles. See Phil. Trans. vol. 80, and the chapter of this volume, which is devoted to Geodesic Operations.

Cor. 3. Since the length of the radius, in any circle, is equal to the length of 57·2957795 degrees, measured on the circumference of that circle; if the *spherical excess* be multiplied by 57·2957795, the product will express the surface of the triangle in square degrees †.

Cor. 4. When $a = 0$, then $A + B + C = 180°$: and when $d = \frac{1}{2}S$, then $A + B + C = 540°$. Consequently the sum of the three angles of a spherical triangle is always between 2 and 6 right angles; which is another confirmation of theor. 3.

Cor. 5. When *two* of the angles of a spherical triangle are right angles, the surface of the triangle varies with its third angle. And when a spherical triangle has *three* right angles, its surface is one-eighth of the surface of the sphere.

Remark. Some of the uses of the spherical excess, in the more extensive geodesic operations, will be hereafter shown. The mode of finding it, and thence the area when the three angles of a spherical triangle are given, is obvious enough; but it is often requisite to ascertain it by means of other data, as, when two sides and the included angle are given, or when all the three sides are given

THEOREM XV.

In every spherical polygon, or surface included by any number of intersecting great circles, the subjoined proportion obtains, viz. as four right angles, or 360°, to the surface of a hemisphere; or, as two right angles, or 180°, to a great circle of the sphere; so is the excess of the sum of the angles above the product of 180° and two less than the number of angles of the spherical polygon, to its area.

For, if the polygon be supposed to be divided into as many triangles as it has sides, by great circles drawn from all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by theor. 5, it will be as $360° : \frac{1}{2}S :: A + B + C - 180° :$ its area. Therefore, putting P for the sum of all the angles of the polygon, n for their number, and V for the sum of all the vertical angles of its constituent triangles, it will be, by composition, as $360° : \frac{1}{2}S :: P + V - 180° . n :$ surface of the polygon. But V is manifestly equal to 360° or $180° \times 2$. Therefore,

as $360° : \frac{1}{2}S :: P - (n - 2)\,180° : \frac{1}{2}S . \frac{P - (n - 2)\,180°}{360°}$, the area of the polygon. Q. E. D.

Cor. 1. If π and d represent the same quantities as in theor 14, cor 2, then the surface of the polygon will be expressed by $\pi d^2 . \frac{P - (n - 2)\,180°}{720°}$.

Cor. 2. If $R° = 57·2957795$, then will the surface of the polygon in square degrees be $= R^2 . [P - (n - 2)\,180°]$

Cor. 3. When the surface of the polygon is 0, then $P = (n - 2)\,180°$; and

† Excess in degrees, i. e. $° = \text{area}° \div 57·2957795$

$$\text{Excess}'' = \frac{\text{area}° \times 3600}{57·2957795}$$
$$= \frac{\text{area in squ. feet} \times 3600}{57·2957795 \times (365154·6)^2}$$

The log. of the factor to these square feet is 9·3267737, which is the logarithm so frequently employed in Geodesic computations.

As a practical rule, the following is very convenient.—Multiply the base into the perpendicular (both taken in miles), and divide the product by 152¼: the quotient is the "excess" in seconds. This may be worked with sufficient accuracy by a common sliding rule.

when it is a maximum, that is, when it is equal to the surface of the hemisphere then $P = (n - 2)\,180° + 360° = n\,180°$: Consequently P, the sum of all the angles of any spheric polygon, is always *less* than $2n$ right angles, but *greater* than $(2n - 4)$ right angles, n denoting the numbers of angles of the polygon.

PROBLEM VI.

To find the area of a spherical triangle in terms of its three sides.

We have just seen that, denoting the spherical excess by E,

$$\frac{E}{4} = \frac{A + B + C - \pi}{4}; \text{ and hence}$$

$$\tan\frac{E}{4} = \frac{2\sin.\dfrac{A+B+C-\pi}{4}}{2\cos.\dfrac{A+B+C-\pi}{4}} \cdot \frac{\cos.\dfrac{A+B-C+\pi}{4}}{\cos.\dfrac{A+B-C+\pi}{4}}; \text{ or by ——}$$

$$= \frac{\sin.\dfrac{A+B}{2} - \sin.\left(\dfrac{\pi}{2} - \dfrac{C}{2}\right)}{\cos.\dfrac{A+B}{2} + \cos.\left(\dfrac{\pi}{2} - \dfrac{C}{2}\right)} = \frac{\sin\dfrac{A+B}{2} - \cos.\dfrac{C}{2}}{\cos.\dfrac{A+B}{2} + \sin.\dfrac{C}{2}}$$

But inserting for these functions of the angles their values from Gauss's formulæ, page 61, this is converted into

$$\tan\frac{E}{4} = \frac{\cos.\dfrac{a-b}{2} - \cos.\dfrac{C}{2}}{\cos\dfrac{a}{2} + \cos\dfrac{C}{2}}\cot.\frac{C}{2} = \frac{\sin.\dfrac{s-a}{2}\,\sin.\dfrac{s-b}{2}}{\cos.\dfrac{s}{2}\,\cos.\dfrac{s-c}{2}}\cot.\frac{C}{2}$$

and for $\cot.\frac{E}{2}$ substituting its value we get

$$\tan\frac{E}{4} = \frac{\sin.\dfrac{s-a}{2}\,.\,\sin.\dfrac{s-b}{2}}{\cos.\dfrac{s}{2}\,.\,\cos.\dfrac{(s-c)}{2}}\sqrt{\frac{\sin.s\,.\,\sin(s-c)}{\sin.(s-a)\sin.(s-b)}}$$

$$= \sqrt{\tan.\frac{s}{2}\tan\frac{s-a}{2}\tan.\frac{s-b}{2}\tan.\frac{s-c}{2}}$$

This remarkable formula was discovered by Simon Lhuilier, of Geneva, and is called by his name. It is by far the best formula yet known for the spherical excess (in terms of any data) and is adapted to logarithms without the aid of a subsidiary angle.*

Formulæ analogous to this for the areas of the *associated* system of triangles in terms of s, $s-a$, $s-b$, $s-c$, of the given fundamental triangle, as well as the polar triangles, may be seen in the *Supplement to Young's Trigonometry*; and several other particulars respecting this class of quantities, to which our limits only allow us to refer the inquiring student

PROBLEM VII.

To find the area of a spherical triangle when two sides and the included angle are given *

$$\frac{E}{2} = \frac{A + B\,(\pi - C)}{2}. \quad \text{Hence,}$$

* When two sides and the angle opposite to one of them is given, it is better to compute the other side, and employ Lhuilier's theorem, and when these are given, two angles and a side opposite to one of them, to compute the third angle, and employ Girard's formula.

$$\cot.\frac{E}{2} = \frac{\cos.\frac{A+B}{2}\cos.\frac{C}{2} - \sin.\frac{A+B}{2}\sin.\frac{C}{2}}{\cos.\frac{A+B}{2}\sin.\frac{C}{2} + \sin.\frac{A+B}{2}\cos.\frac{C}{2}}$$

Inserting in this the values of $\cos.\frac{1}{2}(A+B)$ and $\sin.\frac{1}{2}(A+B)$ from Gauss's theorems, we have, after cancelling $\cos.\frac{1}{2}C$,

$$\cot.\frac{E}{2} = \frac{\cos.\frac{1}{2}(a+b)\sin.^2\frac{1}{2}C + \cos.\frac{1}{2}(a-b)\cos.^2\frac{1}{2}C}{\frac{1}{2}\left\{\cos.\frac{1}{2}(a+b) - \cos.\frac{1}{2}(a-b)\right\}\sin.C}.$$

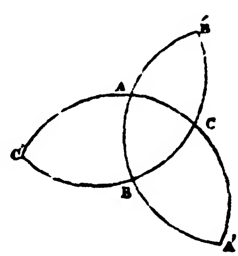
$$= \frac{\cot.\frac{1}{2}a\cot.\frac{1}{2}b + \cos.C}{\sin.C} = \left\{\frac{\cot.\frac{1}{2}a\cot.\frac{1}{2}b + 1}{\cos.C}\right\}\cot.C.$$

[In a manner very similar we should find it in terms of A, c, B, to be $\tan.\frac{E}{2} = \frac{\tan.\frac{1}{2}A\tan.\frac{1}{2}B - \cos.c}{\sin.c}$.]

SECTION IV.

As the symbols S, S — A, S — B, S — C, and s, $s - a$, $s - b$, $s - c$, occur so frequently in the expressions which enter into the formulæ of solution of spherical triangles, it is desirable to point out the particular angles and arcs in the figure which are represented by them.

The associated system of triangles formed by producing the sides of a spherical triangle ABC till they meet again in A′, B′, C′ respectively, have been described at pp. 53—5, to which again the student is referred. If ABC be the *primitive* triangle, the triangles AB′C, BC′A, CA′B are the *supplemental* ones, but if either of the others, as for instance AB′C be taken as the primitive one, then the three remaining ones, ABC, BC′A, CA′B are the *supplemental* ones. It is always most convenient for preserving uniformity of notation (and this in all that relates to formulæ, is of the first consequence) to mark the *primitive* one with plain letters A, B, C, and the opposite vertices of the supplemental angles with the same letter as the equal and opposite angle of the primitive one, with an accent added to it.



The notation here employed is :—

a, b, c denote the sides of the triangle ABC.
$a_,$ $b_,$ $c_,$ those of the triangle opposite the first angle A.
$a_{,,}$ $b_{,,}$ $c_{,,}$ those opposite the second angle B.
$a_{,,,}$ $b_{,,,}$ $c_{,,,}$ those opposite the third angle C.

In which the number of subscribed accents shows the order of the sides a, b, c, to which the triangle is adjacent.

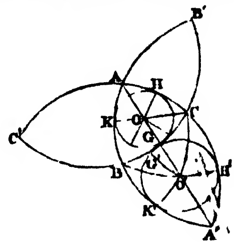
Also in the same manner, for the angles :—

A, B, C the angle of the fundamental triangle ABC
$A_,$ $B_,$ $C_,$ those of the triangle BA′C,
$A_{,,}$ $B_{,,}$ $C_{,,}$ those of CB′A, and
$A_{,,,}$ $B_{,,,}$ $C_{,,,}$ those of BC′A.

The several values of these sides and angles are :—

$$\left.\begin{array}{l|l|l|l} b_, = \pi - b & c_, = \pi - c & B_, = \pi - B & C_, = \pi - C \\ a_{,,} = \pi - a & c_{,,} = \pi - c & A_{,,} = \pi - A & C_{,,} = \pi - C \\ a_{,,,} = \pi - a & b_{,,,} = \pi - b & A_{,,,} = \pi - A & B_{,,,} = \pi - B \end{array}\right\} \ldots (1)$$

Also to designate the radii of the circles inscribed within the several triangles, $r, r_{,}, r_{,,}, r_{,,,}$ are used; and for the radii of the circles circumscribed about the same triangles, the letters $R, R_{,}, R_{,,}, R_{,,,}$, in which r, R, belong to AB'C; $R_{,} r_{,}$ to BA'C, $R_{,,} r_{,,}$ to AB'C, and $R_{,,,} r_{,,,}$ to BC'A.



Let G, H, K be the points of contact of the sides of the triangle with its inscribed circle, and O the centre of that circle. Join AO, BO, CO, and draw the radii to the points of contact G, H, K. Then the tangents AG, AH are equal, and so of the others: that is AK = AH, CH = CG, and GB = BK.

$$\begin{array}{l|l} \text{Put } AK = AH = \alpha & \text{From which } a = \beta + \gamma \\ \phantom{\text{Put }} BK = BG = \beta & \phantom{\text{From which }} b = \gamma + \alpha \\ \phantom{\text{Put }} CG = CH = \gamma & \phantom{\text{From which }} c = \alpha + \beta \end{array} \Bigg\} \quad . \quad (2)$$

By adding the equations of (2) together, and then subtracting each from half the sum of the three, we obtain

$$\left.\begin{array}{r} \alpha + \beta + \gamma = \dfrac{a+b+c}{2} = s \\ \alpha = \dfrac{-a+b+c}{2} = s - a \\ \beta = \dfrac{a-b+c}{2} = s - b \\ \gamma = \dfrac{a+b-c}{2} = s - c \end{array}\right\} \quad . \quad . \quad (3)$$

But in the right angled triangle BOG we have

$$\tan\ OG = \sin\ BG \tan.\ OBG, \text{ that is, } \tan.\ r = \sin.\beta \tan\ \tfrac{1}{2}B,$$

or, substituting the value of β from (3) and of tan. $\frac{1}{2}$B from Th. XII in this, we obtain

$$\tan.\ r = \sin.(s-b) \left\{\frac{\sin\ (s-a) \sin.(s-c)}{\sin\ s \sin.(s-b)}\right\}^{\frac{1}{2}} = \frac{\sqrt{\sin.s \sin\ (s-a) \sin\ (s-b) \sin.(s-c)}}{\sin.s} \quad (4)$$

In the supplemental triangle BA'C, denoting the arcs AK, BG, CH', by $\alpha_{,} \beta_{,} \gamma_{,}$, and the sides of the triangle A'BC by $a_{,} b_{,} c_{,}$, we have in the same manner.

$a_{,} = \beta_{,} + \gamma_{,}$, $b_{,} = \gamma_{,} + \alpha_{,}$, and $c_{,} = \alpha_{,} + \beta_{,}$; from which in the same way as before, we may obtain

$$\left.\begin{array}{l} s_{,} = \alpha_{,} + \beta_{,} + \gamma_{,} = \dfrac{a_{,}+b_{,}+c_{,}}{2} = \dfrac{a+(\pi-b)+(\pi-c)}{2} = \pi - \dfrac{-a+b+c}{2} = \pi-(s-a) \\ s_{,}-a_{,} = \alpha_{,} = \dfrac{-a_{,}+b_{,}+c_{,}}{2} = \pi - \dfrac{-a+(\pi-b)+(\pi-c)}{2} = \dfrac{a+b+c}{2} = s \\ s_{,}-b_{,} = \beta_{,} = \dfrac{a_{,}-b_{,}+c_{,}}{2} = \dfrac{a-(\pi-b)+(\pi-c)}{2} = \dfrac{a+b-c}{2} = s-c \\ s_{,}-c_{,} = \gamma_{,} = \dfrac{a_{,}+b_{,}-c_{,}}{2} = \dfrac{a+(\pi-b)-(\pi-c)}{2} = \dfrac{a-b+c}{2} = s-b \end{array}\right\} (5)$$

In the right-angled triangle BO'G' we have tan. O'G' = sin. BG' tan O'BG; and since from (1) B' π — B, we have tan. $\frac{1}{2}$B' = cot. $\frac{1}{2}$B. Whence from Th. XII. and (5)

$$\tan.\ r' = \sin\ \beta_{,} \tan\ \tfrac{1}{2}B_{,} = \sin\ (s-c) \left\{\frac{\sin.s.\sin\ (s-b)}{\sin.(s-a)\sin.(s-c)}\right\}^{\frac{1}{2}} = \frac{\sqrt{\sin\ s \sin\ (s-a) \sin (s-b)}}{\sin.s-a}$$

Finding the other values for tan $r_{,,}$ and tan. $r_{,,,}$ in the same way, we may tabulate them thus

$$\left.\begin{aligned}
\tan. r &= \frac{\sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)}}{\sin. s}\\
\tan. r_{,} &= \frac{\sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)}}{\sin. (s-a)}\\
\tan. r_{,,} &= \frac{\sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)}}{\sin. (s-b)}\\
\tan. r_{,,,} &= \frac{\sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)}}{\sin. (s-c)}
\end{aligned}\right\} \quad \ldots . (7)$$

Multiply together the four equations marked (7): then $\tan. r \tan. r_{,} \tan r_{,,} \tan. r_{,,,} = \sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)$ . . . . (8)

Divide (8) by the squares of each of the equations in (7): then,

$$\left.\begin{aligned}
\sin.^2 s &= \cot. r \tan. r_{,} \tan. r_{,,} \tan. r_{,,,}\\
\sin.^2 (s-a) &= \tan r \cot r_{,} \tan. r_{,,} \tan. r_{,,,}\\
\sin.^2 (s-b) &= \tan. r \tan. r_{,} \cot. r_{,,} \tan. r_{,,,}\\
\sin.^2 (s-c) &= \tan. r \tan. r_{,} \tan. r_{,,} \cot. r_{,,,}
\end{aligned}\right\} \quad . . (9)$$

Multiply the first and second of equations (7) and the third and fourth, and add the products then $\tan r \tan. r_{,} + \tan. r_{,,} \tan. r_{,,,}$

$$= \sin. (s-b) \sin. (s-c) + \sin. s \sin. (s-a)$$

$$= \sin. \frac{a+b-c}{2} \sin \frac{a-b+c}{2} + \sin. \frac{a+b+c}{2} \sin. \frac{-a+b+c}{2}$$

$$= \sin.^2 \frac{a}{2} - \sin.^2 \frac{b-c}{2} + \sin.^2 \frac{b+c}{2} - \sin.^2 \frac{a}{2}$$

$$= \sin^2 \frac{b+c}{2} - \sin.^2 \frac{b-c}{2} = \sin. b \sin. c$$

Taking each of the other corresponding combinations in the same way, we shall obtain the following set of results —

$$\left.\begin{aligned}
\tan. r \tan r_{,} + \tan. r_{,,} \tan. r_{,,,} &= \sin. b \sin c\\
\tan. r \tan. r_{,,} + \tan r_{,,,} \tan. r_{,} &= \sin a \sin. c\\
\tan r \tan. r_{,,,} + \tan. r_{,} \tan. r_{,,} &= \sin a \sin b
\end{aligned}\right\} \quad . . (10)$$

Or, adding all these together, we get

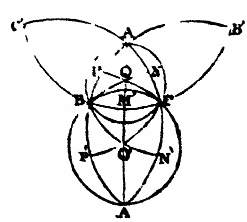
$$\tan. r \tan. r_{,} + \tan. r \tan. r_{,,} + \tan. r \tan. r_{,,,} + \tan. r_{,} \tan. r_{,,} + \tan. r_{,} \tan. r_{,,,} + \tan. r_{,,} \tan. r_{,,,} = \sin. a \sin. b + \sin. b \sin. c + \sin. b \sin. c \quad . . (11)$$

That is, the sum of all the products, two and two, of the tangents of the radii of the associated inscribed circles, is equal to the products, taken two and two, of the sines of the sides of any one of the four triangles

Also, if we multiply two and two the three last of the equations marked (9) and add the results, we get

$$\tan. r_{,} \tan. r_{,,} + \tan. r_{,,} \tan. r_{,,,} + \tan. r_{,,,} \tan. r_{,} = \sin. s \left\{\sin (s-a) + \sin. (s-b) + \sin (s-c)\right\} \quad . \quad (12)$$

We may next take the circumscribed circles, and denote their radii as before described.



Let Q be the centre of the circle circumscribing the primitive triangle ABC. Join AQ, BQ, CQ and draw the perpendiculars QM, QN, QP from Q to the sides of the triangle; which bisect them respectively. Let the angles made by the radii QB, QC with the side BC (or a) be denoted by α, those made by QC QA, with the side AC (or b) be called β, and those made by QA, QB with AB (or c) be called γ Then we have $A = \beta + \gamma$, $B = \gamma + \alpha$, and $C = \alpha + \beta$. from which, as before,

$$\left.\begin{aligned} \alpha+\beta+\gamma &= \frac{A+B+C}{2} = S \\ \alpha &= \frac{-A+B+C}{2} = S-A \\ \beta &= \frac{A-B+C}{2} = S-B \\ \gamma &= \frac{A+B-C}{2} = S-C \end{aligned}\right\} \quad \ldots (13)$$

But the right-angled triangle BQM gives cot. BQ = cot. QB M cot. BM; or

$$\cot. R = \cos. a \cot. \tfrac{1}{2}a.$$

Now, substituting for cos. a, from (13) and for cot. $\frac{1}{2}a$ from Th. VIII. we get

$$\left.\begin{aligned} \cot. R &= \frac{\sqrt{-\cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C)}}{-\cos. S} \\ \text{similarly } \cot. R_{,} &= \frac{\sqrt{-\cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C)}}{\cos. (S-A)} \\ \cot. R_{,,} &= \frac{\sqrt{-\cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C)}}{\cos. (S-B)} \\ \cot. R_{,,,} &= \frac{\sqrt{-\cos S \cos. (S-A) \cos. (S-B) \cos. (S-C)}}{\cos (S-C)} \end{aligned}\right\} \quad \ldots (14)$$

Multiply all these: then

$$\cot R \cot. R_{,} \cot. R_{,,} \cot. R_{,,,} = -\cos. S \cos.(S-A)\cos.(S-B)\cos.(S-C) \quad \ldots (15)$$

Divide (15) by the squares of each of the equations in (14): then

$$\left.\begin{aligned} \cos.^2 S &= \tan. R \cot. R_{,} \cot. R_{,,} \cot. R_{,,,} \\ \cos^2 (S-A) &= \cot. R \tan R_{,} \cot. R_{,,} \cot. R_{,,,} \\ \cos^2 (S-B) &= \cot. R \cot. R_{,} \tan. R_{,,} \cot. R_{,,,} \\ \cos.^2 (S-C) &= \cot. R \cot. R_{,} \cot. R_{,,} \tan. R_{,,,} \end{aligned}\right\} \quad \ldots (16)$$

The first of which is a remarkable expression for the area of a spherical triangle in terms of the four circumscribed radii.

By a process exactly similar to that employed in the derivation of (10) and (11) we may obtain:—

$$\left.\begin{aligned} \cot. R \cot R_{,} + \cot. R_{,,} \cot. R_{,,,} &= \sin. B \sin. C \\ \cot. R \cot R_{,,} + \cot. R_{,,,} \cot R_{,} &= \sin. C \sin. A \\ \cot. R \cot R_{,,,} + \cot. R_{,} \cot. R_{,,} &= \sin. A \sin. B \end{aligned}\right\} \quad \cdot (17)$$

$$\left.\begin{aligned} &\cot. R \cot R_{,} + \cot. R \cot. R_{,,} + \cot. R \cot. R_{,,,} \\ &+ \cot. R_{,,} \cot. R_{,,,} + \cot. R_{,,,} \cot. R_{,} + \cot. R_{,} \cot. R_{,,} \end{aligned}\right\} = \sin. A \sin. B + \sin. B \sin. C + \sin. C \sin. A \quad (18)$$

For a further discussion of this class of properties see Nos. 24 and 27 of *Leybourn's Mathematical Repository;* and the *Supplement to Young's Trigonometry.*

SECTION V.

ON THE NATURE AND MEASURE OF SOLID ANGLES.

A *Solid angle* is defined by Euclid, that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

Others define it the angular space comprised between several planes meeting in one point.

It may be defined still more generally, the *angular space* included between several plane surfaces or one or more curved surfaces, meeting in the point which forms the summit of the angle

According to this definition, solid angles bear just the same relation to the surfaces which comprise them, as plane angles do to the lines by which they are included: so that, as in the latter it is not the magnitude of the lines, but their mutual inclinations, which determines the angle; just so, in the former it is not the magnitude of the planes, but *their* mutual inclinations which determine the angles. And hence all those geometers, from the time of Euclid down to the present period, who have confined their attention principally to the magnitude of the plane angles, instead of their relative positions, have never been able to develope the properties of this class of geometrical quantities; but have affirmed that no solid angle can be said to be the half or the double of another, and have spoken of the bisection and trisection of solid angles, even in the simplest cases, as impossible problems.

But all this supposed difficulty vanishes, and the doctrine of solid angles becomes simple, satisfactory, and universal in its application, by assuming *spherical surfaces* for their measure; just as circular arcs are assumed for the measures of plane angles *. Imagine, that from the summit of a solid angle (formed by the meeting of three planes) as a centre, any sphere be described, and that those planes are produced till they cut the surface of the sphere; then will the surface of the spherical triangle, included between those planes, be a proper measure of the solid angle made by the planes at their common point of meeting: for no change can be conceived in the relative position of those planes, that is, in the magnitude of the solid angle, without a corresponding and proportional mutation in the surface of the spherical triangle. If, in like manner, the three or more surfaces, which by their meeting constitute another solid angle, be produced till they cut the surface of the same or an equal sphere, whose centre coincides with the summit of the angle; the surface of the spherical triangle or polygon, included between the planes which determine the angle, will be a correct measure of *that* angle. And the ratio which subsists between the areas of the spheric triangles, polygons, or other surfaces thus formed, will be accurately the ratio which subsists between the solid angles, constituted by the meeting of the several planes or surfaces at the centre of the sphere.

Hence, the comparison of solid angles becomes a matter of great ease and simplicity: for since the areas of spherical triangles are measured by the excess of the sums of their angles each above two right angles (theor. 6); and the areas of spherical polygons of n sides, by the excess of the sum of their angles above $2n - 4$ right angles (theor. 7); it follows, that the magnitude of a trilateral solid angle will be measured by the excess of the sum of the three angles, made respectively by its bounding planes, above two right angles; and the magnitudes of solid angles formed by n bounding planes, by the excess of the sum of the angles of inclination of the several planes above $2n - 4$ right angles.

* It may be proper to anticipate here the only objection which *can* be made to this assumption; which is founded on the principle, *that quantities should always be measured by quantities of the same kind*. But this, often and positively as it is affirmed, is by no means necessary; nor in many cases is it possible. To measure is to *compare* mathematically: and if by comparing two quantities whose ratio we know or can ascertain, with two other quantities whose ratio we wish to know, the point in question becomes determined; it signifies not at all whether the magnitudes which constitute one ratio, are like or unlike the magnitudes which constitute the other ratio. It is thus that mathematicians, with perfect safety and correctness, make use of space as a measure of velocity, mass as a measure of inertia, mass and velocity conjointly as a measure of force, space as a measure of time, weight as a measure of density, expansion as a measure of heat, a certain function of planetary velocity as a measure of distance from the central body, arcs of the same circle as measures of plane angles, and it is in conformity with this general procedure that we adopt surfaces of the same sphere as measures of solid angles.

As to solid angles limited by curve surfaces, such as the angles at the vertices of cones, they will manifestly be measured by the spheric surfaces cut off by the prolongation of their bounding surfaces, in the same manner as angles determined by planes are measured by the triangles or polygons they mark out upon the same, or an equal sphere. In all cases, the maximum limit of solid angles will be the *plane* towards which the various planes determining such angles approach, as they diverge further from each other about the same summit: just as a right line is the maximum limit of plane angles, being formed by the two bounding lines when they make an angle of 180°. The maximum limit of solid angles is measured by the surface of a hemisphere, in like manner as the maximum limit of plane angles is measured by the arc of a semicircle. The solid right angle (either angle, for example, of a cube) is $\frac{1}{4} = (\frac{1}{2})^2$ of the maximum solid angle: while the plane right angle is half the maximum plane angle.

The analogy between plane and solid angles being thus traced, we may proceed to exemplify this theory by a few instances; assuming 1000 as the numeral measure of the maximum solid angle = 4 times 90° solid = 360° solid.

1. The solid angles of right prisms are compared with great facility. For, of the three angles made by the three planes which, by their meeting, constitute every such solid angle, two are right angles; and the third is the same as the corresponding plane angle of the polygonal base; on which, therefore, the measure of the solid angle depends. Thus, with respect to the right prism with an equilateral triangular base, each solid angle is formed by planes which respectively make angles of 90°, 90°, and 60°. Consequently $90° + 90° + 60° - 180° = 60°$, is the measure of such angle, compared with 360 the maximum angle. It is, therefore, one-sixth of the maximum angle. A right prism with a square base has, in like manner, each solid angle measured by $90° + 90° + 90° - 180° = 90°$, which is $\frac{1}{4}$ of the maximum angle. And thus it may be found, that each solid angle of a right prism, with an equilateral

triangular base	is $\frac{1}{6}$ max angle	=	$\frac{1}{6}$.1000.
square base	is $\frac{1}{4}$.	. =	$\frac{2}{8}$.1000.
pentagonal base	is	. =	$\frac{3}{10}$.1000.
hexagonal	is $\frac{1}{3}$	. =	$\frac{4}{12}$.1000.
heptagonal	is . .	=	$\frac{5}{14}$ 1000
octagonal base	is $\frac{3}{8}$ . .	. =	$\frac{6}{16}$ 1000.
nonagonal	is	. =	$\frac{7}{18}$.1000.
decagonal	is $\frac{2}{5}$ . .	. =	$\frac{8}{20}$ 1000.
undecagonal	is .	. =	$\frac{9}{22}$.1000
duodecagonal	is $\frac{5}{12}$ . .	. =	$\frac{10}{24}$ 1000.
m gonal	is .	. =	$\frac{m-2}{2m}$.1000

Hence it may be deduced, that each solid angle of a regular prism, with triangular base is *half* each solid angle of a prism with a regular hexagonal base. Each with regular

square base	=	$\frac{2}{3}$ of each, with regular	octagonal base,
pentagonal	=	$\frac{3}{4}$ . . .	decagonal,
hexagonal	=	$\frac{4}{5}$	duodecagonal,
$\frac{1}{2}m$ gonal	=	$\frac{m-4}{m-2}$	m gonal base.

Hence again we may infer, that the sum of all the solid angles of any prism of triangular base, whether that base be regular or irregular, is *half* the sum of the solid angles of a prism of quadrangular base, regular or irregular. And the sum of the solid angles of any prism of

tetragonal base is $= \frac{2}{3}$ sum of angles in prism of pentag. base,
pentagonal . . $= \frac{3}{4}$ . . . . hexagonal,
hexagonal . . $= \frac{4}{5}$ . . . . heptagonal,
m gonal . . $= \frac{m-2}{m-1}$ . . . . . . $(m + 1)$ gonal.

2. Let us compare the solid angles of the five regular bodies. In these bodies, if m be the number of sides of each face; n the number of planes which meet at each solid angle; $\frac{1}{2}\bigcirc =$ *half* the circumference or 180°; and A the plane angle made by two adjacent faces: then we have

$$\sin. \tfrac{1}{2}A = \frac{\cos. \frac{1}{2n}\bigcirc}{\sin. \frac{1}{2m}\bigcirc}.$$

This theorem gives, for the plane angle formed by every two contiguous faces of the tetraëdron, 70°31′42″; of the hexaedron, 90°; of the octaedron, 109°28′18″; of the dodecaedron, 116°33′54″; of the icosaedron, 138°11′23″. But, in these polyëdræ, the number of faces meeting about each solid angle, is 3, 3, 4, 3, 5 respectively. Consequently the several solid angles will be determined by the subjoined proportions:

		Solid Angle.	
360° : 3 × 70°31′42″ − 180	:: 1000 :	87·73611	Tetraedron.
360° : 3 × 90 − 180	:: 1000 :	250	Hexaedron.
360° : 4 × 109°28′18″ − 360	:: 1000 :	216·35185	Octaedron.
360° : 3 × 116°33′54″ − 180	:: 1000 :	471·395	Dodecaedron.
360° : 5 × 138°11′23″ − 540°	:: 1000 :	419·30169	Icosaedron.

3. The solid angles at the vertices of cones, will be determined by means of the spheric segments cut off at the bases of those cones; that is, if right cones, instead of having plane bases, had bases formed of the segments of equal spheres, whose centres were the vertices of the cones, the surfaces of those segments would be measures of the solid angles at the respective vertices. Now, the surfaces of spheric segments, are to the surface of the hemisphere, as their altitudes to the radius of the sphere; and therefore the solid angles at the vertices of right cones, will be to the maximum solid angle as the excess of the slant side above the axis of the cone, to the slant side of the cone. Thus, if we wish to ascertain the solid angles at the vertices of the equilateral and the right-angled cones: the axis of the former is $\frac{1}{2}\sqrt{3}$, of the latter, $\frac{1}{2}\sqrt{2}$, the slant side of each being unity. Hence,

		Angle at vertex.	
$1 : 1 - \frac{1}{2}\sqrt{3}$	:: 1000 :	133·97464,	equilateral cone,
$1 : 1 - \frac{1}{2}\sqrt{2}$	:: 1000 :	292·89322,	right-angled cone.

4. From what has been said, the mode of determining the solid angles at the vertices of pyramids will be sufficiently obvious. If the pyramids be regular ones, if N be the number of faces meeting about the vertical angle in one, and A the angle of inclination of each two of its plane faces; if n be the number of planes meeting about the vertex of the other, and a the angle of inclination of each two of its faces: then will the vertical angle of the former, be to the vertical angle of the latter pyramid, as $NA - (N - 2)\ 180°$, to $na - (n - 2)\ 180°$.

If a cube be cut by diagonal planes, into 6 equal pyramids with square bases, their vertices all meeting at the centre of the circumscribing sphere; then each of the solid angles made by the four planes meeting at each vertex, will be $\frac{1}{6}$ of the maximum solid angle; and each of the solid angles at the bases of the pyra-

mids, will be $\frac{1}{16}$ of the maximum solid angle. Therefore, each solid angle at the base of such pyramid, is *one-fourth* of the solid angle at its vertex: and, if the angle at the vertex be bisected, as described below, either of the solid angles arising from the bisection will be double of either solid angle at the base. Hence also, and from the first subdivision of this scholium, each solid angle of a prism, with equilateral triangular base, will be *half* each vertical angle of these pyramids, and *double* each solid angle at their bases.

These angles made by one plane with another, must be ascertained, either by measurement or by computation, according to circumstances. But, the general theory being thus explained, and illustrated, the further application of it is left to the skill and ingenuity of geometers; the following simple example, merely, being added here:

Ex. 1. Let the solid angle at the vertex of a square pyramid be bisected.

1st. Let a plane be drawn through the vertex and any two opposite angles of the base, that plane will bisect the solid angle at the vertex; forming two trilateral angles, each equal to half the original quadrilateral angle.

2ndly. Bisect either diagonal of the base, and draw *any* plane to pass through the point of bisection and the vertex of the pyramid; such plane, if it do *not* coincide with the former, will divide the quadrilateral solid angle into two equal quadrilateral solid angles. For this plane, produced, will bisect the great circle diagonal of the spherical parallelogram cut off by the base of the pyramid; and any great circle bisecting such diagonal is known to bisect the spherical parallelogram, or square; the plane, therefore, bisects the solid angle.

Cor. Hence an indefinite number of planes may be drawn, each to bisect a given quadrilateral solid angle.

Ex. 2. Determine the solid angles of a regular pyramid with hexagonal base, the altitude of the pyramid being to each side of the base as 2 to 1.

Ans. Plane angle between each two faces = 125 22′35′
Plane angle between the base and each face = 66 35 12
Solid angle at the vertex = 89·60648 | the max. solid
Each ditto at the base = 218·19367 | angle being 1000.

Ex. 3. Find the solid angles of the five regular bodies; and shew that if a tetrahedron and an octahedron be formed of *equal* equilateral triangles, the face of the base of the tetrahedron be applied to the face of the octahedron, the planes of their adjacent faces will form continuous planes.

Ex. 4. The three diagonals of a parallelopiped are 3, 4, 5 the plane in which 3, 4 are is inclined to the plane of 4, 5 in angle of 72 10′, the plane 4, 5 is inclined to 5, 3 in angle of 83 4′20″, and the plane 5, 3 to the plane 3, 4 in angle of 63°10′. How many parallelopipeds can be constructed with these data? And assign the solid angles of all that can be so formed.

Ex. 5. A twelve inch, ten inch, nine inch, and six inch wall all touch each the other three (in the manner of the summit of a triangular pile): what are the solid angles of the tetrahedron formed by joining three centres three and three by four planes? And what solid angles does the twelve inch wall subtend when seen from the centres of the other three?

Ex. 6. A crystal of carbonate of lime has its solid angles contained by parallelograms whose angles are 104° 29′ and 75° 31′: and a crystal of quartz by parallelograms whose angles are 94° 4′ and 85° 56′: in which is the sum of all the solid angles the greater, and how much? They both being parallelopipeds. Shew also whether there be any difference between the sum of all the dihedral angles (between the planes of these four) of the one and the other, or not

Ex. 6. The latitudes of three stars A, B, C are 29°10′15″, 55°10′20″, and 62°5′45″, and their longitudes are 152°10′20″, 186°18′10″, and 96°18′11″ respectively: and the right ascensions of three others, *a*, *b*, *c* are 10°18′6″, 29°18′10″, and 65°18′30″; and their declinations are 87°18′30″, 26°15′15″, and 27°18′16″ respectively. How much *more* of the celestial space is comprised within the triangle A B C than within the triangle *a b c*?

Ex. 7. Two cones have their bases on a parallel of latitude of 80° N.: the one has its vertex at the centre of the earth; the other in the surface of the sphere at the latitude of 40° S. Compare their solid vertical angles.

Ex. 8. A cone whose base is 8 feet radius and altitude 15 feet, is seen from a point without it, whose distance from the centre of the cone's base measured on the plane of the base is 12 feet, and its height above the base 5 feet: what portion of the entire sphere surrounding the eye is occupied by the cone? And what is the dihedral angle under which its extreme visible edges appear?

Ex. 9. A copy of Hutton's Tables of Logarithms is laid with its nearer (which is the shorter) edge passing through the same point as the centre of the cone in the last example, and that edge inclined to the line drawn from that centre to the foot of the observer in an angle of 60°: if the edges be 6 inches and 9·7 inches, and its thickness be 1·8 inches, compare the plane angles which its several visible edges and the solid angles which its several visible faces subtend.

Ex. 10. Find the position on the surface of a sphere from which a given triangle shall subtend a given solid angle: the latitudes of the three points being 10°15′, 18°10′, and 16°15′, and the longitudes 18°15′, 25°18′, and 16°18′ reckoned from Greenwich.

Ex. 11. A line is drawn through the centre of a circle and at right angles to its plane: from two points in this line which are at 10 feet distance from each other, the circle is seen to subtend 100·25 and 62·5° solid. Find its diameter

Ex. 12. Give *general expressions* for each solution above required.

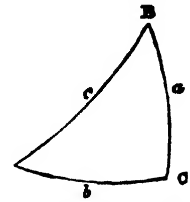
SECTION VI.

Mnemonics in Spherical Trigonometry.

THE great number of cases which occur in Spherical Trigonometry render it in general very difficult to recollect the exact form of the equations of solutions; and to obviate this difficulty, mnemonic rules have been invented which apply to all the cases of a particular class. The first attempt of the kind was made by the inventor of logarithms, and was published in the preface to his work on that subject in 1614. It was adapted to right-angled spherical triangles, and is also applicable to quadrantal triangles, or such as have one side equal to the quadrant or 90°. It is technically known as "Napier's Rules for the Circular Parts." Other methods have been devised for oblique-angled triangles by Gillebrand, Pingré, Mauduit, and Fisher. Delambre has attempted to form rules of a memorial kind by giving to the equations of solutions a form adapted to easy recollection, in his *Astronomie*, vol. i. p. 204—7; but of all the attempts that have been made, that of the Rev. Walter Fisher seems best adapted to answer its purpose—*Edinb. Transac.* vol. ii.

I. Napier's *Rules for the Circular Parts.*

Of the six parts of a triangle, viz. three sides and three angles, any one being fixed in value, there will be five others left. Any one of these being fixed upon as the *middle*, there will of the remaining ones, two *adjacent* to the middle, and the two others *opposite* to the middle: thus, for instance, neglecting the angle C, there will be the five parts a, b, B, A, and c. If any one of these, as, for instance, B, be taken as the middle, the *adjacent* parts will be a and c, and the *opposite* parts will be b and A.



Case. 1. *Right-Angled Triangles.*—Let c be the right-angle, then the circular parts to which the rules refer are the complements of the opposite side (or hypothenuse) and of the other angles, and the sides b and a. Then

$$\text{sin. middle} = \text{tan. adjacent} \times \text{tan. other adjacent}$$
$$= \text{cos. opposite} \times \text{cos. other opposite.}$$

Now in each of these equations, both factors are the same functions of the parts; viz. both tangents or both cosines; and it so happens that the first vowel in *tangent* is the first in *adjacent*, and the first in *cosine* is the first in *opposite;* and hence the recollection of the function, cosine, or tangent can never create a moment's hesitation when it has been determined how the three parts, the one given and the one sought, are related to one another.

In the use of this method, mark the three parts in question, and consider whether any two of them are *adjacent* to the third, or opposite to the third, and form the equation accordingly. Thus if a and b were given to find c, it is at once seen that they are *opposite* to c, and hence

$$\sin\left(\frac{\pi}{2} - c\right) = \cos a \cos. b:$$

or if A were sought, then A and a are *adjacent* to b, and $\sin. b = \tan a \tan\left(\frac{\pi}{2} - \text{A}\right)$ from which $\cot. \text{A} = \sin b \cot. a$, both which accord with the usual formulæ for these cases

In the some way, the student should *verify* the truth of the rule by its application *to all* the cases that can occur in the right-angled triangle A B C

Case 2. *Quadrantal Triangles*—Instead of taking C a right-angle take c a quadrant. Then the circular parts are A, B, $\frac{\pi}{2} - a$, $\frac{\pi}{2} - b$, and $-\left(\frac{\pi}{2} - \text{C}\right)$, and the same rules apply here as before.

Fisher's *Rules for Oblique-Angled Triangles.*

Theorem 1.—*Given two parts and an opposite one.*

$$\text{S . A : S . O :: S . } a \text{ : S } o.$$

Theorem 2.—*An included part given or sought.*

$$\text{S} . \frac{\text{A} - a}{2} \cdot \text{sin.} \frac{\text{A} + a}{2} :: \text{T} \frac{\text{O} - o}{2} : \text{T} \frac{\text{M}}{2}$$

Theorem 3.—$\text{T} . \frac{\text{A} - a}{2} : \text{T} . \frac{\text{A} + a}{2} :: \text{T} . \frac{\text{O} - o}{2} : \text{T} . \frac{\text{O} + o}{2}.$

Theorem 4.—*Given three sides or angles of an oblique-angled triangle.*

$$\text{S . A} \times \text{S} . a : r^2 :: \text{S} . \frac{(\text{A} + a) + l}{2} \times \text{S} . \frac{(\text{A} + a) - l}{2} . \text{S.}^2 \frac{\text{M}}{2}.$$

In explanation of these analogies, it must be remarked, that S and T prefixed signify the words sine and tangent respectively, A and a designate the two *adjacent* parts to the middle part M, and O, o the *opposite* parts.

In Napier's rules there were only five parts to be considered, the sixth or last part being always of a given magnitude. In Fisher's rules (and indeed in all others that have been framed) the six parts must be taken in order from the part fixed upon as the *middle*.

The *middle* is either a *side* or the *supplement of angle*.

The *adjacent* parts (A and a) are those *immediately contiguous to the middle part*, and of a kind always different from it. Thus if the middle be a side, the adjacent parts are angles, and if the middle be the supplement of an angle, the adjacent parts are sides.

The *opposite parts* are the two contiguous to the adjacent ones, and of *the same denomination as the middle part*.

The letter l denotes the *last* or most distant part from the middle, and is always of a different kind from the middle.

That these four theorems may be called to mind with greater facility, the following words, formed by abbreviating the terms of the respective analogies, should be committed to memory: viz.

Sao, satom, tao, sarsalm

III. Cagnoli's *Rule for finding whether there be double solutions in the Problems* VI. *and* VII. (See Delambre's *Astronomy*, vol 1 page 197, where it is examined by means of its application to all the cases.)

When b *is greater than* a, *and less than* $\pi - a$, *or,*

when B *is greater than* A, *and less than* $\pi - $ A,

the whole triangle is determinate, and the unknown quantity is always of the same species as that to which it is opposite.

In all other cases there will be two solutions.

SECTION VII.

Questions for Exercise in Spherical Trigonometry.

Ex. 1 In the right-angled spherical triangle BAC, right-angled at A, the hypothenuse $a = 78°\ 20'$, and one leg $c = 76°\ 52'$, are given, to find the angles B, and C, and the other leg b.

Here, by table I, case 1, $\sin. C = \dfrac{\sin\ c}{\sin\ a}$.

$$\cos\ B = \frac{\tan\ c}{\tan.\ a} \qquad \cos\ b = \frac{\cos\ a}{\cos.\ c}$$

Or, $\log.\sin.\ C = \log\ \sin.\ c - \log\ \sin\ a + 10$

$\log.\cos.\ B = \log\ \tan.\ c - \log.\tan.\ a + 10$

$\log\ \cos\ b = \log\ \cos.\ a - \log.\cos.\ c + 10.$

Hence, $10 + \log.\sin\ c = 10 + \log.\sin.\ 76°\ 52' = 19{\cdot}9884891$

$\log.\sin.\ a = \log\ \sin.\ 78°\ 20' = 9{\cdot}9909338$

Remains, $\log\ \sin.\ C = \log\ \sin.\ 83°\ 56' = 9{\cdot}9975553$

Here C is acute, because the given leg is less than 90°.

Again, $10 + \log\ \tan.\ c = 10 + \log\ \tan\ 76°\ 52' = 20{\cdot}6320468$

$\log.\tan.\ a = \log.\tan\ 78°\ 20' = 10{\cdot}6851149$

Remains, $\log.\cos.\ B = \log.\cos\ 27°\ 45' = 9{\cdot}9469319$

B is here acute, because a and c are of like affection.

Lastly, $10 + \log\ \cos\ a = 10 + \log.\cos.\ 78°\ 20' = 19{\cdot}3053189$

$\log\ \cos\ c = \log.\cos.\ 76°\ 52' = 9{\cdot}3564426$

Remains, $\log.\cos\ b = \log\ \cos.\ 27°\ 8' = 9{\cdot}9493763$

where b is less than 90°, because a and c both are so.

***Ex* 2** In a right-angled spherical triangle, denoted as above, are given $a = 78°20'$, $B = 27°45'$; to find the other sides and angle

Ans $b = 27°8'$, $c = 76°52'$, $C = 83°56'$

***Ex* 3** In a spherical triangle, with A a right angle, given $b = 117°34'$, $C = 31°51'$; to find the other parts Ans $a = 113°55'$, $c = 28°51'$, $B = 104°8'$

***Ex*. 4** Given $b = 27°6'$, $c = 76°52'$, to find the other parts

Ans $a = 78°20'$ $B = 27°45'$, $C = 83°56'$.

***Ex* 5** Given $b = 42°12'$, $B = 48°$, to find the other parts.

Ans $a = 64°40\frac{1}{2}'$, or its supplement,
$c = 54°44'$, or its supplement,
$C = 64°35'$, or its supplement.

***Ex* 6** Given $B = 48°$, $C = 64°35'$, required the other parts?

Ans $b = 42°12'$, $c = 54°44'$, $a = 64°40\frac{1}{2}'$

***Ex* 7** In the quadrantal triangle ABC, given the quadrantal side $a = 90°$, an adjacent angle $C = 42°12'$, and the opposite angle $A = 64°40'$, required the other parts of the triangle?

***Ex* 8.** In an oblique-angled spherical triangle are given the three sides, viz. $a = 56°40'$, $b = 83°13'$, $c = 114°30'$, to find the angles

Here, by the fifth case of table 2, we have

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}$$

Or, 2 log sin $\frac{1}{2}$A = log sin. $(s-b)$ + log. sin $(s-c)$ + ar comp log sin b + ar comp log sin c where $2s = a + b + c$

log sin $(s-b)$ = log sin $43°58\frac{1}{2}'$ = 9.8415749
log sin $(s-c)$ = log sin $12°41\frac{1}{2}'$ = 9.3418385
A C log sin b = A C log sin $83°13'$ = 0.0030508
A C log sin c = A C log sin $114°30'$ = 0.0409771
Sum of the four logs 19.2274413
Half sum = log sin $\frac{1}{2}$A = log sin $24°15\frac{1}{2}'$ = 9.6137206

Consequently the angle A is $48°31'$

Then, by the common analogy,

As, sin a	sin $56°40'$	log	9.9219401
To, sin A	sin $48°31'$	log	9.8745679
So is, sin b	sin $83°13'$	log =	9.9969492
To, sin B	sin $62°56'$	log	9.9495770
And so is, sin c	sin $114°30'$	log	9.9590229
To, sin C	sin $125°19'$	log	9.9116507

So that the remaining angles are, B = $62°56'$, and C = $125°19'$

2ndly By way of comparison of methods, let us find the angle A, by the analogies of Napier, according to case 5, table 3 In order to which, suppose a perpendicular demitted from the angle C on the opposite side c Then shall we have

$$\tan \tfrac{1}{2} \text{ diff seg of } c = \frac{\tan \frac{1}{2}(b+a) \tan \frac{1}{2}(b-a)}{\tan \frac{1}{2}c}$$

This, in logarithms, is

log tan $\frac{1}{2}(b+a)$ = log tan $69°56\frac{1}{2}'$ 10.4375601
log tan $\frac{1}{2}(b-a)$ = log tan $13°16\frac{1}{2}'$ 9.3727819
Their sum 19.8103420
Subtract log. tan $\frac{1}{2}c$ = log tan $57°15'$ = 10.1916394
Rem log cos dif seg = log cos $22°34'$ = 9.6187026

Hence, the segments of the base are $79°49'$ and $34°41'$.

Therefore, since $\cos A = \tan. 79°49' \times \cot. b$:
To log. tan. adja. seg. $=$ log. tan. 79°49′ $=$ 10·7456257
Add log. tan. side b $=$ log. tan. 83°13′ $=$ 9·0753563
The sum, rejecting 10 from the index $=$ log. cos. A $=$ log. cos. 48°32′ $\}$ $=$ 9·8209820

The other two angles may be found as before. The preference is, in this case, manifestly due to the former method.

Ex. 9. In an oblique-angled spherical triangle, are given two sides, equal to 114°30′ and 56°40′ respectively, and the angle opposite the former equal to 125°20′; to find the other parts. Ans. Angles 48°30′ and 62°55′; side, 83°12′

Ex. 10. Given, in a spherical triangle, two angles, equal to 48°30′ and 125°20′, and the side opposite the latter; to find the other parts.

Ans. Side opposite first angle, 56°40′; other side, 83°12′; third angle, 62°54′

Ex. 11. Given two sides, equal 114°30′ and 56°40′, and their included angle 62°54′; to find the rest.

Ex. 12. Given two angles, 125°20′ and 48°30′, and the side comprehended between them 83°12′; to find the other parts.

Ex. 13. In a spherical triangle, the angles are 48°31′, 62°56′, and 125°20′; required the sides?

Ex. 14. Given two angles, 50°12′ and 58°8′, and a side opposite the former 62°42′; to find the other parts

Ans. The third angle is either 130°54′33″ or 156°16′32″,
Side betw. giv angles, either 119°3′32″ or 152°14′14″
Side opp. 58°8′ either 72°12′13″ or 100°47′37″

Ex. 15. The excess of the three angles of a triangle, measured on the earth's surface, above two right angles, is 1 second; what is its area, taking the earth's diameter at 7957¾ miles? Ans. 76·75299, or nearly 76¾ square miles.

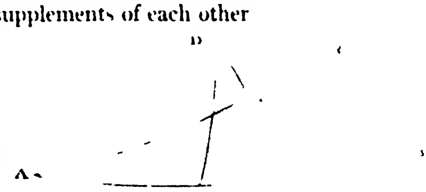
Ex. 16 Determine the solid angles of a regular pyramid with hexagonal base, the altitude of the pyramid being to each side of the base as 2 to 1.

Ans Plane angle between each two lateral faces 125°22′35″
between the base and each face 66°35′12″.
Solid angle at the vertex 89·60648 } The max angle
Each ditto at the base 218·19367 } being 1000.

Ex. 17. Find the distance between the Observatories of Paris and Greenwich, the latitudes being 48°50′14″ and 51°28′40″, and their longitudes 2°20′24″ E and 0°.

Ex. 18. Find the inclinations of the faces of the regular tetrahedron and octahedron; and show that they are the supplements of each other

Ex. 19. Two great circles of the heavens AB, AC making a given angle α, and the point D such that AE $= \beta$, ED $= \gamma$ what are the arcs AF, FD, the angles at E and F being right angles? Apply it to the case



$\alpha = 23°28'$, $\beta = 42°18'$ and $\gamma = 29°6'$. Find also the distances EF, and AD.

[*Note.*—If AB be the celestial equator, and AC the ecliptic, then AE, ED are called the ***right ascension*** and ***declination*** of a heavenly body at D; and AF, FD are called its ***longitude and latitude***.]

Ex 20 The right ascensions of two stars are observed to be 26° 10′ 15″ E and 12° 15′ 18″ W, and their declinations to be 33° 10′ 15″ N and 12° 16′ 45″ S respectively Find their differences of latitude and longitude, and their distances from each other and from a third star whose latitude and longitude are 48° 10′ 15″ N and 108° 18′ 30″ W

Ex 21 Given the altitudes of two stars 33° 10′ 15″ and 28° 18′ 45″, and their azimuths (or angles made by vertical planes passing through them with the meridian) equal to 42° 10′ 18″ E and 31° 15′ 18″ W to find their difference of longitude and difference of right ascension the inclination of the equator to the ecliptic being supposed 23° 28′

Ex 22 The crown of an arch of Aurora Borealis was observed from Woolwich Common to have the altitude 15° 10′, and the azimuth 18° 12′ W The same arch was seen at the same time from Bath under an azimuth of 3° 2′ E What was its altitude in the Bath observation and to what place would it appear vertical, and how high was it above the earth

[*Note*—The latitude of Bath is 51° 22′ 30″ N and its longitude 2° 21′ 31″ W whilst those of Woolwich are 51° 28′ 20″ N and 0° 3′ 34″ E]

Ex 23 It is required to assign the position of a place on the earth's surface in lat and long which is equidistant from three other places whose latitudes and longitudes are given—say London St Petersburgh and Cape Horn

Ex 24 Demonstrate the following properties —

1 $\sin^2\frac{1}{2}(A-B)\sin^2\frac{1}{2}c + \sin^2\frac{1}{2}(A+B)\cos^2\frac{1}{2}c = \cos^2\frac{1}{2}C$,

2 $\cos^2\frac{1}{2}(A-B)\sin^2\frac{1}{2}c + \cos^2\frac{1}{2}(A+B)\cos^2\frac{1}{2}c = \sin^2\frac{1}{2}C$,

3 $\sin^2\frac{1}{2}(a-b)\cos^2\frac{1}{2}C + \sin^2\frac{1}{2}(a+b)\sin^2\frac{1}{2}C = \sin^2\frac{1}{2}c$, and

4 $\cos^2\frac{1}{2}(a-b)\cos^2\frac{1}{2}C + \cos^2\frac{1}{2}(a+b)\sin^2\frac{1}{2}C = \cos^2\frac{1}{2}c$,

5 $\sin^2 C\sin(a+b)\sin(a-b) = \sin^2 c\sin(A+B)\sin(A-B)$

6 $\dfrac{\sin(A+B)}{\sin C} = \dfrac{\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{\cos^2\frac{1}{2}c}$,

7 $\dfrac{\sin(A-B)}{\sin C} = \dfrac{\sin\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b)}{\sin^2\frac{1}{2}c}$,

8 $\dfrac{\sin(a+b)}{\sin c} = \dfrac{\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{\sin^2\frac{1}{2}C}$,

9 $\dfrac{\sin(a-b)}{\sin c} = \dfrac{\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{\sin^2\frac{1}{2}C}$.

Ex 25 Demonstrate the properties geometrically and trigonometrically enunciated in the scholium on theorem III

Ex 26 If from the angles A B C of any spherical triangle great circles be drawn through any point P on the surface of the sphere, and divide the opposite sides into segments which, estimated from its extremities, are a_1 and a_2, b_1 and b_2 and c_1 and c_2 it is required to prove that

$$\sin a_1 \sin b_1 \sin c_1 = \sin a_2 \sin b_2 \sin c_2$$

none of the segments forming either of these products being contiguous to one another

Also, if any one great circle divide the sides of the triangle so as to form such segments, the same equation between the sines of those segments holds good

Show likewise conversely, that if the sides be so divided, the three points of

section shall either be in the same great circle, or that the circles drawn to the opposite angles will all meet in a point.*

Ex. 27. To find expressions for the parts of a spherical triangle, there are given the following several triads of data. A, B, $a-b$; A—B, a, b; $a-b$, $b-c$, B; C, c, r; R, r, c; R, r, c; a, A, p_1; p_1, p_2, p_3; l_1, l_2, l_3; p_1, l_1, a; p_1, l_1, A; $a-b$, $b+c$, B; sin. a cos. b, sin. a, C; tan. a — tan. b, tan. A — tan. B, cot. C; sin. a sin. b sin. c, tan. A tan B tan. C, sin. a ÷ sin A; sin a sin. A, sin. b sin B, sin. c sin. C; and lastly, S, s, p_1. In this notation p_1, p_2, p_3 designate the perpendicular arcs from A upon a, B upon b, and C upon c, and l_1, l_2, l_3 the arcs from A, B, C, to bisect a, b, c respectively. R and r as already used.

Ex. 28. If a spherical triangle be inscribed in a circle, and the radius of the inscribed circle be also given, the centre of the inscribed circle will always be at the same distance from the centre of the circumscribing circle, and if about a given circle triangles be described, so that the circles circumscribing the triangles shall be always of the same magnitude, the distance of their centres will be always the same.

Ex. 29. If a spherical quadrilateral figure be inscribed in a circle on the sphere, shew that the sum of one pair of opposite angles is equal to the sum of the other pair; and find expressions for those angles, the diagonals, their angles of intersection, and the area of the figure.

Ex. 30. Demonstrate the following equations:

$$\frac{\text{tan. } \frac{1}{2}(\text{A}+\text{B})}{\text{tan. } \frac{1}{2}(\text{A}-\text{B})} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)},$$

$$\text{cot. } b = \frac{\cot \text{B} \sin \text{A}}{\sin c} + \text{cot. } c \cos \text{A} = \frac{\cot \text{B} \sin \text{C}}{\text{sin. } a} + \cot a \cos \text{C},$$

$$\text{sin. } (a-c) = \text{sin. } b \frac{\text{cos. C} - \cos \text{A}}{2 \text{ cos.}^2 \frac{1}{2}\text{B}}, \text{ and } \sin (a+c) = \text{sin. } b \frac{\cos \text{C} + \cos \text{A}}{2 \sin^2 \frac{1}{2}\text{B}}$$

$$\frac{\cot \text{A}}{\cot a} = \frac{\text{cos. } a \sin b - \cos \text{C} \sin a \text{ cos. } b}{\text{cos. A sin. B} + \text{cos. } c \sin \text{A cos. B}}; \text{ and}$$

$$\sin \text{B sin. C} - \text{cos. B cos. C cos. } a = \text{sin. } b \sin c + \text{cos. } b \text{ cos. } c \cos \text{A}$$

Ex. 31. Also the following

$$\text{tan. } \tfrac{1}{2}(\text{A}+\text{B}+\text{C}) = \frac{-\text{cos. } a - \cos b - \cos c}{m}$$

$$\tan \tfrac{1}{2}(-\text{A}+\text{B}+\text{C}) = \frac{1+\text{cos. } a - \cos b - \text{cos. } c}{m}$$

$$\text{tan. } \tfrac{1}{2}(\text{A}-\text{B}+\text{C}) = \frac{1-\text{cos. } a + \text{cos. } b - \text{cos. } c}{m}$$

$$\text{tan. } \tfrac{1}{2}(\text{A}+\text{B}-\text{C}) = \frac{1-\cos a - \cos b + \cos c}{m}$$

(where m denotes the radical expression in the note on theor. VI. p 56), and find the analogous values of cot. $\frac{1}{2}(a+b+c)$, &c.

Ex. 32. In a tetrahedron whose six edges are 7, 8, 9, 10, 11, 12, it is required

* By means of these theorems, a considerable number of curious and important properties of spherical triangles are easily proved: as, for instance, that the perpendiculars from A, B, C upon a, b, c, meet in one point, the arcs bisecting A, B, C meet in a point, the arcs bisecting a, b, c meet in a point, and so on.

to find the plane angles formed by the several edges, each to *all* the others*, the several dihedral angles formed by the faces two and two, and the several solid angles formed by the faces three and three.

Ex. 33. Prove the following property: that in every tetrahedron as ABCD in the figure below, we have

AB . CD sin. (AB, CD) = BC . AD sin. (BC, AD) = AC . BD sin. (AC, BD)

where sin. (AB, CD) signifies the sine of the inclination of the opposite edges AB, CD to one another in the sense explained in the note *.

Ex 34. Given two of the adjacent triangular faces of a tetrahedron and their mutual inclination, to find all the other angles, plane, dihedral, and solid.

Ex. 35. Find the radii of the spheres inscribed within and circumscribed about the tetrahedron ABCD, as well as the radii of those which touch each face externally and the other three faces produced.

Ex. 36. Determine all the equal and regular figures which can be described on the surface of a sphere so as to exactly cover it.

Ex 37 Given the three diagonals of a parallelopiped equal to $2a$, $2b$, $2c$, and the inclinations of their planes respectively equal to C, B, and A, to find the edges, the volume, and the plane, dihedral, and solid angles of the figure.

Ex. 38 Given the three adjacent edges equal to a, b, c, and the three plane angles which they mutually form equal to C, B, A respectively, to find the diagonals and all the other angles of the parallelopipedon.

Ex. 39 Given the three perpendiculars from the angles of a spherical triangle to the opposite sides, to determine its area, sides, and angles

Ex 40. Given the three arcs of great circles bisecting the internal angles and terminated in the opposite sides, to find the sides, angles, and area.

Ex. 41 Given the three arcs bisecting the angles *externally*, and terminating in the opposite sides, to find the sides, angles, and area.

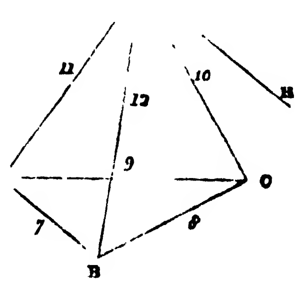
Ex 42 Of the four circles inscribed within the associated system of spherical triangles, any three are given to find the fourth.

Ex 43 Of the circumscribing circles, any three are given to find the fourth.

Ex. 44 If through the vertex and through the extremities of the base of a spherical triangle, two equal and parallel less circles be described: then all the spherical triangles which have the same base, and their vertices in the circle through the vertex of the given one will have the same area.

Ex 45 If the vertical angle of a spherical triangle, and the sum of the three sides be given, the base will always be a tangent to a circle which may be found.

* In the annexed figure the *order* in which these sides are to be taken is exhibited otherwise, with the same conditions a considerable number of tetrahedrons might be formed differing either in some or in all the above particulars. The angle formed by AB and CD is the angle formed by CD and DH, DH being drawn through D parallel to AB and in the same way the inclinations of BC to AD and of AC to BD are to be understood. These three pairs of edges are called *opposite*



Ex. 46. A person is going from London to Constantinople, and intends to touch at the equator. Which is the shortest route and distance he must travel to effect it?

Ex. 47. A cone has its base on a given circle of the sphere, and its vertex at a given point within or without the sphere. Show that it will cut the sphere a second time in a circle, and find position of its centre and the magnitude of its radius, by a general formula.

Ex. 48. Find each circumscribed radius in terms of the inscribed radii, and each inscribed radius in terms of the circumscribed radii.

PRINCIPLES OF POLYGONOMETRY.

The theorems and problems in Polygonometry have an intimate connexion and close analogy to those in plane trigonometry; and are in great measure deducible from the same common principles. Each comprises three general cases.

1. A triangle is determined by means of two sides and an angle; or, which amounts to the same, by its sides except one, and its angles except two. In like manner, any rectilinear polygon is determinable when all its sides except one, and all its angles except two, are known.

2. A triangle is determined by one side and two angles; that is, by its sides except two, and all its angles. So likewise, any rectilinear figure is determinable, when all its sides except two, and all its angles, are known.

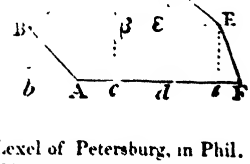
3. A triangle is determinable by its three sides; that is, when all its sides are known, and all its angles, but three. In like manner, any rectilinear figure is determinable by means of all its sides, and all its angles except three.

In each of these cases, the three unknown quantities may be determined by means of three independent equations; the manner of deducing which may be easily explained, after the following theorems are duly understood.

THEOREM I.

In any polygon, any one side is equal to the sum of all the rectangles of each of the other sides drawn into the cosine of the angle made by that side and the proposed side *.

Let ABCDEF be a polygon: then will $AF = AB \cdot \cos. A + BC \cdot \cos. CB \wedge FA + CD \cdot \cos. CD \wedge AF + DE \cdot \cos. DE \wedge AF + EF \cdot \cos. EF \wedge AF$ †



For, drawing lines from the several angles, respectively parallel and perpendicular to AF; it will be

* This theorem and the following one were announced by Mr. Lexel of Petersburg, in Phil. Trans. vol. 65, p. 282; but they were first demonstrated by Dr. Hutton, in Phil. Trans. vol. 66, p. 600.

† When a caret or other angular symbol is put between two letters or pairs of letters denoting lines, the expression altogether denotes the angle which would be made by those two lines if they were produced till they met: thus CB ∧ FA denotes the inclination of the line CB to FA.

$$Ab = AB \,.\, \cos .\, BAF,$$
$$bc = B\beta = BC \,.\, \cos\, CB\beta = BC \,.\, \cos .\, CB \wedge AF,$$
$$cd = \delta D = CD \,.\, \cos .\, CD\delta = CD \,.\, \cos\, CD \wedge AF,$$
$$de = \epsilon E = DE \,.\, \cos .\, DE\epsilon = DE \,.\, \cos\, DE \wedge AF,$$
$$eF = \,.\;\;.\;\; EF \,.\, \cos\, EFe = EF \,.\, \cos\, EF \wedge AF.$$

But $AF = bc + cd + de + eF - Ab$, and Ab, as expressed above, is in effect subtractive, because the cosine of the obtuse angle BAF is negative. Consequently,

$AF = Ac + cd + de + eF = AB \,.\, \cos\, BAF + BC \,.\, \cos\, CB \wedge AF + \&c.$ as in the proposition. The same demonstration will apply, *mutatis mutandis*, to any other polygon.

Cor. When the sides of the polygon are reduced to three, this theorem becomes the same as the fundamental theorem (1), page 18, as we ought to expect should be the case.

The perpendicular let fall from the highest point or summit of a polygon, upon the opposite side or base, is equal to the sum of the products of the sides comprised between that summit and the base, into the sines of their respective inclinations to that base.

Thus, in the preceding figure, $Cc = CB \,.\, \sin\, CBAF + BA \,.\, \sin\, A$, or $Cc = CD \,.\, \sin\, CDAF + DE \,.\, \sin\, DEAF + EF \,.\, \sin\, F$. This is evident from an inspection of the figure.

Cor. 1. In like manner $Dd = DE \,.\, \sin\, DEAF + EF \,.\, \sin\, F$, or $Dd = CB \,.\, \sin\, CBFA + BA \,.\, \sin\, A - CD \,.\, \sin\, CDAF$.

Cor. 2. Hence, the sum of the products of each side, into the sine of the sum of the *exterior* angles (or into the sine of the sum of the supplements of the interior angles), comprised between those sides and a determinate side, is = + perp − perp or = 0. That is to say, in the preceding figure,

$$AB \,.\, \sin\, A + BC \,.\, \sin\, (A + B) + CD \,.\, \sin\, (A + B + C) + DE \,.\, \sin\, (A + B + C + D) + EF \,.\, \sin\, (A + B + C + D + E) = 0.$$

Here it is to be observed, that the sines of angles greater than 180° are negative (chap. ii. equa. vii.).

Cor. 3. Hence again, by putting for $\sin\, (A + B)$, $\sin\, (A + B + C)$, their values $\sin A \cos B + \sin B \cos A$, $\sin A \cos (B + C) + \sin (B + C) \cos A$, &c. (page 18, equa. v.), and recollecting that $\text{tang} = \frac{\sin}{\cos}$, we shall have,

$\sin A\, (AB + BC \cos B + CD \cos (B + C) + DE \cos (B + C + D) + \&c.) + \cos A\, (BC \sin B + CD \sin (B + C) + DE \cos (B + C + D) + \&c.) = 0$, and thence finally, $\tan\, (180° - A)$, or $\tan\, BAF =$

$$\frac{BC \sin B + CD \sin (B+C) + DE \sin (B+C+D) + EF \sin (B+C+D+E)}{AB + BC \cos B + CD \cos (B+C) + DE \cos (B+C+D) + EF \cos (B+C+D+E)}.$$

A similar expression will manifestly apply to any polygon, and when the number of sides exceeds four, it is highly useful in practice.

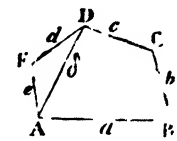
Cor. 4. In a triangle ABC, where the sides AB, BC, and the angle ABC, or its supplement B, are known, we have

$$\tan\, CAB = \frac{BC \,.\, \sin\, B}{AB + BC \,.\, \cos\, B}, \qquad \tan\, BCA = \frac{AB \sin\, B}{BC + AB \cos\, B},$$

in both which expressions, the second term of the denominator will become subtractive whenever the angle ABC is acute, or B obtuse.

THEOREM III.

The square of any side of a polygon is equal to the sum of the squares of all the other sides, minus twice the sum of the products of all the other sides multiplied two and two, and by the cosines of the angles they include.



For the sake of brevity, let the sides be represented by the small letters which stand against them in the annexed figure. then, from theor. 1, we shall have the subjoined equations, viz

$$a = b \cos a'b + c \cos ac + \delta \cos a'\delta,$$
$$b = a \cos a'b + c \cos. b'c + \delta \cos b\delta,$$
$$c = a \cos a'c + b \cos b'c + \delta \cos c'\delta,$$
$$\delta = a \cos. a\delta + b \cos b\delta + c \cos. c\delta.$$

Multiplying the first of these equations by a, the second by b, the third by c, the fourth by δ; subtracting the three latter products from the first, and transposing b^2, c^2, δ^2, there will result

$$a^2 = b^2 + c^2 + \delta^2 - 2\,(bc \cos bc + b\delta \cos b\delta + c\delta \cos c\delta)$$

In like manner,

$$c^2 = a^2 + b^2 + \delta^2 - 2\,(ab \cos ab + a\delta \cos a\delta + b\delta \cos. b\delta)$$

&c. &c.

Or, since $bc = C$, $b\delta = C + D - 180°$, $c\delta = D$, we have

$$a^2 = b^2 + c^2 + \delta^2 - 2\,(bc \cos C - b\delta \cos (C + D) + c\delta \cos D),$$
$$c^2 = a^2 + b^2 + \delta^2 - 2\,(ab \cos B - b\delta \cos (A + B) + a\delta \cos A)$$

&c. &c.

The same method applied to the pentagon ABCDE, will give

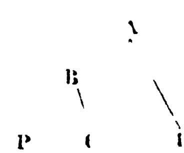
$$a^2 = b^2 + c^2 + d^2 + e^2 - 2 \left\{ \begin{array}{l} bc \cos C - bd \cos (C + D) + be \cos (C + D + E) \\ \quad + cd \cos D - ce \cos. (D + E) + de \cos E \end{array} \right\}$$

And a like process is obviously applicable to any number of sides, whence the truth of the theorem is manifest.

Cor. The property of a plane triangle expressed in equa. 1, page 15, is only a particular case of this general theorem.

THEOREM IV.

Twice the surface of any polygon, is equal to the sum of the rectangles of its sides, except one, taken two and two, by the sines of the sums of the *Exterior* * angles contained by those sides.



1. For a trapezium, or polygon of four sides. Let two of the sides AB, DC, be produced till they meet at P. Then the trapezium ABCD is manifestly equal to the difference between the triangles PAD and PBC. But twice the surface of the triangle PAD is (Mens. of Planes, pr. 2 rule 2) AP . PD . sin P = (AB + BP) . (DC + CP) . sin P, and twice the surface of the triangle PBC is = BP . PC . sin. P: therefore their difference, or twice the area of the trapezium, is = (AB . DC + AB . CP + DC . BP) sin P. Now in Δ PBC,

* The *exterior* angles here meant, are those formed by producing the sides in the same manner as in theor. 20 Geometry, and in cors. 1, 2 theor. 2 of this chap.

$$\sin P : \sin B :: BC : PC, \text{ whence } PC = \frac{BC \cdot \sin. B}{\sin P}.$$

$$\sin P : \sin C :: BC : PB, \text{ whence } PB = \frac{BC \cdot \sin C}{\sin P}.$$

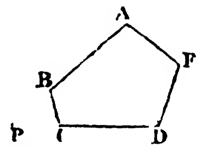
Substituting these values of PB, PC, for them in the above equation, and observing that $\sin P = \sin (PBC + PCB) = \sin$ sum of *exterior* angles B and C, there results at length,

$$\text{Twice surface of trapezium} = \left\{ \begin{array}{l} AB \cdot BC \sin B, \\ + AB \cdot DC \sin (B + C), \\ + BC \cdot DC \sin C. \end{array} \right.$$

Cor. Since $AB \cdot BC \cdot \sin B =$ twice triangle ABC, it follows that twice triangle ACD is equal to the remaining two terms, viz

$$\text{twice area } ACD = \left\{ \begin{array}{l} AB \cdot DC \sin (B + C), \\ + BC \cdot DC \sin C. \end{array} \right.$$

2. For a pentagon, as ABCDE. Its area is obviously equal to the sum of the areas of the trapezium ABCD, and of the triangle ADE. Let the sides AB, DC, as before, meet when produced at P. Then, from the above, we have



$$\text{Twice area of the trapezium } ABCD = \left\{ \begin{array}{l} AB \cdot BC \sin B, \\ + AB \cdot DC \sin (B + C), \\ + BC \cdot DC \sin C. \end{array} \right.$$

And, by the preceding corollary,

$$\text{Twice triangle } DAE = \left\{ \begin{array}{l} AP \cdot DE \sin (P + D) \text{ or } \sin (B + C + D) \\ + DP \cdot DE \sin D. \end{array} \right.$$

$$\text{That is, twice triangle } DAE = \left\{ \begin{array}{l} AB \cdot DE \sin (B + C + D) \\ + DC \cdot DE \sin D \\ + BP \cdot DE \sin (B + C + D) \\ + CP \cdot DE \sin D. \end{array} \right.$$

Now, $BP = \dfrac{BC \sin C}{\sin (B + C)}$, and $CP = \dfrac{BC \sin B}{\sin (B + C)}$, therefore the last two terms become $\dfrac{BC \cdot DE \sin C \sin (B + C + D)}{\sin (B + C)} + \dfrac{BC \cdot DE \sin B \sin D}{\sin (B + C)} =$

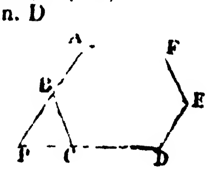
$BC \cdot DE \dfrac{\sin B \sin D + \sin C \sin (B + C + D)}{\sin (B + C)}$ and this expression, by means of the formula for 4 arcs (art 30, page 27), becomes $BC \cdot DE \sin (C + D)$. Hence, collecting the terms, and arranging them in the order of the sides, they become

$$\begin{array}{l} \text{Twice the area of the pen-} \\ \text{tagon } ABCDE \end{array} = \left\{ \begin{array}{l} AB \cdot BC \sin B \\ + AB \cdot DC \sin (B + C) \\ + AB \cdot DE \sin (B + C + D) \\ + BC \cdot DC \sin C \\ + BC \cdot DE \sin (C + D) \\ + DC \cdot DE \sin D \end{array} \right.$$

Cor. Taking away from this expression the 1st, 2d, and 4th terms, which together make double the trapezium ABCD, there will remain

$$\text{Twice area of the triangle } DAE = \left\{ \begin{array}{l} AB \cdot DE \sin (B + C + D) \\ + BC \cdot DE \sin (C + D) \\ + DC \cdot DE \sin. D \end{array} \right.$$

3. For a hexagon, as ABCDEF. The double area will be found, by supposing it divided into the pentagon ABCDE, and the triangle AEF. For, by the last rule, and its corollary, we have



$$\left.\begin{array}{l}\text{Twice area of the penta-}\\ \text{gon ABCDE}\end{array}\right\} = \left\{\begin{array}{l} AB \cdot BC \sin. B \\ + AB \cdot CD \sin. (B + C) \\ + AB \cdot DE \sin. (B + C + D) \\ + BC \cdot CD \sin. C \\ + BC \cdot DE \sin. (C + D) \\ + CD \cdot DE \sin. D. \end{array}\right.$$

$$\left.\begin{array}{l}\text{Twice area of the tri-}\\ \text{angle ABF}\end{array}\right\} = \left\{\begin{array}{l} AP \cdot EF \sin (B + C + D + E) \\ + DP \cdot EF \sin. (D + E) \\ + DE \cdot EF \sin. E. \end{array}\right.$$

$$\left.\begin{array}{l}\text{Or, twice area of the}\\ \text{triangle AEF}\end{array}\right\} = \left\{\begin{array}{l} AB \cdot EF \sin. (B + C + D + E) \\ + DC \cdot EF \sin. (D + E) \\ + DE \cdot EF \sin. E \\ + BP \cdot EF \sin. (B + C + D + E) \\ + CP \cdot EF \sin. (D + E). \end{array}\right.$$

Now, writing for BP, CP, their respective values, $\frac{BC \sin. C}{\sin. (B + C)}$ and $\frac{BC \sin. B}{\sin (B + C)}$, the sum of the last two expressions, in the double areas of AEF, will become

$$BC \cdot EF \cdot \frac{\sin. C \sin. (B+C+D+E) + \sin. B \sin. (D+E)}{\sin. (B + C)}:$$

and this, by means of the formula for five arcs (art. 30. page 28), becomes $BC \cdot EF \cdot \sin. (C + D + E)$. Hence, collecting and properly arranging the several terms as before, we shall obtain

$$\left.\begin{array}{l}\text{Twice the area of the hex-}\\ \text{agon ABCDEF}\end{array}\right\} = \left\{\begin{array}{l} AB \cdot BC \sin. B \\ + AB \cdot CD \sin. (B + C) \\ + AB \cdot DE \sin. (B + C + D) \\ + AB \cdot EF \sin (B + C + D + E) \\ + BC \cdot CD \sin. C \\ + BC \cdot DE \sin. (C + D) \\ + BC \cdot EF \sin. (C + D + E) \\ + CD \cdot DE \sin. D \\ + CD \cdot EF \sin. (D + E) \\ + DE \cdot EF \sin. E. \end{array}\right.$$

4. In a similar manner may the area of a heptagon be determined, by finding the sum of the areas of the hexagon and the adjacent triangle; and thence the area of the octagon, nonagon, or of any other polygon, may be inferred; the law of continuation being sufficiently obvious from what is done above, and the number of terms $= \frac{n-1}{1} \cdot \frac{n-2}{2}$, when the number of sides of the polygon is n; for the number of terms is evidently the same as the number of ways in which $n - 1$ quantities can be taken, two and two; that is, (by the nature of Permutations) $= \frac{n-1}{1} \cdot \frac{n-2}{2}$.

Scholium.

This curious theorem was first investigated by *Simon Lhuillier*, and published in 1789. Its principal advantage over the common method for finding the areas of irregular polygons is, that in this method there is no occasion to construct the figures, and of course the errors that may arise from such constructions are avoided.

In the application of the theorem to practical purposes, the expressions above become simplified by dividing any proposed polygon into two parts by a diagonal, and computing the surface of each part separately.

Thus, by dividing the trapezium ABCD into two triangles, by the diagonal AC, we shall have

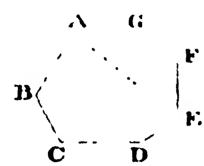
$$\text{twice area trapezium} \left.\right\} = \left\{ \begin{array}{l} AB \,.\, BC \sin. B \\ +CD \,.\, AD \sin. D \end{array} \right.$$

The pentagon ABCDE may be divided into the trapezium ABCD, and the triangle ADE, whence

$$\text{twice area of pentagon} \left.\right\} = \left\{ \begin{array}{l} AB \,.\, BC \sin. B \\ +AB \,.\, DC \sin. (B+C) \\ +BC \,.\, DC \sin. C \\ +DE \,.\, AE \sin. E. \end{array} \right.$$

Thus again, the hexagon may be divided into two trapeziums, by a diagonal drawn from A to D, which is to be the line excepted in the theorem, then will

$$\text{Twice area of hexagon} \left.\right\} = \left\{ \begin{array}{l} AB \,.\, BC \sin B \\ +AB \,.\, DC \sin (B+C) \\ +BC \,.\, DC \sin C \\ +DE \,.\, EF \sin E \\ +DE \,.\, AF \sin. (E+F) \\ +EF \,.\, AF \sin F \end{array} \right.$$



And lastly, the heptagon may be divided into a pentagon and a trapezium, the diagonal, as before, being the excepted line; so will the double area be expressed by 9 instead of 15 products, thus

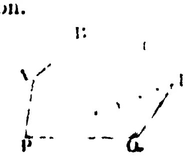
$$\text{Twice area of heptagon} \left.\right\} = \left\{ \begin{array}{l} AB \,.\, BC \sin B \\ +AB \,.\, CD \sin (B+C) \\ +AB \,.\, DE \sin. (B+C+D) \\ +BC \,.\, CD \sin C \\ +BC \,.\, DE \sin. (C+D) \\ +CD \,.\, DE \sin. D \\ +EF \,.\, FG \sin F \\ +EF \,.\, GA \sin (F+G) \\ +FG \,.\, GA \sin G \end{array} \right.$$

The same method may obviously be extended to other polygons, with great ease and simplicity.

It often happens, however, that only one side of a polygon can be measured, and the distant angles be determined by intersection; in this case the area may be found, independent of construction, by the following problems.

PROBLEM I.

Given the length of one of the sides of a polygon, and the angles made at its two extremities by that side and lines drawn to all the other angles of the polygon; to find an expression for the surface of that polygon.



Here we suppose known PQ; also
$APQ = a'$, $BPQ = b'$, $CPQ = c'$, $DPQ = d'$,
$AQP = a''$, $BQP = b''$, $CQP = c''$, $DQP = d''$.
Now, $\sin. PAQ = \sin. (a' + a'')$, $\sin PBQ = \sin (b' + b'')$.

Therefore, $\sin. (a' + a'') : PQ :: \sin. a'' : PA = \dfrac{\sin. a''}{\sin. (a' + a'')} PQ.$

And, . . . $\sin (b' + b'') : PQ :: \sin. b'' : PB = \dfrac{\sin b''}{\sin. (b' + b'')} PQ$

But, triangle $APB = AP \,.\, PB \,\tfrac{1}{2}\sin. APB = \tfrac{1}{2}AP \,.\, PB \sin. (a' - b')$.

Hence, surface $\triangle APB = \tfrac{1}{2}PQ^2 \dfrac{\sin a'' \sin. b'' \sin. (a' - b')}{\sin (a' + a'') \sin (b' + b'')}$

In like manner, $\triangle BPC = \tfrac{1}{2}PQ^2 \dfrac{\sin b'' \sin. c'' \sin (b' - c')}{\sin (b' + b'') \sin (c' + c'')}$

$\triangle CPD = \tfrac{1}{2}PQ^2 \,.\, \dfrac{\sin c'' \sin d'' \sin (c' - d')}{\sin (c' + c'') \sin (d' + d'')}$. &c. &c. &c.

$$\triangle DPQ = QP \cdot PD\,\tfrac{1}{2}\sin. DPQ = PQ \cdot \frac{\sin. d''}{\sin. (d' + d'')} \cdot \tfrac{1}{2}PQ \sin. d' = \tfrac{1}{2}PQ^2 \cdot \frac{\sin. d' \sin. d''}{\sin. (d' + d'')}. \text{ Consequently,}$$

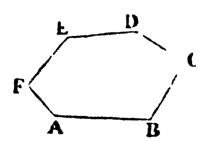
$$\text{Surface } PABCDQ = \tfrac{1}{2}PQ^2 \left\{ \begin{array}{l} \dfrac{\sin. a'' \sin. b'' \sin. (a' - b')}{\sin. (a' + a'') \sin. (b' + b'')} \\ + \dfrac{\sin. b'' \sin. c'' \sin. (b' - c')}{\sin. (b' + b'') \sin. (c' + c'')} \\ + \dfrac{\sin. c'' \sin. d'' \sin. (c' - d')}{\sin. (c' + c'') \sin. (d' + d'')} \\ + \dfrac{\sin. d' \sin. d''}{\sin. (d' + d'')} \end{array} \right.$$

The same method manifestly applies to polygons of *any* number of sides: and all the terms except the last are perfectly symmetrical, while that last term is of so obvious a form, that there cannot be the least difficulty in extending the formula to any polygon whatever.

PROBLEM II.

Given in a polygon, all the sides and angles, except three; to find the unknown parts.

This problem may be divided into three general cases, as shown at the beginning of this chapter: but the analytical solution of all of them depends on the same principles, and these are analogous to those pursued in the analytical investigations of plane trigonometry. In polygonometry, as well as trigonometry, when three unknown quantities are to be found, it must be by means of three independent equations, involving the known and unknown parts. These equations may be deduced from either theorem 1, or 3, as may be most suited to the case in hand; and then the unknown parts may each be found by the usual rules of extermination.



For an example, let it be supposed that in an irregular hexagon ABCDEF, there are given all the sides except AB, BC, and all the angles except B; to determine those three quantities.

The angle B is evidently equal to $(2n - 4)$ right angles $-(A + C + D + E + F)$; n being the number of sides, and the angles being here supposed the interior ones.

Let $AB = x$, $BC = y$, then, by theor. 1,

$$x = y \cos. B + DC \cos. AB^{\wedge}CD + DE \cos. AB^{\wedge}ED$$
$$+ EF \cos. AB^{\wedge}EF + AF \cos. AB^{\wedge}AF;$$
$$y = x \cos. A + AF \cos. BC^{\wedge}AF + FE \cos. BC^{\wedge}FE$$
$$+ DE \cos. BC^{\wedge}DE + DC \cos. BC^{\wedge}CD.$$

In the first of the above equations, let the sum of all the terms after $y \cos. B$, be denoted by c; and in the second the sum of all those which fall after $x \cos. B$, by d; both sums being manifestly constituted of known terms: and let the known co-efficients of x and y be m and n respectively. Then will the preceding equations become

$$x = ny + c \quad . \quad . \quad y = mx + d.$$

Substituting for y, in the first of the two latter equations, its value in the second, we obtain $x = mnx + nd + c$. Whence there will readily be found

$$x = \frac{nd + c}{1 - mn}, \text{ and } y = \frac{mc + d}{1 - mn}.$$

Thus AB and BC are determined. Like expressions will serve for the determination of any other two sides, whether contiguous or not: the co-efficients of x and y being designated by different letters for that express purpose; which would have been otherwise unnecessary in the solution of the individual case proposed.

Remarks. Though the algebraic investigations commonly lead to results which are apparently simple, yet they are often, especially in polygons of many sides, inferior in practice to the methods suggested by subdividing the figures. The following examples are added for the purpose of explaining those methods: the operations, however, are merely indicated; the detail being omitted to save room.

EXAMPLES.

Ex. 1. In a hexagon ABCDEF, all the sides except AF, and all the angles except A and F, are known. Required the unknown parts. Suppose we have

	Ext ang.	Whence	
AB = 1284			
BC = 1782	B = 32°	B + C	= 80°
CD = 2400	C = 48°	B + C + D	= 132°
DE = 2700	D = 52°	B + C + D + E	= 198°
EF = 2860	E = 66°	A + F	= 162°.

Then, by cor. 3 th. 2, tan. BAF =

$$\frac{\mathrm{BC\,sin.\,B+CD\,sin.\,(B+C)+DE\,sin.\,(B+C+D)+EF\,sin.\,(B+C+D+E)}}{\mathrm{AB+BC\,cos.\,B+CD\,cos.\,(B+C)+DE\,cos.\,(B+C+D)+EF\,cos.\,(B+C+D+E)}}$$

$$= \frac{\mathrm{BC\,sin.\,32^\circ + CD\,sin.\,80^\circ + DE\,sin.\,132^\circ + EF\,sin.\,198^\circ}}{\mathrm{AB + BC\,cos.\,32^\circ + CD\,cos.\,80^\circ + DE\,cos.\,132^\circ + EF\,cos.\,198^\circ}}$$

$$= \frac{\mathrm{BC\,sin.\,32^\circ + CD\,sin.\,80^\circ + DE\,sin.\,48^\circ - EF\,sin.\,18^\circ}}{\mathrm{AB + BC\,cos.\,32^\circ + CD\,cos.\,80^\circ - DE\,cos.\,48^\circ - EF\,cos.\,18^\circ}}.$$

Whence BAF is found 106°31′38″; and the other angle AFE = 91°28′22″. So that the exterior angles A and F are 73°28′22″, and 88°31′38″ respectively: all the exterior angles making four right angles, as they ought to do. Then, all the angles being known, the side AF is found by theor. 1. = 4621·5.

If one of the angles had been a re-entering one, it would have made no other difference in the computation than what would arise from its being considered as subtractive.

Ex. 2. In a hexagon ABCDEF, all the sides except AF, and all the angles except C and D, are known: viz.

	Ex. Ang.
AB = 2400	
BC = 2700	A = 54°
CD = 3200	B = 62°
DE = 3500	E = 64°
EF = 3750	F = 72°

We shall have, by theor. 2 cor. 1,

$$\left.\begin{array}{l}\mathrm{AB\,sin.\,A}\\ +\,\mathrm{BC\,sin\ (A+B)}\\ +\,\mathrm{CD\,sin.\,(A+B+C)}\end{array}\right\} = \left\{\begin{array}{l}\mathrm{DE\,sin.\,(E+F)}\\ \quad +\,\mathrm{EF\,sin.\,F.}\end{array}\right.$$

$$\text{Therefore, CD sin. }(116^\circ + \mathrm{C}) = \left\{\begin{array}{l}-\,\mathrm{AB\,sin.}\ \ 54^\circ\\ -\,\mathrm{BC\,sin.}\ 116^\circ\\ +\,\mathrm{DE\,sin.}\ 136^\circ\\ +\,\mathrm{EF\,sin.}\ \ 72^\circ\end{array}\right.$$

$$\text{Or, } 116^\circ + \mathrm{C} = \left\{\begin{array}{l}\quad 149^\circ 23'26'',\\ +\ \ 33^\circ 36'34''.\end{array}\right.$$

The second of these will give for C, a re-entering angle; the first will give exterior angle C = 33°23′26″, and then will D = 14°36′34″. Lastly,

$$\mathrm{AF} = \left\{\begin{array}{l}-\,\mathrm{AB\,cos.}\ 54^\circ\\ +\,\mathrm{BC\,cos.}\ 64^\circ\\ +\,\mathrm{CD\,cos.}\ 30^\circ 36'34''\\ +\,\mathrm{DE\,cos.}\ 44^\circ\\ -\,\mathrm{EF\,cos.}\ 72^\circ\end{array}\right\} = 3885{\cdot}905.$$

Ex. 3. In a hexagon ABCDEF, are known all the sides except AF, and all the angles except B and E; to find the rest.

Given AB	$= 1200$	Exterior angles A	$= 64°$
BC	$= 1500$		
CD	$= 1600$	C	$= 72°$
DE	$= 1800$	D	$= 75°$
EF	$= 2000$		
		F	$= 84°$.

Suppose the diagonal BE drawn, dividing the figure into two trapeziums. Then, in the trapezium BCDE, the sides except BE, and the angles except B and E, will be known; and these may be determined as in example 1. Again, in the trapezium ABEF, there will be known the sides except AF, and the angles except the adjacent ones B and E. Hence, first for BCDE: (cor. 3 theor. 2).

$$\tan. \text{CBE} = \frac{\text{CD} \sin. \text{C} + \text{DE} \sin. (\text{C} + \text{D})}{\text{BC} + \text{CD} \cos. \text{C} + \text{DE} \cos (\text{C} + \text{D})} =$$

$$\frac{\text{CD} \sin. 72° + \text{DE} \sin. 147°}{\text{BC} + \text{CD} \cos. 72° + \text{DE} \cos. 147°} = \frac{\text{CD} \sin. 72° + \text{DE} \sin. 33°}{\text{BC} + \text{CD} \cos. 72° - \text{DE} \cos. 33°}.$$

Whence CBE $= 79°2'1''$; and therefore DEB $= 67°57'59''$.

$$\text{Then EB} = \left\{ \begin{array}{l} \text{BC} \cos. 79°\ 2'\ 1'' \\ + \text{CD} \cos.\ \ 7°\ 2'\ 1'' \\ + \text{DE} \cos. 67°57'59'' \end{array} \right\} = 2548{\cdot}581.$$

Secondly, in the trapezium ABEF,

AB sin. A + BE sin. (A + B) = EF sin. F: whence

$$\sin. (\text{A} + \text{B}) = \frac{\text{EF} \sin. \text{F} - \text{AB} \sin. \text{B}}{\text{BE}} = \sin. \left\{ \begin{array}{l} 20°\ 55'\ 54'' \\ 159°\ \ 4'\ \ 6''. \end{array} \right.$$

Taking the lower of these, to avoid re-entering angles, we have B (exterior ang.) $= 95°4'6''$; ABE $= 84°55'54''$; FEB $= 63°4'6''$: therefore ABC $= 163°57'55''$; and FED $= 131°2'5''$: and consequently the exterior angles at B and E are $16°2'5''$ and $48°57'55''$ respectively.

Lastly, AF $= -$ AB cos. A $-$ BE cos. (A $+$ B) $-$ EF cos. F $= -$ AB cos. $64°$ $+$ BE cos. $20°55'54''$ $-$ EF cos. $84°$ $= 1645{\cdot}292$.

[*Note.*—The preceding three examples comprehend all the varieties which can occur in Polygonometry, when all the sides except one, and all the angles but two, are known. The unknown angles may be about the unknown side; or they may be adjacent to each other, though distant from the unknown side; and they may be remote from each other, as well as from the unknown side.]

Ex. 4. In a hexagon ABCDEF, are known all the angles, and all the sides except AF and CD: to find those sides.

Given AB	$= 2200$	Ext. Ang. A	$= 96°$
BC	$= 2400$	B	$= 54°$
		C	$= 20°$
DE	$= 4800$	D	$= 24°$
EF	$= 5200$	E	$= 18°$
		F	$= 148°$.

Here, reasoning from the principle of cor. 2, theor. 2, we have

$$\left. \begin{array}{l} \text{AB} \sin.\ 96° \\ + \text{BC} \sin.\ 150 \\ + \text{CD} \sin.\ 170 \end{array} \right\} = \left\{ \begin{array}{r} \text{DE} \sin. 160° \text{ or AB} \sin. 84° \\ + \text{EF} \sin. 148° + \text{BC} \sin. 30° \\ + \text{CD} \sin. 10° \end{array} \right\} = \left\{ \begin{array}{l} \text{DE} \,.\, \sin. 14° \\ + \text{EF} \,.\, \sin. 32°. \end{array} \right.$$

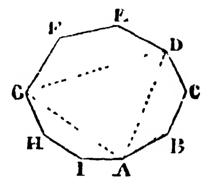
$$\text{Whence CD} = \left\{ \begin{array}{l} \text{DE} \sin. 14° \text{ cosec. } 10° - \text{AB} \sin. 84° \text{ cosec. } 10° \\ + \text{EF} \sin. 32° \text{ cosec. } 10° - \text{BC} \sin. 30° \text{ cosec. } 10° \end{array} \right\} = 3045{\cdot}58.$$

$$\text{And AF} = \left\{ \begin{array}{l} \text{DE} \sin. 24° \text{ cosec. } 10° - \text{CB} \sin. 20° \\ + \text{EF} \sin\ 42° \text{ cosec. } 10° - \text{BA} \sin\ 74° \end{array} \right\} = 14374{\cdot}98$$

Ex. 5. In the nonagon ABCDEFGHI, all the sides are known, and all the angles except A, D, G: it is required to find those angles.

Given AB = 2400	FG 3800	Ext. Ang.	B = 40°
BC = 2700	GH = 4000		C = 32°
CD = 2800	HI = 4200		E = 36°
DE = 3200	IA = 4500.		F = 45°
EF = 3500			H = 48°
			I = 50°

Suppose diagonals drawn to join the unknown angles, and dividing the polygon into three trapeziums and a triangle; as in the marginal figure. Then,



1st. In the trapezium ABCD, where AD and the angles about it are unknown; we have (cor. 3, theor. 2).

$$\tan.\mathrm{BAD} = \frac{\mathrm{BC}\sin.\mathrm{B}+\mathrm{CD}\sin.(\mathrm{B}+\mathrm{C})}{\mathrm{AB}+\mathrm{BC}\cos.\mathrm{B}+\mathrm{CD}\cos.(\mathrm{B}+\mathrm{C})} = \frac{\mathrm{BC}\sin.40^\circ+\mathrm{CD}\sin.72^\circ}{\mathrm{AB}+\mathrm{BC}\cos.40^\circ+\mathrm{CD}\cos.72^\circ}.$$

Whence BAD = 39° 30′ 42″, CDA 32° 29′ 18″.

$$\text{And AD} = \left\{\begin{array}{l} \mathrm{AB}\cos.39^\circ\ 30\ 42 \\ +\mathrm{BC}\cos.\ \ 0\ \ 29\ 18 \\ +\mathrm{CD}\cos.32\ \ 29\ 18 \end{array}\right\} = 6913{\cdot}292$$

2dly. In the quadrilateral DEFG, where DG and the angles about it are unknown; we have

$$\tan.\mathrm{EDG} = \frac{\mathrm{EF}\sin.\mathrm{E}+\mathrm{FG}\sin.(\mathrm{E}+\mathrm{F})}{\mathrm{DE}+\mathrm{EF}\cos.\mathrm{E}+\mathrm{FG}\cos.(\mathrm{E}+\mathrm{F})} = \frac{\mathrm{EF}\sin.36^\circ+\mathrm{FG}\sin 81^\circ}{\mathrm{DE}+\mathrm{EF}\cos.36^\circ+\mathrm{FG}\cos.81^\circ}.$$

Whence EDG = 41° 14′ 53″, FGD = 39° 45′ 7″.

$$\text{And DG} = \left\{\begin{array}{l} \mathrm{DE}\cos.41^\circ\ 14'\ 53'' \\ +\mathrm{EF}\cos.\ \ 5\ \ 14\ \ 53 \\ +\mathrm{FG}\cos.39\ \ 45\ \ \ 7 \end{array}\right\} = 8812{\cdot}803.$$

3dly. In the trapezium GHIA, an exactly similar process gives HGA = 50° 46′ 53″, IAG = 47° 13′ 7″, and AG = 9780·591.

4thly. In the triangle ADG, the three sides are now known, to find the angles: viz. DAG = 60° 53′ 26″, AGD = 43° 15′ 54″, ADG = 75° 50′ 40″. Hence there results, lastly,

IAB = 47° 13′ 7″ + 60° 53′ 26″ + 39° 30′ 42″ = 147° 37′ 15″,
CDE = 32 29 18 + 70 50 40 + 41 14 53 = 149 34 51,
FGH = 39 45 7 + 43 15 54 + 50 46 53 = 133 47 54.

Consequently, the required exterior angles are A = 32° 22′ 45″, D = 30° 25′ 9″, G = 46° 12′ 6″.

Ex. 6. Required the area of the hexagon in ex. 1. Ans. 16530191.

Ex. 7. In a quadrilateral ABCD, are given AB = 24, BC = 30, CD = 34; angle ABC = 92° 18′, BCD = 97° 23′. Required the side AD, and the area.

Ex. 8. In prob. 1, suppose PQ = 2538 links, and the angles as below; what is the area of the field ABCDQP?

APQ = 89° 14′, BPQ = 68° 11′, CPQ = 36° 24′, DPQ = 19° 57′;
AQP = 25 18′, BQP = 69 24′, CQP = 94 6′, DQP = 121 18′:

Ex. 9. It is required to inscribe a polygon in a given polygon, so that its sides shall be parallel to the same number of given lines.

Ex. 10. It is required to inscribe a polygon in a given circle to fulfil the same conditions.

Ex. 11. It is required to describe a polygon through any number of given points, so that its angles shall be situated in the same number of given lines.

Ex. 12. It is required to inscribe a polygon in a given circle, so that its sides shall pass through the same number of given points.

THE CONIC SECTIONS.

DEFINITIONS.

1. The Conic Sections are the figures made by a plane cutting a cone, either right or oblique.

[Whether the cone be right or oblique, the sections made in it by a plane have precisely the same properties, but as the investigation is more simple when they are considered in the right cone, it is usual to employ this instead of the oblique cone.]

2. According to the different positions of the cutting plane there arise five different species of figures or sections; namely, *two right lines intersecting in the vertex of the cone*, the *circle*, the *ellipsis*, the *hyperbola*, and the *parabola*. The three last are peculiarly called *conic sections*, the properties of the right line and the circle having been previously laid down in Plane Geometry.

3. If the cutting plane pass through the vertex of the cone, and also cut the circular base, its section with the conical surface will be *two straight lines*, as VA, VB. This the student will readily prove.

[Since the line which generates the conical superficies (Geom. def. 105) may be of indefinite length, the cone itself is of indefinite length, the figure delineated being only part of it. Hence the lines in which the plane cuts the surface are of indefinite length.

Also, as by producing the line indefinitely on the other side of the vertex, an equal and opposite cone will be produced, the same plane section, or the lines VA, VB are also indefinitely continued beyond the vertex.]

4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a *circle*, as ABD. See theor. 117. Geom.

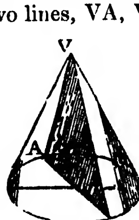
5. The section will be an *ellipse* when the cone is cut obliquely through both sides, or when the plane of section is inclined in a less angle to the plane of the base than the tangent plane to the cone is.

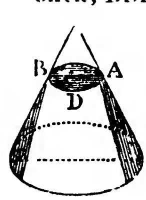
[The student is required to prove that the section is comprised within finite limits, and that the curve is one continuous line.]

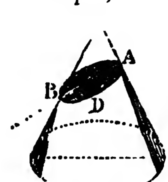
6. The section is a *parabola* when the cutting plane is parallel to a tangent plane of the cone, or when it makes the same angle with the plane of the base that the tangent plane parallel to it does.

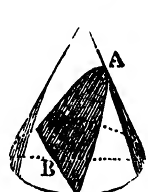
[It is to be proved that this curve is of *infinite extent* on the side BE estimated *from* A, the figure delineated being only a part of it.]

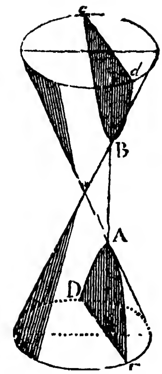
Two lines, VA, VB. Circle, BDA. Ellipse, BDA. Parabola, ABE.











7. The section is an *hyperbola*, when the cutting plane makes a greater angle with the base than the tangent plane to the cone makes with it.

8. And if all the sides of the cone be continued through the vertex, forming the *opposite cone*, and the plane be also continued to cut the opposite cone, this latter section will be the *opposite hyperbola* to the former; as dBe. The two opposite hyperbolas are also called, and more properly, the two *opposite branches of the hyperbola*, considered as one curve.

[It will appear hereafter, cor 4, pr. II. Hyperbola, that whatever be the position of the plane of the hyperbolic section, the two figures are in all respects equal to one another.

It is required to prove that the opposite hyperbolic branches cut from the two opposite cones are of infinite extent from A and B in the directions of DE and de respectively.

All sections made by *planes parallel to the axis are hyperbolas.*]

9. The *transverse plane* is that which is perpendicular both to the plane of the conic section and to the plane of the cone's base.

10. The *transverse axis* of any conic section is the portion of the line of intersection of the transverse plane with the plane of the conic section, and indefinitely produced

11. The *transverse diameter* of the section is the portion of the transverse axis which lies between its two points of intersection with the curve; and is represented in the foregoing figures by AB.

[The axis of a parabola is infinite in length, AB being only a part of it]

12. The points in which the transverse axis cuts the curve, are called the *transverse vertices* of the curve; and often simply the *vertices.*

[The ellipse and hyperbola have two vertices, but the parabola only one]

13. The middle of the transverse diameter is called the *centre of the conic section.*

[The centre of a parabola is infinitely distant from the vertex. The centre of the ellipse lies *within*, or on the *concave side* of the curve: and the centre of the hyperbola *without*, or on the *convex side* of the curve]

14. The *vertical angle of a right* cone is the angle included by the two lines which constitute the section (def. 2) made by a plane passing through the axis of the cone: and the angle contained between the axis and one of these lines is called the *generating angle* of the right cone, it being the constant angle made by the generating line with the fixed line or axis.

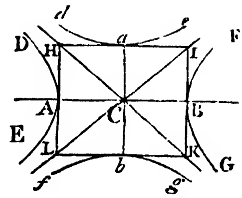
15. Two right cones, whose vertical angles are the supplements of each other, are called *supplementary* or *conjugate cones.*

[If two pairs of such cones be so placed as to have their vertices coincident, their axes will be at right angles to one another, and in the same plane with their lines of contact: and the axes will bisect the angles made by those lines of contact.

The common tangent plane to the conjugate cones at their lines of contact will be perpendicular to the plane in which the axes and lines of contact are situated]

16. Let LCI and HCK be the lines of contact of the conjugate cones, and AB, ab, their axes; take any point I in one of the lines of contact, and draw IH, IK cutting the axes in a and B, and being themselves consequently bisected in those points. Then if the cones whose axial sections are HCI and ICB,

with their opposite sheets, be cut by planes parallel to the plane of the axes, and at distances from it equal to I*a* and IB respectively, two pairs of opposite hyperbolic branches will be produced in the cones by these planes. If now these planes, with their sections, be moved down till they coincide with the plane of the axes, and their intersections with the common tangent planes coincide with HR and LI respectively, they will form the annexed figure, in which the pairs of branches are called the *conjugate hyperbolas*. That is, the branches FBG, DAE are *conjugate* to the branches *fbg*, *dae*; and mutually *fbg*, *dae* are called *conjugate* to FBG, DAE.

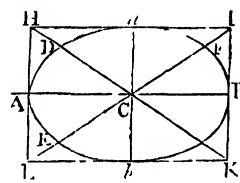


[These four branches of the curve, though not cut from the cone by the *same* plane, are yet to be viewed as forming part of the same system. Indeed, though for convenience of verbal description they have been cut by means of two planes, it is easy to see that by a different method of placing the axes of the conjugate cones the conjugate sections might have been made by a single plane—namely, by depressing the axis of the obtuse angled cones below the present plane of the axis (the cone still having the same tangent planes perpendicular to the plane of the paper) by a distance equal to the excess of IA above IB.]

17. The lines AB, *ab*, will be the *conjugate rectangular diameters*, (see prop. 2, Hyperbola): and the rectangle HIKL is called the *conjugate rectangle*, or the *rectangle of the axes*. These are sometimes called the *principal diameters*.

18. The lines HK and LI are called the *asymptotes of the hyperbola*.

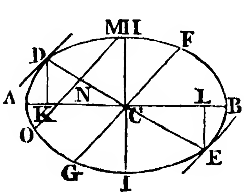
19. The perpendicular (to the transverse axis of the ellipse), from the centre C, and terminated both ways by the curve, the *conjugate rectangular diameter* (Ellipse, pr. 2.) of the curve, and the rectangle formed of AB and *ab*, viz. HIKL is the *conjugate rectangle*, or *rectangles of the axes*.



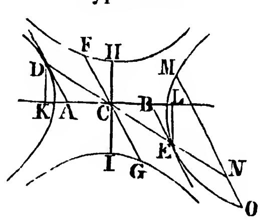
20. A *tangent to a conic section* is a straight line which meets the curve, but being produced both ways does not cut it.

21. A *diameter of a conic section* is a line drawn through the centre and terminated both ways by the curve, and the extremities of such a diameter are called *its vertices*.

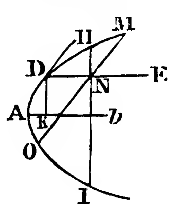
Ellipse.



Hyperbolas.



Parabola.



22. If a tangent and a diameter of a conic section be drawn through the same point in the curve, they are said to be *conjugate to each other*: and *any lines parallel to these* are said also to be conjugate to each other. If they be lines terminated by the curve, and meet at a point *within* it, they are called *conjugate chords;* if they meet the curve and intersect at a point *without* it, they are *conjugate secants;* if they both touch it, they are *conjugate tangents;* if they both pass through the centre, they are *conjugate diameters;* and if one be a

diameter, and the other a chord, the latter is called an *ordinal chord to that diameter;* and the half-chord is called an *ordinate* to that diameter*. Thus FG, DE are congregate diameters.

23. The portions of the diameter estimated from the vertices to the intersection of an ordinate chord are called the *abscisses* of the points where the chord intersects the curve.

[Hence the ellipse and hyperbola have two determinate abscisses for every ordinate, but the parabola only one, the other vertex of the diameter being infinitely distant.]

24. The absciss and ordinate when spoken of together without any peculiar specification of either, are called *co-ordinates.*

25. The *parameter of any diameter* is a third proportional to that diameter and its conjugate in the ellipse and hyperbola, and to any absciss and its ordinate in the parabola †.

26. The *focus* is the point in the transverse axis at which the ordinate is equal to the parameter of the transverse ‡.

27. The *subtangent* is the portion of the diameter of a conic section intercepted between the tangent and the ordinate.

28. The *normal* is a line drawn from a point in the curve perpendicular to the tangent at that point, and limited by its intersection with the transverse diameter.

29. The *subnormal* is the portion of the diameter intercepted between the ordinate and the normal.

30. If tangents be drawn at the extremities of any two diameters conjugate to each other, they are shown in the subsequent part of the work to form a parallelogram; and this is called the *conjugate parallelogram.* When it is right-angled it is called the *conjugate rectangle* Also, when a parallelogram is formed by joining the points of contact of the conjugate parallelogram, this is also conjugate, and the two are distinguished by the terms *circumscribed* and *inscribed* conjugate parallelograms.

31. The *focal distance,* or *eccentricity,* is the distance of the focus from the centre.

32. If in the ellipse and hyperbola we take a fourth proportional to the semi-transverse, the eccentricity, the distance of the ordinate from the centre, and place it in the transverse axis also reckoned from the centre, its extremity is called the *dividing point.*

33. If to the point of the curve of a conic section where the right ordinate passes through the focus, a tangent be drawn to the curve and produced to cut the transverse axis, and from this point of intersection a perpendicular of inde-

* It will be shown, in the course of the work, that the point of bisection of the *ordinal chord* is its point of intersection with the diameter.

† This is also called the *latus rectum* by nearly all the older writers, and by many modern ones. The term *parameter* is used also to signify any constant quantity or datum upon which, wholly or conjointly with other data, the form and magnitude of a previously defined figure depends.

Curves are said to be *similar* where their definitions are the same, and the *same ratios* subsist between the several parameters of the one and the corresponding parameters of the other.

Thus circles depend on one datum, the radius, and hence they are similar figures It will be shown hereafter that all parabolas are similar figures · but that all ellipses and all hyperbolas are similar when two data are given for their construction.

‡ This point is called *punctum comparationis* in the older writers.

finite length be drawn to the transverse, this perpendicular is called the *directrix* of the section.

[As there are two foci in the ellipse there will be two directrices: and for a similar reason each conjugate pair of hyperbolas involving two foci, the complete system of conjugate hyperbolas will have four directrices: whilst the parabola having but one focus will have but one directrix.]

34. If straight lines be drawn from any point in the curve to the adjacent focus, and also parallel to the transverse axis, to meet the directrix, they shall for all points in the same curve have the same ratio. This is shewn farther on, and the ratio itself is called the *determining ratio.*

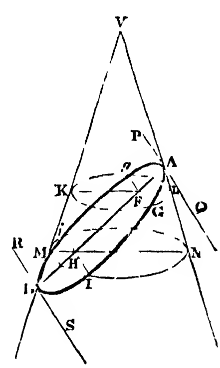
[In the parabola the *determining ratio* is a ratio of equality (paral. prop. 2); in the ellipse a ratio of *lesser inequality* (Ellipse, pro. 8); and in the hyperbola, a ratio of *greater inequality* * (hyp. pr. 8): the antecedent of the ratio in each case being the line drawn to the focus.]

35. The distance of any point in the curve of a conic section from the focus is called the *focal distance:* any chord passing through the focus and terminated by the curve, is called a *focal chord:* if the chord be an ordinate, it is called the *focal ordinate*; and the tangents to the curve at the extremities of such an ordinate, the *focal tangents.*

THE ELLIPSE.—SECTION I.

PROPERTIES OF LINES RELATED TO THE TRANSVERSE AND CONJUGATE DIAMETERS.

PROP. I.—*The squares of the ordinates of the transverse axis are to each other as the rectangles of their abscisses.*



Let AVB be the vertical transverse plane, AGVB the plane of the elliptic section; AB the transverse axis; KGLg, MINi, sections of the cone perpendicular to its axis; and KL, MN, the intersections of these planes with the transverse plane.

Then, because KGLg and MINi are sections perpendicular to the axis of the cone they are circles: and because AVB, the vertical plane, passes through the axis of the cone, the lines KL and MN are diameters of these circles.

Also, since the planes KGL, AGB are each perpendicular to the vertical plane AVB, their line of section gFG is perpendicular to that plane, and therefore to the lines AB, KL, which are in it. Similarly iHI is perpendicular to AB and MN.

Also, KL, MN being diameters of the circles, and at right angles to GFg and IHi, these lines are bisected in F and H respectively.

Again, the plane AGB, and the tangent planes at AV and BV being perpendicular to the vertical transverse plane, the intersections PAQ, RBS are also perpendicular to the plane AVB; and being in the same plane are, hence, parallel to the lines GFg and IHi.

* The two pairs of conjugate hyperbolas have the determining ratios different, but each opposite branch of the same pair has the same ratio. This will be shown in treating of the hyperbola.

But QAP, being in the tangent plane, touches the ellipse at A, and meets it in no other point, for the plane meets it in no other point. It is also in the plane of the ellipse, and is hence a tangent to the ellipse at the point A. Similarly the line SBR is a tangent at B, the other extremity of the transverse axis.

The lines FGg, HIi being, then, in the plane of the ellipse and parallel to the tangents of the extremities of the diameter AB: and being bisected in F and H respectively, are therefore double ordinates to the diameter AB.

Now, by the similar triangles AFL, AHN and BFK, BHM, we have

$$AF : AH :: FL : HN, \text{ and}$$
$$FB : HB :: KF : MH.$$

hence by compounding these ratios we obtain*

$$AF \,.\, FB : AH \,.\, HB :: KF \,.\, FL : MH \,.\, HN.$$

But, by the circle, $KF \,.\, FL = FG^2$, and $MH \,.\, HN = HI^2$; and hence

$$AF \,.\, FB : AH \,.\, HB :: FG^2 : HI^2. \qquad \text{Q. E. D.}$$

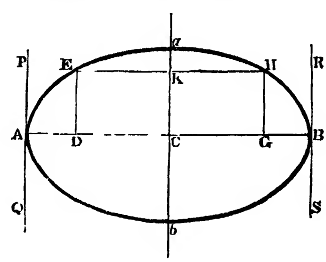
Cor. 1. The transverse axis AB bisects its conjugate chords, Gg and Ii.

Cor. 2. The two tangents at the extremities of the transverse axis are parallel to one another, and perpendicular to that axis.

Cor. 3. The ordinates of the transverse axis are perpendicular to that axis.

Cor. 4. The conjugate axis is at right angles to the transverse.

Prop. II.—*As the square of the transverse axis is to the square of the conjugate,*† *as the rectangle of the abscisses is to the square of their ordinate.*



That is, $AB^2 : ab^2$ (or $AC^2 : aC^2$) :: $AD \,.\, DB : DE^2$.

For (theor. 1) $AC \,.\, CB : AD \,.\, DB :: aC^2 : DE^2$.

But since C is the centre, $AC \,.\, CB = AC^2$, and aC is the semi-conjugate.

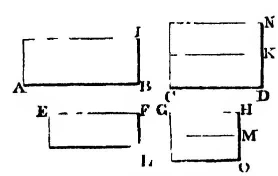
Therefore $AC^2 : AD \,.\, DB :: aC^2 : DE^2$.

Or permu. $AC^2 : aC :: AD \,.\, DB : DE^2$.

Or, doubling, $AB^2 : ab^2 :: AD.DB : DE^2$.

Q. E. D.

* Let four straight lines be proportionals, and any other four be also proportionals; then the rectangle contained by the first and fifth has the same ratio to the rectangle contained by the second and sixth, that the rectangle contained by the third and seventh has to the rectangle contained by the fourth and eighth.



Let AB have to CD the same ratio that EF has to GH, and BI the same ratio to DK that FL has to HM; and let AI be the rectangle under AB and BI, CK the rectangle under CD and DK, EL the rectangle under EF and FL, and GM the rectangle under GH and HM, then AI is to CK as EL to GM.

For in DK, HM (produced if necessary) take DN and HO such that AB is to CD as DN to BI, and EF to GH as HO to FL, and complete the rectangles CN and GO.

Then the rectangle CN is equal to the rectangle AI and the rectangle GO to the rectangle EL. But EF is to GH as AB to CD, and AB to CD as DN to BI, also we have DN to BI as EF to GH, that is as HO to FL. But BI is to DK as FL is to HM; and therefore DN is to DK as HO to HM.

Again, the rectangle CN is to CK as DN to DK, and GO to GM as HO to HM; hence CN is to CK as GO to GM, and, therefore, finally, the rectangle AI is to the rectangle CK as the rectangle AI is to the rectangle EL is to the rectangle GM. Q. E. D.

See, also, Geom. theor. 85.

† By the *transverse and conjugate axes* is here meant, the portion intercepted by the curve on the two sides of the centre.

Cor. 1. Since by definition AB : ab :: ab : parameter (denoted by P), AB has to P the duplicate ratio of AB to ab, and (Geom. theor. 78, or Euc. VI 20. cor. 2.) AB : P :: $AB^2 : ab^2$; hence AB : P :: $AC^2 - CD^2$ (= AD . DB, Geom. theor. 32, or Euc. II. 5, cor.) : DE^2. Or, in words,

As the transverse is to its parameter, so is the rectangle of the abscisses to the square of the ordinate.

Cor. 2. Ordinates DE, GH equi-distant from the centre are equal to one another.

For AD . DB : AG . GB :: $DE^2 : GH^2$.

But AD = GB and DB = AG; hence AD . DB = AG . GB, and therefore $DE^2 = GH^2$, or DE = GH.

Cor. 3. Join EH. Then it is parallel to the transverse AB, perpendicular to the conjugate ab, and bisected by the latter in K.

Cor. 4. Ca is greater than any other ordinate on either side of it. For in the same ellipse the square of the ordinate varies as the rectangle of the abscisses: and it has been shown (see Max. et Min. pr. 1.) that this rectangle is the greatest when the axis is bisected. Hence, then, the ordinate is also greatest at that point.

Cor. 5. Draw LaM through a the extremity of the conjugate axis, parallel to the transverse AB: it will be a tangent to the ellipse at a.

For, if not let it cut the ellipse at H: then since (cor. 4.) aC is greater than GH, the line aH is not parallel to AB. But by construction it is parallel: which is impossible. Hence aH is not parallel to AB. Hence also the line parallel to AB passing through a is a tangent to the ellipse at a.

In like manner a line through b parallel to AB is a tangent to the ellipse at b.

Cor. 6. The tangents at the extremities of the conjugate axes are parallel to one another, and perpendicular to the conjugate axis. They are also (Cor. 3., 5.) parallel to the line HE: and hence HE is a double ordinate.

Cor. 7. The axes AB, ab are conjugate diameters each to the other: and the tangents at their extremities constitute the conjugate rectangle. (Def. 31.)

Prop. III.—*As the square of the conjugate axis is to the square of the transverse; so is the rectangle of the abscisses of the conjugate to the square of their ordinate.*

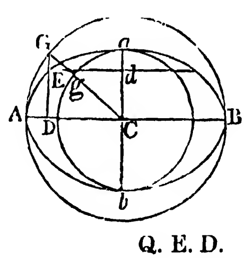
That is,

$Ca^2 : CA^2$ :: ad . db (or $Ca^2 - Cd^2$) : dE^2.

For draw the ordinate ED to the transverse AB. Then, Prop. II.

$Ca^2 : CA^2$ :: DE^2 (=Cd^2) : AD . DB.

Or, *alternando et dividendo*, $Ca^2 : CA^2$ :: $Ca^2 - cd^2$ (or ad . db) : CA^2 — AD . DB (Geom. theor. 32, or Euc. II. 6) = $CD^2 = dE^2$.



Q. E. D.

Cor. 1. If two circles be described, one on each axis of the ellipse, as diameters, the one will be inscribed within the ellipse, and the other circumscribed about it.

Cor. 2. An ordinate in either circle is to the corresponding ordinate in the ellipse as the axis of this ordinate is to the other axis. That is,

$$CA : Ca :: DG : DE, \text{ and}$$
$$Ca : CA :: dg : dE.$$

For by the circle, $AD \,.\, DB = DG^2$; and by the ellipse

$$CA^2 : Ca^2 :: AD \,.\, DB \; (= DG^2) : DE^2$$
$$\text{Or } CA : Ca :: DG : DE.$$
$$\text{Similarly, } Ca : CA :: dg : dE.$$

Cor. 3. By equality of ratios,

$$DG : DE \; (= cd) :: dE \; (= DC) : dg.$$

Cor. 4. Hence, also, as the ellipse and circle be conceived to be generated by the motion of a line perpendicular to the transverse axis, and by points whose distance from that axis are in a constant ratio, the spaces which they describe simultaneously will also be in the same constant ratio. Hence the area of the whole ellipse is to the area of the circle, as also any corresponding parts of them, to one another in the same ratio as the two axes of the ellipse are to one another: or as the square of the diameter to the rectangle of the two axes; or as the axis along which the ordinate moves is to the other axis; or, again, the areas of the two circles, and of the ellipse, are as the square of each axis, and the rectangle of the two. Hence, moreover, the ellipse is a mean proportional between the two circles.

Cor. 5. If the two axes become equal, the ellipse becomes a circle; which, therefore, is a particular case of the ellipse.

Cor. 6. If two ellipses have the same line for the transverse axis, their ordinates to corresponding abscisses are in a constant ratio: and likewise if they have the same conjugate axes, the ordinates to them are in the same constant ratio.

EXERCISES.

1. *Problem.* Resolve the questions given at p. 426, vol. I. on the Mensuration of the Ellipse. Also,

Find the area of a segment of an ellipse cut off by an *oblique ordinate,* the abscisses of the two extremities of the oblique ordinate being given.

2. *Theorem.* The points C, g, G are in the same straight line.

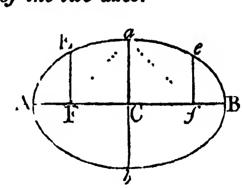
3. *Problem.* By means of the preceding theorem, show how to find any number of points in the curve of the ellipse, whose transverse and conjugate diameters are given.

4. *Problem.* The transverse axis is 8, the conjugate 6; find the ordinates corresponding to the abscisses, 1, 2, 3, 4, 5, 6, 7; and the abscisses corresponding to the ordinates ·5, 1·5, 2·5 and 3.

Prop. IV. *The square of the distance of the focus from the center is equal to the difference of the squares of the semiaxes; or, the square of the distance between the foci is equal to the difference of the squares of the two axes.*

That is, $CF^2 = CA^2 - Ca^2$

$$Ff^2 = AB^2 - ab^2.$$



For to F, the focus, draw the ordinate FE. This (by definition 27) will be the semi-parameter. Then, by prop. II.

$CA^2 : Ca^2 :: CA^2 - CF^2 : FE^2$; and by definition of the parameter, (def. 26).

$CA^2 : Ca^2 :: Ca^2 : FE^2$. Hence (Euc. v. 11.)

$$Ca^2 = CA^2 - CF^2, \text{ or } CA^2 - Ca^2 = CF^2.$$

Or, by doubling the lines in question, $AB^2 - ab^2 = Ff^2$ Q. E. D.

Cor. 1. The two semiaxes and the focal distance may be made to constitute a right angled triangle.

Cor. 2. The distance Fa, or fa, is equal to the semi-transverse axis AC.

Cor. 3. The semi-conjugate axis Ca is a mean proportional between AF, FB, (or between Af, fB) the distances of either focus from the two vertices. For

$$Ca^2 = CA^2 - CF^2 = AF \cdot FB, \text{ or}$$
$$Ca^2 = CA^2 - Cf^2 = Af \cdot fB.$$

EXERCISES.

1. *Problem* Given the two axes of an ellipse, to find the distances of a given point from the two foci.

2. *Problem.* Given the ratio and sum of the transverse diameter and its conjugate to find the eccentricity.

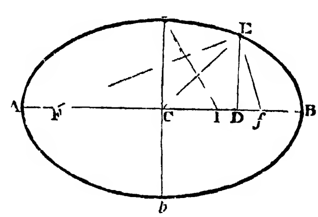
3. *Problem.* Given the distance of the foci from their vertex, and the length of the line joining the extremities of the transverse and conjugate diameter: to find the focus and the diameters themselves.

4. *Problem.* The transverse and conjugate are 8 and 6, find the angle subtended at the extremity of the conjugate by the eccentricity; and the length of a line which bisects that angle, and is terminated by the axis.

PROP. V. *Lines drawn from the foci and centre to a point in the ellipse are equal, severally, to the distances of the extremities of the transverse and conjugate diameter from the corresponding dividing point:*

Also, the sum of the lines drawn from the foci to the point in the curve equal to the transverse axis, and their difference is equal to the difference of the segments of the transverse made by the dividing point.

That is, if I be the dividing point corresponding to E,



1 . . . . $a\mathrm{I} = \mathrm{CE}$,
2 . . . . $\mathrm{EF} = \mathrm{AI}$,
3 . . . . $\mathrm{E}f = \mathrm{IB}$,
4 . . . . . . $\mathrm{EF} + \mathrm{E}f = \mathrm{AB}$,
5 . . . . . . $\mathrm{EF} - \mathrm{E}f = 2\mathrm{CI} = \mathrm{II}'$, if $\mathrm{I'C} = \mathrm{CI}$.

For, by def. of dividing point (def. 33, and Geom. theor. 74, or Euc. VI. 22.)

$\mathrm{CB}^2 : \mathrm{C}f^2 :: \mathrm{CD}^2 : \mathrm{CI}^2$, or convertendo, (Geom. theor. 69, or Euc. V. E.)

$\mathrm{CB}^2 : \mathrm{C}a^2 :: \mathrm{CD}^2 : \mathrm{CD}^2 - \mathrm{CI}^2$.

But, $\mathrm{CB}^2 : \mathrm{C}a^2 :: \mathrm{CB}^2 - \mathrm{CD}^2 : \mathrm{DE}^2$ (pr. 2.)

Ex equali, $\mathrm{CB}^2 : \mathrm{C}a^2 :: \mathrm{CB}^2 : \mathrm{CD}^2 - \mathrm{CI}^2 + \mathrm{DE}^2 = \mathrm{CE}^2 - \mathrm{CI}^2$;

Or, (Euc. V. 9) $\mathrm{C}a^2 = \mathrm{CE}^2 - \mathrm{CI}^2$; or $\mathrm{CE}^2 = \mathrm{C}a^2 + \mathrm{CI}^2 = a\mathrm{I}^2$.

Or $a\mathrm{I} = \mathrm{CE}$ . . . . . . Q. E. 1mo D.

Again, $\mathrm{FE}^2 = \mathrm{FC}^2 + \mathrm{CE}^2 + 2\mathrm{FC} \cdot \mathrm{CD}$, (Geom. theor. 37, or Euc. 13. II.

$= \mathrm{FC}^2 + a\mathrm{C}^2 + \mathrm{EI}^2 + 2\mathrm{CB} \cdot \mathrm{CI}$ (preceding case)

$= \mathrm{AC}^2 + \mathrm{CI}^2 + 2\mathrm{AC} \cdot \mathrm{CI}$, (def. 33.)

$= (\mathrm{AC} + \mathrm{CI})^2$, (Geom. theor. 31, or Euc II. 4.)

Or $\mathrm{EF} = \mathrm{AI}$ . . . . Q. E. 2do D.

Also, $f\mathrm{E}^2 + 2f\mathrm{C} \cdot \mathrm{CD} = f\mathrm{C}^2 + \mathrm{CE}^2$

$= f\mathrm{C}^2 + \mathrm{C}a^2 + \mathrm{CI}^2$

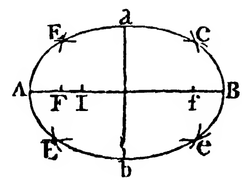
$= \mathrm{BC}^2 + \mathrm{CI}^2$

$= 2\mathrm{BC} \cdot \mathrm{CI} + \mathrm{BI}^2$

But $\mathrm{BC} \cdot \mathrm{CI} = f\mathrm{C} \cdot \mathrm{CD}$, and hence

$fE^2 = BI^2$, or $fE = BI$ . . . . Q. E. 3^tio D.
Hence $FE + Ef = 2BC = AB$, ... Q. E. 4^to D.
And $FE - Ef = 2CI = II'$ . . . . . Q. E. 5^to D.

Cor. 1. Hence is derived the common mechanical construction of the curve by means of points, or of a string, thus:—



In the transverse take the foci F, *f*, and any point I. Then with the radii AI, BI, and centres F, *f*, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points I, as many other points will be found in the curve. Then with a steady hand, the curve line may be drawn through all the points of intersection E.

Or, take a thread of the length AB of the transverse axis, and fix its two ends in the foci E, *f*, by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

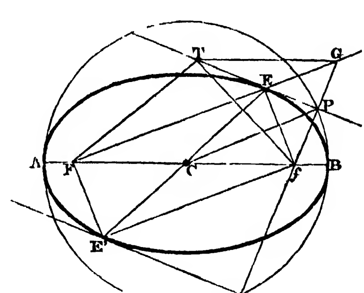
Cor. 2. Let a point E′ be taken below AB and at the same distance from F that E is from *f*, then joining E′*f*, it is equal to EF; and the figure EFE′*f* is a parallelogram, whose diagonals bisect one another. Hence EE′ passes through C the middle of F*f* that is the centre of the ellipse, and is bisected there. That is, EE′ is a diameter, it passing through the centre; and the diameter is bisected in the centre.

EXERCISES.

1. *Problem.* Find the greatest and least diameters of an ellipse, and the position and magnitude of its two *equal diameters*.

2. *Theorem.* Two concentric ellipses whose transverse diameters are perpendicular to one another, if they intersect at all, intersect in four points, situated in the circumference of a circle.

Prop. VI. *Let the interior angle made by producing one of the focal lines be bisected; the bisecting line is a tangent to the ellipse at that point.*



Thus, let FE be produced, and the angle FEG be bisected by EP. Then EP will touch the ellipse at E.

For if EP be not a tangent, it must meet the curve again at some point, as T. Take $EG = Ef$; and join FT, *f*T, GT, and *f*G meeting ET in P.

Then since E*f* and EG are equal, and the angle *f*EG is bisected, the sides *f*P, PG are equal, and likewise the angles *f*PE, GPE. Hence also the sides *f*T, TG are equal to one another: that is, the sum of the two lines FT, T*f*, is equal to the sum of the two lines FT, TG. But $FT + TG$ is greater than FG, that is, than $FE + EG$, that is, again, than $FE + Ef$.

But since T is in the curve of the ellipse, we have $FE + Ef = FT + Tf$, which is absurd. The point T is not therefore in the ellipse.

In the same way it can be shown that no other point of EP but E is in the ellipse, and hence EP is a tangent. Q. E. D.

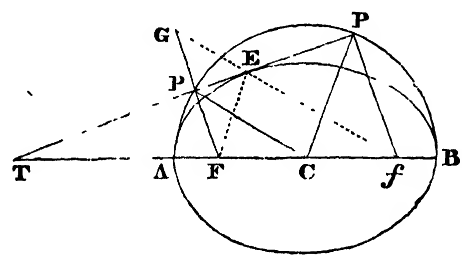
Schol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this theorem, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from those points to the other focus. So the ray fE is reflected into FE. And this is the reason why the points F, f, are called the *foci*, or burning points.

Cor. 1. The tangents at the opposite extremities of any diameter are parallel to one another: and parallel tangents are at the extremities of the same diameter.

Cor. 2. The distance of the point of intersection of the perpendicular from the focus upon the tangent with the tangent, that is, CP, is equal to the semi-transverse axis.

For since FG and Ff are bisected in P and C, the line CP is also parallel to fG and equal to half FG, that is to half AB, is to CA or CB.

Cor. 3. Every diameter is bisected at the centre. For EE′ and Ff are the diameters of parallelogram. Hence EC = CE′.



Cor. 4. The intersection of the perpendiculars with the tangent will always be situated in the circumference of a circle described on the transverse axis.

Cor. 5. The perpendiculars have the same ratio as the focal distances; as likewise have the intercepted portions of the tangent between the point of contact; or FP : fP :: FE : fE :: PE : Ef.

Cor. 6. The rectangle contained by the perpendiculars FP, fp from the foci on the tangent is equal to the square of the semi-conjugate diameter.

For conceive the circle on AB to be completed and fP produced to meet it in H, and CH joined. Then PpH being a right angle is in a semicircle, of which C is the centre; and hence PCH is a diameter. Also by parallels fH = PE. But $pf.f$H = Af.fB = (cor. 3, prop. iv.) aC^2

Cor. 7. The lines CP, CF are parallel to fE, FE. (Euc. VI. 2.)

Cor. 8. If perpendiculars be drawn from foci and centre to the tangent, that from the centre is half the sum of those drawn from the foci.

EXERCISES.

1. *Prob.* From the point T to draw a tangent to a given ellipse, both when T coincides with E a point in the curve, and when it lies without the curve.

This admits of two very elegant constructions: One dependent on the process which was employed in the demonstration of the general proposition; and the other dependent upon the third corollary. The student is recommended to attempt the construction by both methods.

In the latter case, the point of contact can be assigned without tracing the curve: in the former not.

2. *Theor.* There cannot be more than two tangents drawn from the same point to an ellipse.

PROP. VII. *If a tangent and an ordinate be drawn from any point in the curve to meet the transverse axis, the semi-transverse will be a mean proportional between the distances of the two intersections from the centre.*

That is, $CD : CB :: CB : CT$

For, draw FE and produce it to G, and draw fE; and let I be the dividing point.

Then $FT : Tf :: FE : Ef$, or componendo, et dividendo.

$FT + Tf : FT - Tf :: FE + Ef : FE - Ef$,

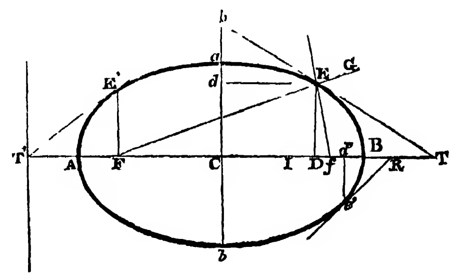
that is,

$2CT : 2CF :: 2BC : 2CI$,

or $CT : CF :: AC : CI$,

Alt° $CT : CB :: CF : CI$

$:: CB : CI$. Q. E. D.



Cor. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every ellipse described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

Cor. 2. When the outer ellipse, by enlarging, becomes a circle, then by drawing ET perpendicular to CE, and joining T to the point E, the tangent to the point E at the ellipse is obtained.

Cor. 3. $AD : DB :: AT : TB$; for componendo et dividendo of the proposition

$$CT + CB : CT - CB :: CB + CD : CB - CD$$
$$\text{or } AD : DB :: AT : TB$$

Cor. 4. $AD \cdot DB = CD \cdot DT$.

For each rectangle is equal to $CB^2 - CD^2$.

Cor. 5. Let another tangent be drawn, viz. eR: then also $CB^2 = cd \cdot CR$.

Hence $Cd \cdot CR = CD \cdot CT$, and $Cd : CD :: CT : CR$.

Cor. 6. $CD : CT :: BD^2 : BT^2$.

For $CT : CB :: CB : CD$, or dividendo

$BD : BT :: CD : CB$, or squaring

$BD^2 : BT^2 :: CD^2 : CB^2$, or by continued proportionals,

$:: CD : CT$.

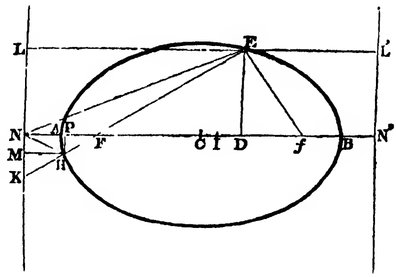
Cor. 7. If D be the focus, then TH being drawn perpendicular to ABT it is the *directrix* (def. 34); and by this theorem the semi-transverse is a mean proportional between the distance of the focus and directrix from the centre.

EXERCISES.

Theor. Let the tangent meet the conjugate axis at t, and an ordinate also be drawn to meet it at d. Then

1. $Cd : Ca :: Ca : Ct$.
2. $da^2 : at^2 :: Cd : Ct$.
3. $bd : da :: bt : ta$.
4. $bd \cdot da = cd \cdot dt$.

Prop. VIII. *The distance of the directrix from a point in the ellipse has to the distance of the focus from the same point the same ratio, wherever in the curve that point be taken.*



That is, FE : EL :: fE : EL′ :: CF : CA = constant ratio.

By property of directrix, (cor. pr. vii.) CF : CA :: CA : CN; and by def. of dividing point CF : CA :: CI : CD. Hence Ex. equali. and Compon , and ex equali again,

CF : CA :: AC+CI : NC+CD
:: AI : DN
:: FE : EL.

In like manner, dividendo, &c , we get,

Cf : CB :: fE : EL′. Q. E. D.

Cor. 1. If a line HE be drawn through the focus F, and the lines HN, EN be drawn, they will make equal angles with the axis at the point N.

For, draw the ordinates ED, HP to meet the axis in D and P; and likewise the lines HM, EL parallel to the axis.

Then by the proposition HM (=NP) : EL (=DN) : HF : FE
: HP : DE by sim. triangles

Hence the right angled triangles NPH, NDE are similar, (Geom. theor. 86, Ex. vi. 4,) and the angles HNP, DNE are equal.

Cor. 2. The chord HE is so divided by the focus and directrix in F and R, that EF : FH :: EK : KH, or KH : HF :: KE : EF.

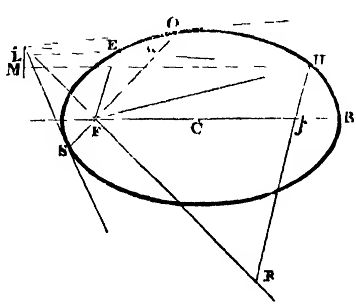
For by similar triangles KH : KE :: MH : EL, or by the proposition,
:: HF : FE

And the second follows from this at once by alternation.

Schol.—This is the ratio which (def. 35.) has been called the *determining ratio.* In many treatises on the conic sections this property has been made the *definition*, and from it the properties of the curves are readily and elegantly derived. It, however, requires us *afterwards* to show that the figure is a conic section.

It is obvious as stated in the note on def. 35, that the ratio is in this case one of lesser inequality, since F being *within* the curve, CF is less than CA.

Prop. IX. *If a two lines be drawn from any point in the directrix to cut an Ellipse, one of which passes through the Focus, then lines drawn from the points of intersection of the other line with the ellipse to the focus will make equal angles with the first-named line through the focus.*



That is, EFT = HFR.

Draw the lines EL, HM parallel to the axis, and HR parallel to EF.

Then by parallels, EF : HR :: TE : TH
:: LE : HM

But by prop. viii , LE : HM :: EF : FH

Hence (Euc. v. 11) EF : HR :: EF : FH;

And (Euc. v. 9) HR=HF.

Hence the angle HFR=HRF=EFT.

Q. E. D.

Cor. 1. If the secant TEH become a

tangent TQ, the points E and H coalesce, and the angles QFR, QFT are equal, or the angle TFQ is a right angle.

In like manner if the tangent TS be drawn, TFS is a right angle.

Cor. 2. If from any point T in the directrix, tangents TQ, TS be drawn to the ellipse, the chord joining the points of contact, QS, will pass through the focus. For by the best corollary, TFS and TFQ are two right angles, and hence the points Q, F, S, are in one right line, or QS passes through the focus F.

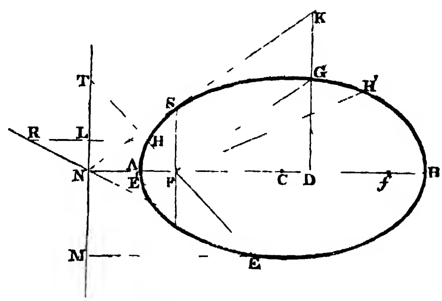
Cor. 3. The line QF bisects the angle EFH.

Cor. 4. The line HE is so divided that EK : KH :: ET : TH.

Prop. X. *A straight line drawn through the focus is divided in that point, so that four times the rectangle contained by its segments is equal to the rectangle contained by the line itself and the parameter of the ellipse.*

That is, 4 HF . FE=P . HE

Draw the lines HL, EM, and HN, EN as before, and produce EN to meet HL in R; and draw FS, the semi-parameter.



Then, (Cor. 1, prop. viii.) the angles HNL, ENM are equal; and RNL = ENM. Hence RL = LH, and RH = 2HL.

By similar triangles EH : EF :: 2 HL : NF, or NF . EH = 2 HL . EF.

Also, by prop. viii. HL : HF :: NF : FS,

And by equality 2FE : 2FE :: HE : HE,

And compounding (note pa. 102) 2HL . FE : 2HF . FE :: NF . HE : FS . HE.

But it has been shown that the first term is equal to the third; hence (Euc. v. 9) we have also the second equal to the fourth, that is

2HF . FE = FS . HE, or doubling

4HF . FE = 2FS . HE = P . HE. Q. E. D.

Cor. 1. If another line H′E′ also pass through the focus, the rectangles of the segments have the same ratio as the lines themselves, or

HF . FE : H′F . FE′ :: HE : H′E′.

Cor. 2. Draw the tangent from N; and produce any ordinate DG to meet it in K, and join FG; then FG = DK

For, (prop. IX.) SF is perpendicular to NF, and therefore parallel to DK. Hence by similar triangles NFS, NDK, NF : FS :: ND : DK.

And by (VIII.) NF : FS :: ND : FG. Hence DK=FG.

EXERCISES.

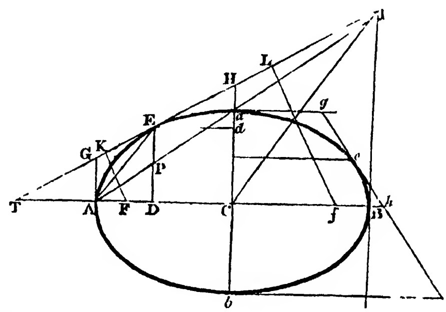
1. *Theorem.* The chords drawn through the focus are proportional to the squares of their parallel diameters.

2. *Problem.* Given the position of a line passing through the given focus, the position of the conjugate axis, and the ratio of the two axes, to find where it will intersect the ellipse.

3. *Problem.* The perpendicular from the centre C intercepted by the focal tangent is equal to the semi-transverse axis.

4. *Problem.* Given the positions of the directrix, the focal tangent, and the magnitude of the transverse *or* conjugate axis, to construct the elements of the curve.

Prop. XI. *If there be any tangent meeting four perpendiculars to the axis drawn from these four points, namely, the centre, the two extremities of the axis, and the point of contact; those four perpendiculars will be proportionals.*



That is,
AG : DE :: CH : BI.
For, by prop. 7.
TC : AC :: AC : DC;
therefore dividendo
TA : AD :: TC : AC (= CB);
and, componendo
TA : TD :: TC : TB,
and by sim. triangles
AG : DE :: CH : BI.
Q. E. D.

Cor. 1. Hence TA, TD, TC, TB and TG, TE, TH, TI } are also proportionals.

For these are as AG, DE, CH, BI, by similar triangles.

Cor. 2. Draw AI to intersect DE in P; then since TA : TE :: TC : TI, the triangles TAE, TCI are similar, as well as the triangles AED, CBI, and ADP, ABI.

Hence AD DE CB : BI,
And AD DP AB : BI;
Therefore DE DP AB : CB :: 2 : 1; *which suggests another simple practical method of drawing a tangent to an ellipse.*

Cor. 3. The rectangle AG . BI = Ca^2. For draw E*d* parallel to AB. Then C*d* = DE: and by the theorem and Euc. vi. 16. ED . HC = C*d* . CH = Ca^2 (Ex. 4, on prop. vii.)

Cor. 4. Draw the perpendiculars from the foci: then FK, *f*L = AG . BI For by last Cor. AG . BI = Ca^2, and by (Cor. 6, pr. vi.) FK . *f*L = Ca^2. Hence, &c.

Cor. 5. In precisely the same may corresponding properties be deduced respecting lines *parallel* to the transverse diameter from four corresponding points, viz. that *ag* : *de* :: C*h* : *bi*, &c. and *ag* . *bi* = Cb^2. These are left for the student's investigation.

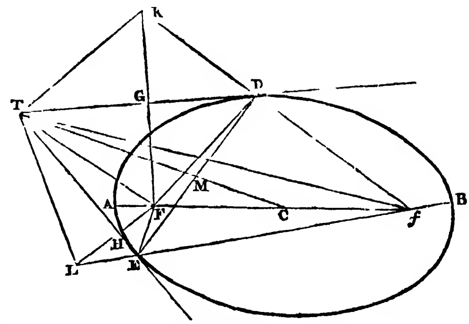
SECTION II.

ON THE PROPERTIES OF THE ELLIPSE DEPENDING UPON OBLIQUE CONJUGATE DIAMETERS.

Prop. XII. *If lines be drawn from the points of contact of any two tangents to an ellipse to the focus, and also from the intersection of those tangents, the angle formed by the two former lines is bisected by the last: and the lines drawn to the two foci from the intersection of the tangents, make equal angles*

with the tangents: the line drawn from the intersection of the tangents to the centre bisects the chord of contact.

Let AB be the transverse diameter; F, f the foci; C the centre; and DT, ET the tangents meeting in T. Join TF, TC, Tf, FE, FD, fE, fD, CE, DF, and DE; and let TC meet DE in M. Then,



1. EFT = DFT, and EfT = DfT.
2. ETF = DTf, and ETf = DTF.
3. EM = MD.

1. Draw FG, FH perpendicular to the tangents meeting fD, fE in K and L, and join TK, TL. Then (prop. vi) FDG = KDG, and the angles at G right angles by construction, and GD common: hence DK = DF, and fK = fD + DF = AB (prop. v.) Similarly fL = AB; and therefore fK = fL.

Again, FG = GK and GT is common to the two right angled triangles FTG and KTG. Hence TK = TF. Similarly TL = TF. Hence TK = TL.

Hence in the two triangles TKf, TLf; TK = TL, Kf = Lf, and Tf common; and therefore angles TfK and TfL are equal: that is DfT = EfT. In exactly the same way it may be proved that DFT = EFT.

Q. E. 1mo D.

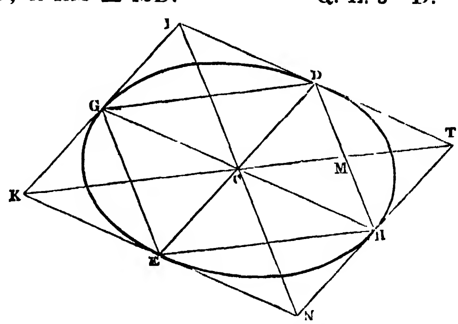
2. FTf = $\frac{1}{2}$ (FTK − FTL) = FTD − FTE. Hence FTE = FTD − FTf = fTD. And, adding FTf to both, we have, also, FTD = fTE.

Q. E. 2do D.

3. The triangle TEC is composed of the three triangles TEF, FTC, and CEF, which are respectively equal to the three TEL, TCf and fEC. But the whole six constitute the triangle TLf. Hence TEC is the half of TLf. And similarly TCD is the half of TKf. Hence since TKf is equal to TLf, the halves TEC, TDC are equal.

But they stand upon the same base, (and on opposite sides) TC: and hence are between parallels equi-distant from the base TC. The line joining their vertices is, therefore, bisected by the base; or EM = MD. Q. E. 3do D.

Cor. 1. If one diameter of an ellipse (def 23.) be conjugate to another, the second is conjugate to the first: that is, if GH be parallel to the tangent DT, at the extremity of DE; then DE will be parallel to the tangent at H, the extremity of GH.

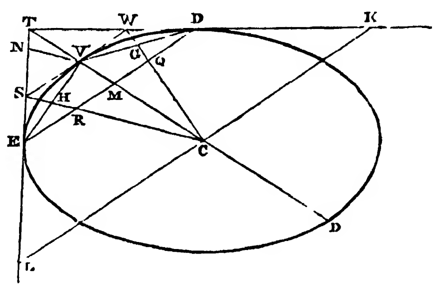


For, join EH, HD, and CT cutting HD in M. Then DH is bisected in M, and DE in C. Hence CT is parallel to EH. Consequently, CEH = DCT, and because by hypothesis, CH is parallel to DT, DCT = HCE, and DC = CE, the triangles DCT, ECH are in all respects equal, and CT = EH and parallel to it. Hence (Geom. theor. 24, or Euc. I. 33), HT is equal and parallel to CD.

Cor. 2. Hence any circumscribed *conjugate parallelogram* (def. 31) can be drawn when one of the points of contact is given: and it will be double the corresponding *inscribed conjugate parallelogram*. That the inscribed figure is a parallelogram, is evident, since GH, DE are bisected at their intersection C, (prop. vi. cor. 3.)

Cor. 3. The diagonals of the conjugate parallelograms are conjugate to one another. Thus LN and KT are conjugate; they being each parallel to the tangents at the other's extremities.

PROP. XIII. *Every ordinal chord (def. 23) is bisected by its conjugate diameter: the tangents to every ordinal chord meet in that conjugate diameter produced: and that semi diameter itself is a mean proportional between the segments, estimated from the centre, cut off by the chord and tangent.*



Let WS be a tangent and VD the diameter from V: then DE, parallel to WS, is the ordinal chord. Then DE is bisected in M; the tangents at D and E meet in CV produced: and CM : CV :: CV : CT.

Let the tangents at D and E meet the tangent at V in W and S; join SC, WC cutting the chords DV and EV in G and H, and the chord DE in R and Q. Draw, also, LK parallel to DE through C.

First. By def. 23 and construction, the three lines WS, DE and KL are parallel: hence the triangle VWG is similar to DGQ, and SVH to REH. But by the last proposition VG = GD and VH = HE. Hence the same pair of triangles are also equal, and VW = DQ, and SV = ER.

Again, by similar triangles, LCS, RES, and CWK, QWD, we have

CL : RE CS : SH, or, by parallels,
CW : WG, or, by the latter pair of triangles,
CK : QD.

But by parallels DE, LK, LC = CK (cor. pr. xii.) and hence RE = QD. But RE = SV, and QD = VW; therefore, SV = VW.

Also, by similar triangles SCV, RCM, and VCW, MCQ

SV : RM :: VC : CM
:: VW : MQ.

But SV = VW, and hence RM = MQ; and therefore also EM = MD.

Q. E. 1mo D.

Second. It was proved in prop. xii. that C, M, T are in one straight line; and in the first part of the present, that C, M, V are in one straight line: hence C and M are in the line CV, and the four points, C, M, V, P are in one straight line. That is, the tangents at D and E meet at the same point in the diameter conjugate to DE, produced. Q. E. 2do D.

Third. Draw VN parallel to CS. Because SH is parallel to NV, and EH = HV, (Geom. theor. 80, or Euc. vi. 2.) ES = SN.

Then by the similar triangles TNV, TSC and MET, VST, we have

CV : CT :: SN (= SE) : ST,
:: MV : VT; or, *alternando et dividendo,*
CV − MV : CV :: CT − VT : CT; that is, CM : CV :: CV : CT.
Q. E. 3[io] D.

Cor. 1. The line DV is *harmonically** *divided* in M and T: for *comp. et div.*
CV + CM : CV − CM :: CT + CV : CT − CV;
that is, DM : MV :: DT : TV.

Cor. 2. The rectangle DM . MV = CM . MT.
For $CV^2 - CM^2$ = VM . MD (Geom. theor. 33, or Euc. ii. 5.cor.); and
$CV^2 - CM^2$ = CT . CM − CM^2 = CM (CT − CM) = CM . MT.
Hence VM . MD = CM . MT.

Cor. 3. DM . MV : DT . TV :: $CM^2 : CV^2$. For, by cor. 1.
$CM^2 : CV^2 : CV^2 : CT^2$, and hence, *div. et altern.*
$CV^2 - CM^2 : CT^2 - CV^2 :: CM^2 : CV^2$, that is, as stated above,
DM . MV : DT . TV :: $CM^2 : CV^2$.

Cor. 4. Every *ordinate* to a diameter is parallel to the tangent at its vertex.

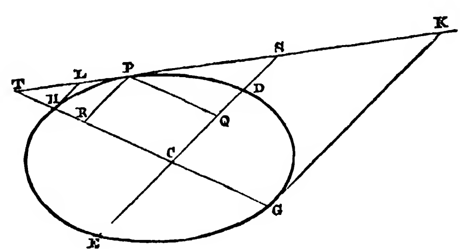
Cor. 5. All ordinates to the same diameter are parallel to one another.

Cor. 6. Any straight line which bisects two parallel chords is a diameter, and is conjugate to those chords.

EXERCISE.

1. *Problem.* Given an ellipse to find its centre, foci, and rectangular conjugate diameters; and to find the conjugate diameters, one of which makes a given angle with a given line.

Prop. XIV. *As the square of any diameter is to the square of its conjugate, so is the rectangle of the abscisses to the square of the ordinate.*



Let DE, GH be two conjugate diameters, and PQ an ordinate to the former, and P a point in the ellipse: then
$CD^2 : CH^2$ :: EQ . QD : DP^2, and
$CH^2 : CD^2$ :: HR . RG : RP^2

Draw the tangent at P to meet the conjugate diameters in S and T.

Then (prop. xiii.) CQ : CD :: CD : CS. Hence (Geom. theor. 82, cor. 2, or Euc. vi. 20. cor. 2.)
$CQ^2 : CD^2$ :: CQ : CS; or, *dividendo.*
$CD^2 - CQ^2 : CD^2$ :: SQ : SC; or by parallels,

* When two points are taken in a given straight line, such that their distances from the two extremities have the same ratio, the division is called *harmonical.* This may be easily shown to coincide with the definition (in numbers) given at page 165., vol. i.: and is left for the student's exercise.

The properties of harmonical lines are amongst the most interesting, and are certainly the most fruitful in both curious and important practical results, that occur in the whole range of Plane Geometry, and on all accounts are well worthy of the student's attention, and of more attention from Geometers generally than has usually been given to them.

$$\begin{aligned}&:: PQ : CT,\\ &\text{Or since } PQ\ (= CR) : CH :: CH : CT\\ &:: PQ^2 : CH^2, \text{ or quadrupling,}\\ &:: ED^2 : GH^2.\end{aligned}$$

In the same manner may the other analogy be proved. Q. E. D.

Cor. 1. The rectangles of the abscisses of any diameter are as the squares of the corresponding ordinates: for each ratio is the same as that of the squares of the conjugate diameters.

Cor. 2. Any diameter is to its parameter, as the rectangle of its abscisses to the square of the corresponding ordinate. It is shown as in cor. 1. prop. i.

Cor. 3. Draw the tangents GK, HL at the extremities of the diameter GH: then HL : PR :: CS : GK (see dem. of prop. ix.)

Cor. 4. If *any* tangent LK be drawn to intersect two parallel tangents GK, HL, it will cut off segments HL, GK, whose rectangle is equal to the square of the semi-diameter CD parallel to them.

Cor. 5. The tangent HL is harmonically divided in P and T.

Cor. 6. In the same way as in cor. 1, prop. ix. it may be proved that HK and LG will bisect PR, and hence that the three lines pass through the same point.

EXERCISES.

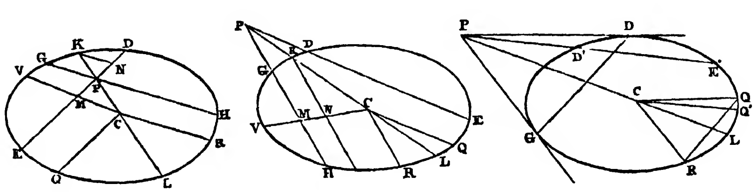
1. *Theorem.* The chords which join the extremities of conjugate diameters are conjugate to one another.

2. *Theorem.* The diameters which bisect these chords are conjugate to one another.

3. *Prob.* Given the ratio of two conjugate diameters of a given ellipse to find their positions; and determine the limits of the ratio.

4. *Theorem.* The rectangle under the segments LP, PK of the tangent at P is equal to the square of the semi-diameter parallel to LK.

Prop. XV. *If any two straight lines which intersect one another and the ellipse be drawn parallel to two given diameters, the rectangle under the segments of the one, shall be to the rectangle under the segments of the other, as the square of the diameter to which it is parallel is to the square of the other.*



Let the two lines PDE, PGH meet in P and cut the ellipse in D, E and G, H; and let the semi-diameters CQ, CR be parallel to them: then

$$DP \cdot PE : HP \cdot PG :: CQ^2 : CR^2.$$

Draw CV to bisect GH in M, and draw KN parallel to GH; and through P draw the diameter KL. Then CV is conjugate to CR.

Hence by the last proposition,

$CR^2 - KN^2 : CR^2 - GM^2 :: CN^2 : CM^2$; or *dividendo*,
$GM^2 - KN^2$* $: CR^2 - KN^2 :: CN^2 - CM^2 : CN^2$; or, by similar triangles
$:: KN^2 - PM^2 : KN^2$; or *alto. et componendo*,
$GM^2 - PM^2 : CR^2 :: KN^2 - PM^2 : KN^2$; or by sim. trans.
$:: CK^2 - CP^2 : CK^2$.

But GH, KL are bisected in M and C respectively; and hence (Euc. ii. 5, cor.) $GP . PH :: CR^2 : KP . PL : CK^2$.

In like manner

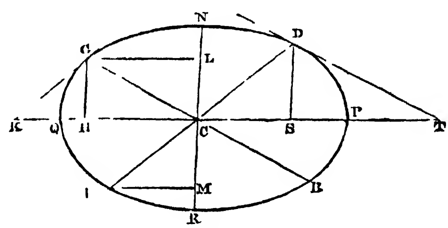
$DP . PE : CQ^2 :: KP . PL : CK^2$.

And (Euc. v. 14), and *alto*.

$GP . PH : DP . PE :: CR^2 : CQ^2$. Q. E. D.

Cor. If one of the lines be a tangent, the rectangle of its segments becomes the square of the tangent; and if both be tangents, (fig. 3.) then these tangents have the same ratio as their parallel diameters, and GD is parallel to QR.

Prop. XVI. *If from the extremities of any two conjugate diameters of an ellipse ordinates be drawn to any third diameter, the rectangle of the abscisses made by one of the ordinates is equal to the square of the distance from the centre (estimated on that diameter) of the other ordinate.*



Let DE and GH be the conjugate diameters, and from the extremities D and G to any other diameter PQ draw the ordinates GH and DS. Then the rectangle contained by QH, HP is equal to the square of CS.

Let the tangents at D and G meet the diameter produced in T and K. Then by the similar triangles KGH, CDS and GHC, DPT, we have

KH : CS :: KG : DE by the similar triangles KGH, CDS.
KG : DC :: KC : CT by the similar triangles KGC, CDT.

Hence (Euc. v. 14) KH : CS :: KC : CT.

Again, (prop. xiii.) $KC . CH = CQ^2$ $CS . CT$. Hence (theor. 81, or Euc. vi. 16.)

KC : CT :: CS : CH.

And (Euc. v. 14) $CS : CH :: KH : CS$; or $CS^2 = CH . HK$.

But (xiii. cor. 2) $CH . HK = QH . HP$, whence $QH . HP = CS^2$

Similarly $CH^2 = QS . SP$. Q. E. D.

Cor. 1. $KC . CH = TC . CS$.

Cor. 2. $CH . HK = CS^2$, and $CS . ST = CH^2$.

both of which have been proved in the above demonstration.

Cor. 3. $CS^2 + CH^2 = CQ^2$; and $CP^2 = CL^2 + CM^2$, if NR be the diameter conjugate to PQ. For $CS^2 = CH . HK$, or adding CH^2 to these equals, $CS^2 + CH^2 = CH^2 + CH . HK = CH . CK = CQ^2$.

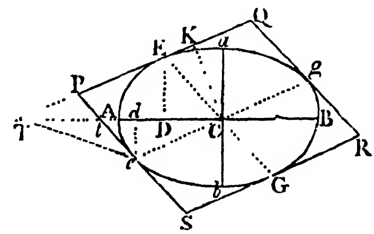
Cor. 4. $QC : CN :: CS : HG$, or $QC . HG = CS . CN$.

For, (last prop.) $QC^2 : CN^2 :: QH . HP : GH^2$, or last corollary,
$QC^2 : CN^2 :: CS^2 : GH^2$; and, Euc. vi.20, cor 2,
and hence $QC : CN :: CS : GH$.

* When the point P is *without* the ellipse, the step dividendo is differently made. The student should be required to point out *how* it is made.

PROP. XVII. *The sum of the squares of the pairs of conjugate diameters is always the same: and the areas of the conjugate parallelograms (whether inscribed or circumscribed) is always the same.*

Let PQRS be a conjugate circumscribed parallelogram; EG, eg the conjugate diameters at its points of contact; and Ee Gg the inscribed conjugate parallelogram: also let AB and ab be the transverse and conjugate diameters. Then,



$EG^2 + eg^2 = AB^2 + ab^2$,
$PQRS = AB \cdot ab$
$EeGg = AC \cdot Ca$.

Draw the transverse diameter AB, and DE, de perpendicular to it, and CK perpendicular to PQ; and produce PQ, PS to meet in AB in T and t. Then,

First. By the last proposition

$$CA^2 = CD^2 + Cd^2 \text{ and } Ca^2 = DE^2 + de^2,$$

Hence $CA^2 + Ca^2 = CD^2 + DE^2 + Cd^2 + de^2 = CE^2 + Ce^2$.

Or, $EG^2 + eg^2 = AB^2 + ab^2$. Q. E. 1$^{mo.}$ D.

Second. By cor. 5, last theorem,

$$CA : Ca :: CD : de$$

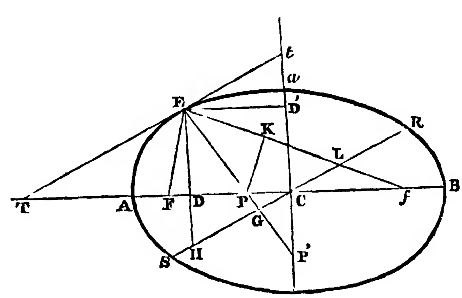
And prop. xiii. $CT : CA :: CA : CD$,

Hence in equali $CT : Ca :: CA : de$, or $CT \cdot de = Ca \cdot CA$.

But $CT \cdot de$ is double the triangle TeC; as is likewise the parallelogram CEPe upon the same base Ce and between the same parallels Ce, TQ. Hence parallelogram EPeC = rectangle of the semi-diameters CA, Ca: and quadrupling these equals, the parallelogram PQRS is equal to the conjugate rectangle. See def. 17. Q. E. 2do D.

PROP. XVIII. *If from any point in the curve there be drawn an ordinate, and a perpendicular to the curve, or to the tangent at that point: then, the*

Dist. on the trans. between the centre and ordinate, CD :
Will be to the dist. PD ::
As sq of the trans. axis :
To sq. of the conjugate.



That is,

$CA^2 : Ca^2 :: DC : DP$.

For, by theor. 2,

$CA^2 : Ca^2 :: AD \cdot DB : DE^2$,

But, by rt. angled Δs,

the rect. $TD \cdot DP = DE^2$;

and, by cor. 2, theor. 16, $CD \cdot DT = AD \cdot DB$;

therefore $CA^2 : Ca^2 :: TD \cdot DC : TD \cdot DP$,

Or (Euc. vi. 1.) $AC^2 : Ca^2 :: DC : DP$. Q. E. D.

Cor. 1. $AC^2 - Ca^2 : Ca^2 :: PC : PD$; that is $CF^2 : Ca^2 :: PC : PD$, F being the focus.

Cor. 2. $AB : P :: DC : DP$, by definition of the parameter, and Euc. vi. 21.

Cor. 3. In the same manner, if we draw ED′ parallel to AB, and produce EP to meet Ca in P′, we have $P : AB :: D'P' : CD'$.

Cor. 4. DP . D′P′ = DC . CD′, from cors. 3 and 4.

Cor. 5. Draw a diameter parallel to the tangent TEt, meeting the normal in G. Then EG . EP = Ca^2.

For produce ED to meet CG in H. Then the angles at D and G are right angles, and hence the triangles EDP, EGH are similar, and

$$\text{ED } (= \text{CD}') : \text{EP} :: \text{EG} : \text{EH } (= \text{C}t), \text{ that is}$$
$$\text{EP} . \text{EG} = \text{CD}' . \text{C}t = \text{C}a^2, \text{ (by prop. xi. exercise.)}$$

Cor. 6. Draw PK perpendicular to Ef: then EK is half the parameter.

Let CG meet Ef in L: then the triangles EGF, EPK are similar; and EG : EL :: EK : EP, or LE . EK = PE . EG = Ca^2.

But EP bisects the angle FEf (since it is perpendicular to the tangent, which bisects the exterior angle made by FE and Ef, prop. vii.) and hence EL = $\frac{1}{2}$ (EF + Ef) = AC (prop. V.) We have hence

$$\text{AC} . \text{EK} = \text{C}a^2, \text{ or AC} : \text{C}a :: \text{C}a : \text{EK, or EK} = \tfrac{1}{2}\text{P}.$$

EXERCISES.

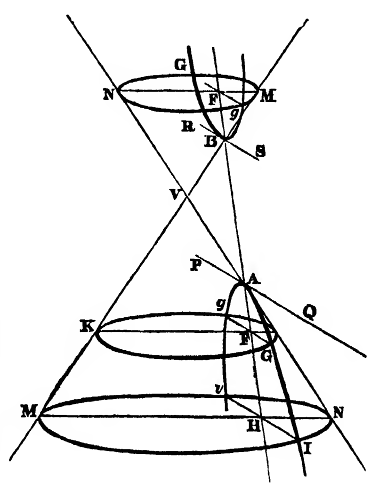
1. *Theorem.* The rectangle of the focal distances FE . Ef is equal to the square of the semi-diameter RS conjugate to E.

2. *Problem.* From a given point in either of the axes to draw a normal to the ellipse.

3. *Theorem.* The segment of the normal intercepted by the axes of the ellipse is equal to the eccentricity.

THE HYPERBOLA.

Prop. I. *The squares of the ordinates of the transverse are to each other as the rectangles of the corresponding abscisses.*



Let AVB be the vertical transverse plane; AGIB the plane of the hyperbolic section; AB the transverse axis; PQ, RS, the tangents at the vertices of the hyperbolic sections; KGLg and MINi sections of the cone perpendicular to its axis; and KL, MN the intersections of these planes with the plane of the hyperbola.

Then, because KGLg and MINi are sections perpendicular to the axis of the cone, they are circles: and because AVB, the vertical plane passes through the axis of the cone, the lines KL and MN are diameters of these circles.

Also, since the planes KGL, AGB, are each perpendicular to the vertical plane AVB, their line of section gFG is perpendicular to that plane, and therefore to the lines AB and KL which are in it. Similarly iHI is perpendicular to AB and MN.

Also, KL, MN, being diameters of the circles and at right angles to GFg and IHi, these lines are bisected in F and H respectively.

Again, the plane AGB, and the tangent planes to the cone at VA, VB, being perpendicular to the vertical transverse plane, the intersections PAQ, RBS are also perpendicular to the plane AVB, and are, hence, parallel to the lines G F g and IH i.

But QAP, being in the tangent plane, touches the hyperbola at A, and meets it in no other point. It is also in the plane of the hyperbola, and is hence a tangent to the hyperbola at the point A. Similarly the line SBR is a tangent at the other extremity of the transverse axis.

The lines GF g, IH i are therefore in the plane of the hyperbola and parallel to the tangents at the extremities of the diameter AB; and being bisected in F and H respectively, they are the double ordinates to the diameter AB.

Now, by the similar triangles, AFL, AHN and BFK, BHM, we have

$$AF : AH :: FL : HN, \text{ and}$$
$$FB : HB :: KF : MH.$$

Hence, by compounding these ratios there results (Note, page 102)

$$AF \cdot FB : AH \text{ I } HB :: KF \cdot FL : MH \cdot HN.$$

But by the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$; and hence

$$AF \cdot FB : AH \cdot HB :: FG^2 : HI^2. \qquad \text{Q. E. D.}$$

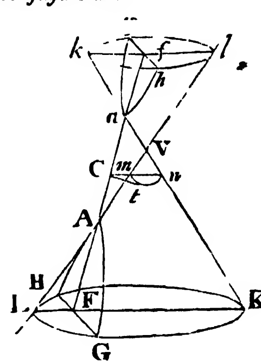
Cor. 1. The transverse axis AB, bisects its conjugate chords, Gg and Ii.

Cor. 2. The two tangents at the extremities of the transverse axis are parallel to one another and perpendicular to that axis.

Cor. 3. The ordinates of the transverse are perpendicular to that axis.

Cor. 4. The conjugate axis is at right angles to the transverse.

PROP. II. *As the rectangle of the abscisses is to the square of the ordinate, so is the square of the transverse axis to the square of the conjugate axis.*



Let a continued plane cut, from the two opposite cones, the two mutually connected opposite hyperbolas HAG, *hag*, whose vertices are A, *a*, and bases HG, *hg*, parallel to each other, falling in the planes of the two parallel circles LGK, *lgk*. Through C, the middle point of A*a*, let a plane be drawn parallel to that of LGK, it will cut the cone LVK in a circular section whose diameter is *mn;* to which circular section, let C*t* be a tangent at *t*.

Then, by sim. tri. AC*m*, AFL $AC : Cm :: AF : FL$;
And, by sim. tri. *a*C*n*, *a*FK $aC : Cn :: aF : FK$;
$\therefore AC \cdot Ca : Cm \cdot Cn :: AF \cdot Fa : LF \cdot FK$,
Or, $AC^2 : Ct^2 :: AF \cdot Fa : FH^2$.

In like manner, for the opposite hyperbola $AC^2 : Ct^2 :: Af \cdot fa : fg^2$.

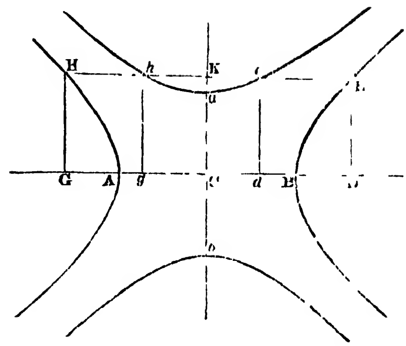
Here C*t* is the semi-conjugate to the opposite hyperbolas HAK, *hak*, by def. 30. Q. E. D.

Cor. 1. Since by the definition, $AB : ab :: ab : P$, AB has to P the duplicate ratio of AB to *ab* and (Geom. theor. 83, or Euc. vi. 20, cor. 2.) $AB : P :: AB^2 : ab^2$. Hence
$AB : P :: AC^2 - CD^2$ (= AD . DB, Euc. ii. 5, cor.) : DE^2. Or in words, *as the transverse is to its parameter, so is the rectangle of the abscisses to the square of their ordinate.*

Cor. 2. Ordinates DE, GH equi-distant from the centre, in the two branches of the hyperbola, are equal to one another.

For in each case the first three terms are equal in the above proportion, and hence also the fourth terms in each; or the squares of the ordinates are equal, and hence the ordinates themselves.

Cor. 3. Draw EH. Then it is parallel to the transverse AB, perpendicular to its conjugate *ab*, and is bisected by the latter in K.



SCHOLIUM

If the plane of the hyperbolas be parallel to the axis, it will cut the transverse plane AV*a* so that the triangle AV*a* will be isosceles, and hence the plane through C, the middle of its base, will pass through the vertex, and the line CV will be the conjugate semi-diameter, the circle *mCt* being then *reduced* to a point, and C*t*, C*m*, and C*n* all equal.

If now we suppose the two conjugate cones (def. 15.) to have their axes passing through V and at right angles to the transverse plane AVB, the plane of the axes will be parallel to the plane of the hyberbolic sections.

Let us suppose, then, that a second plane parallel to the plane of the axes, and at a distance from V equal CA be made to cut the pair of conjugate cones. This will give another pair of opposite sections G′A′H′ and *g′a′h′*, which (by def. 16.) are the hyperbolas conjugate to the original ones GAH and *gah*: and it is necessary to show that the relation of these two pairs of hyperbolas is mutual, or that the former pair GAH and *gah* are conjugate to G′A′H′ and *g′a′h′*.

Conceive the vertical transverse planes to be drawn to the two pairs of conjugate sections: then since these pass through the axes, they make in the conical surfaces linear sections, whose inclinations to the respective axes are equal to the generating conical angles. Also since these angles are (by def. 14.) the complements of each other, and by construction the perpendicular CV in the right angled triangle CVA is equal to the base C′V in the other C′V′A′, the triangles are equal, and the hypothenuse of the one VA is equal to the hypothenuse in the other VA′; and the base CA in the former equal to the perpendicular CA′ in the latter. The relation between the two diameters AB, A′B′ and the two distances CV and C′V is therefore the same, whichever be taken first. That is, if the section made through C′ be conjugate to that through C, the latter section will be in the same sense (def. 16.) conjugate to the former one.

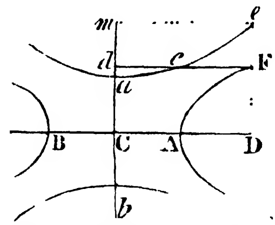
Cor. 4. The tangents at the vertices of the conjugate hyperbolas are parallel to the transverse and perpendicular to the conjugate axes.

Cor. 5. The ordinates to the conjugate hyperbolas at equal distances from the centre, and estimated from the transverse axis are equal. For (fig. cor. 1.) since C*a* is the transverse axis of the conjugate hyperbolas, hK $=$ Ke, and *he* is parallel to AB: hence $gh = de$.

Cor. 6. C*a* is the least ordinate that can be drawn to the conjugate hyperbola, estimating from the transverse axis AB.

For no point of the curve *hae* lies between the tangent at *a* and the centre of the curve C.

PROP. III. *As the square of the conjugate axis is to the square of the transverse, so is the sum of the square of that semi-conjugate diameter and distance of the ordinate to the square of that ordinate.*



That is,

$Ca^2 : CA^2 :: (Ca^2 + Cd^2) : dE^2$.

For, draw the ordinate ED to the transverse AB. Then Prop. II.

$Ca^2 : CA^2 :: DE^2 : (= Cd^2)\ AD \cdot DB$.

Or, alto. et compo. $Ca^2 : CA^2 :: Ca^2 + Cd^2 : AD \cdot DB + AC^2 = CD^2 = dE^2$.

In the same manner, taking *ab* as the transverse axis (or AB as the conjugate) we find $CA^2 : Ca^2 :: CA^2 + CD^2 : DE^2$.

Q. E. D.

Cor. 1. By prop. ii., $CA^2 : Ca^2 \quad CD^2 - CA^2 \quad DE^2$.
And by prop. iii., $CA^2 : Ca^2 \quad CD^2 + CA^2 \quad De^2$.
And Euc. v. 11., $DE^2 : De^2 \quad CD^2 - CA^2 \quad CD^2 + CA^2$.

Cor. 2. By similar reasoning, and taking *ab* as the transverse axis

$$De^2 : dE^2 :: Cd^2 - Ca^2 : Cd^2 + Ca^2.$$

Cor. 3. If the hyperbola be equilateral the square of the ordinate is equal to the rectangle of the abscisses; for in this case, the axes being equal, the first ratio (that of the squares of the axes) is a ratio of equality, and hence the second is also a ratio of equality.

Cor. 4. If two pairs of hyperbolas be described having the same transverse axis, their ordinates are in a constant ratio; and likewise if they have the same conjugate axis, their ordinates to it are in the same constant ratio.

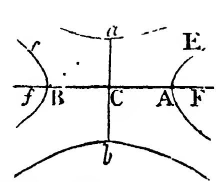
EXERCISE.

Problem. The transverse axis is 8, the conjugate 6; find the ordinates corresponding to the abscisses (in the primitive and conjugate hyperbolas) 0, 1, 2, 4, 8, 12, 16, and 20 on each side of the centre; and the abscisses answering to the ordinates 0, 1, 2, 3, 4, 10, 15, in *both* pairs of curves.

[*Note.*—Some of these will take an imaginary form.]

PROP. IV. *The square of the distance of the focus from the centre is equal to the sum of the squares of the semi-axes: or,*

The square of the distance of the foci is equal to the sum of the squares of the two axes.



That is, $CF^2 = CA^2 + Ca^2$, or

$Ff^2 = AB^2 + ab^2$.

For to F the focus, draw the ordinate FE. This, (by def. 27.) will be the semi-parameter. Then by prop. ii.

$CA^2 : Ca^2 :: CF^2 - CA^2 : FE^2$, and by def. 26.

$CA^2 : Ca^2 :: Ca^2 \qquad : FE^2$.

Hence (Euc. v. 11.) $Ca^2 = CF^2 - CA^2$, or $CA^2 + Ca^2 = CF^2$. Or, by doubling the lines in question, $AB^2 + ab^2 = Ff^2$. Q. E. D.

Cor. 1. The two semi-axes and the focal distance being made the sides of a triangle, it will be right-angled.

Cor. 2. The distance Aa is equal to the focal distance.

Cor. 3. The semi-conjugate Ca is a mean proportional between the distances of either focus from the vertices of the curve. For

$$Ca^2 = CF^2 - CA^2 = AF \cdot FB,$$
$$\text{And } Ca^2 = Cf^2 - CA^2 = Af \cdot fB.$$

Cor. 4. The foci of the four hyperbolas are in the same circle described about C, with the distance Aa.

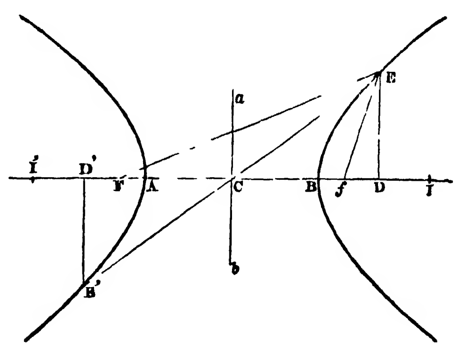
EXERCISES.

Solve for the hyperbola the four problems proposed respecting the ellipse at page 105.

PROP. V. *The square of a line drawn from the centre of an hyperbola to a point in the curve is equal to the difference of the squares of the semi-conjugate and distance of the dividing point from the centre:—*

Lines drawn from the foci of an hyperbola are equal, severally, to the distances of the extremities of the transverse diameter from the dividing point:—

Also, the difference of the lines drawn from the foci to any point in the hyperbola is equal to the transverse axis, and their sum to the sum of the segments of the transverse, made by the dividing point.



That is, if I be the dividing point in the axis, corresponding to the point E in the hyperbola, then,

1. $CI^2 = Ca^2 + CE^2$.
2. $EF = AI$,
3. $Ef = IB$,
4. $EF - Ef = AB$
5. $EF + Ef = 2CI = II'$, if $I'C = IC$.

For by the def. of dividing point (33.) and (Geom. theor. 89, or Euc. vi. 22.)

$CB^2 : Cf^2 :: CD^2 : CI^2$, or convertendo (Geom. theor., or Euc. v. E.)

$CB^2 : Ca^2 :: CD^2 : CI^2 - CD^2$;

But $CB^2 : Ca^2 :: CD^2 - CB^2 : DE^2$ (prop. ii.): hence,

En equali $CB^2 : Ca^2 :: CB^2 : CI^2 - CD^2 + DE^2 = CI^2 - EC^2$; or (Euc. v. 9.) $Ca^2 = CI^2 - CE^2$. Hence,

$$CI^2 = Ca^2 + CE^2. \qquad \text{Q. E. 1}^{\text{mo}} \text{ D.}$$

Again, $FE^2 = FC^2 + CE^2 + 2FC \cdot CD$ (Geom. theor. 36, or Euc. ii. 13.)
$= FC^2 + CI^2 - Ca^2 + 2CB \cdot CI$, def. (33.)
$= IC^2 + CA^2 + 2CB \cdot CI$ $(2CA \cdot CI)$
$= (AC + CI)^2$, (Geom. theor. 33, or Euc. ii. 4.)

Or $EF = AI$. Q. E. 2ndo D.

Also, $fE^2 = fC^2 + CE^2 - 2fC \cdot CD$. (Geom. theor. 36, or Euc. II. 13.)

$= fC^2 + CI^2 - Ca^2 - 2Cf \cdot CD,$

$= CA^2 - 2Cf \cdot CD + CI^2$ (prop. IV. cor. 3.)

$= CA^2 - 2CA \cdot CI + CI^2$ (def. 33.)

$= (CI - CA)^2$

Or $fE = CI - CA = CI - CB = BI$. Q. E. 3^tio^ D.

Hence $fE - fE = AB$ . . . . Q. E. 4^to^ D.

$FE + fE = AI + BI = 2CI = II'$ . . . Q. E. 5^to^ D.

Cor. 1. Similar properties obviously exist amongst the corresponding lines in the conjugate hyperbolas.

Cor. 2. Let a point D′ be taken in the hyperbola opposite to EB, such that CD′ = CD. Then CE′ = CE, and E′CE is one straight line bisected in C. For D′E′ = DE (as is easily proved) and CD′ = CD, and the angles at D′ and D right angles. Hence CE′ = CE, and the angles D′CE′, DCE equal. Hence, &c.

Cor. 3. Hence every diameter is bisected by the centre of the curve.

Cor. 4. Hence if by any contrivance (and there are several) a string can be fixed at two points F and *f* and kept stretched, so that the difference between its parts are always of the same length, a pencil moving in the intersection will trace an hyperbola.

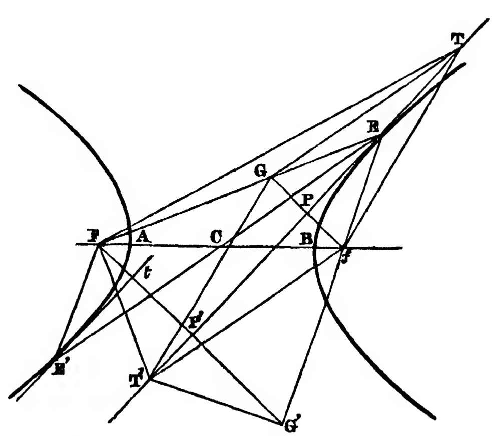
Cor. 5. Or, it may be done by points as in the ellipse, see prop. v. page 106.

EXERCISES.

1. *Problem.* Find the position of two diameters of an hyperbola, which shall be equal to one another and contain a given angle.

2. *Theorem.* If a circle be described on the conjugate axis, a tangent drawn to it from the dividing point I is always equal to CE.

Prop. VI. *Let the angle made by the focal lines be bisected: then the bisecting line is a tangent to the hyperbola at that point.*



Thus, let FE*f* be the lines drawn from the foci to any point E in either of the opposite hyperbolas, the line TET′ bisecting FE*f* is a tangent to the curve at E.

For, if ET be not a tangent, it will cut the curve in some other point, let it be at T. Join FT, *f*T, and from one of the foci, as *f*, draw *f*P perpendicular to the line TT′ and produce it to meet FE at G, and join GT.

Then, since the angles at P are right angles, they are equal; and by hypothesis the angles GEP, *f*EP are equal; and the side PE common to the two triangles GEP, *f*EP: hence the sides GE, E*f* are also equal, and GF their difference is (prop. v.) equal to AB the transverse axis.

Also, because $fP = PG$, and PT common to the two right-angled triangles PTf and PTG, we have $GT = Tf$. Hence the difference of FT and Tf is equal to the difference of FT and TG, and therefore (Geom. theor. 11.) less than the third side FG of the triangle FTG.

But by the admission T is a point in the curve, and hence the difference between FT and Tf (that is between FT and TG) is equal to AB (prop. v.) that is, as has been shown, to FG. Hence FG is equal to AB and greater than AB, which is absurd. That is, T is not a point in the curve.

Neither does PT meet the *opposite* hyperbola.

For, suppose it does meet it in some point T′, and draw FG′ perpendicular to ET, and produce FP′, fE to meet at G′, and join G′T′.

Then by a reasoning similar to that employed in the last case, it will readily appear that $FT' = T'G'$, and that $fG' = FG = AB$. But since by the admission T′ is a point in the hyperbola, the difference between fT′ and T′F, that is between fT′ and T′G′ is equal to fG′: whilst (by Geom. theor. 11.) that difference is less than fG′. And the point T′, it hence follows is not in the opposite hyperbola.

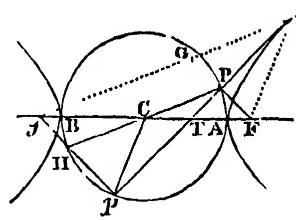
The line TE is therefore a tangent to the hyperbola at the point E. Q. E. D.

Cor. 1. The tangents at the opposite extremities of any diameter are parallel; and parallel tangents are at the opposite extremities of a diameter.

For draw EC and produce it till $CE' = CE$. Then (prop. v. cor. 4.) the point E′ is in the opposite hyperbola. Hence the diagonals EE′ and fF being bisected at their intersection, the figure Ef E′F is a parallelogram, and the angle $FEf = FE'f$. Draw the tangent E′t, which bisects FE′f by the proposition; and hence is parallel to ET.

The converse follows, *ex absurdo.*

Cor. 2. The distance of the point where the perpendicular from the focus to any tangent, meets that tangent, is equal to the semi-transverse axis.



For draw CP: then since CP bisects the sides of the triangle fFG, it is parallel to FE and equal to half FG, that is to half AB. Hence $CP = CA$ or CB.

Cor. 3. The intersections of the perpendiculars upon the tangents with the tangents themselves will always be in the circumference of a circle on the transverse axis.

Cor. 4. The perpendiculars have the same ratio as the focal distances, and so likewise have the intercepted portions of the tangents. For

$$FP : fp :: FE : Ef :: EP : Ep,$$ by similar triangles FPE, fpE.

Cor. 5. The rectangle contained by the perpendiculars from the foci upon any tangent is equal to the square of the semi-conjugate diameter.

For join HC (H being the intersection of fp with the circle): then since HpP is a right angle, it is in a semicircle, and hence HP passes through the centre C, and coincides with HC; and FHPG is a parallelogram, and $fH = GP = PF$. Hence, also, $FP . fp = fH . fp = fB . fA = fC^2 - fA^2 = Ca^2$, Ca being the semi-conjugate (prop. iv.)

Cor. 6. If perpendiculars be drawn from the foci and centre to a tangent, that from the centre is half the difference of those drawn from the foci.

Cor. 7. A tangent to the hyperbola always cuts the transverse axis between the centre and the vertex of that curve to which it is a tangent. This is evident from the reasoning in Cor. 1, above.

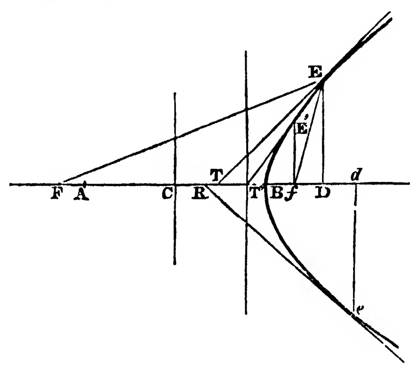
EXERCISES.

1. *Problem.* Draw a tangent to the hyperbola from given point without it. See Ellipse, page 107.

2. *Theorem.* There can only be two tangents drawn to the pair of opposite hyperbolas from the same point; nor to their conjugate hyperbolas more than two.

Distinguish between the cases when both tangents can be drawn to the same, and when one to each of the opposite hyperbolas: and shew what takes place when the point is in the asymptotes.

PROP. VII. *If a tangent and an ordinate be drawn from any point in the curve of an hyperbola, the semi-transverse axis a mean proportional between the distances of the tangent and the ordinate with the transverses.*



That is,
$CD : CB :: CB : CT$, or
$CD \cdot CT = CB^2$.
For draw fE, EF;
Then $FT : Tf :: FE : Ef$,
Or, $FT + Tf : FT - Tf ::$
$FE + Ef : FE - Ef$; that is,
$2Cf : 2CT ::$
$FE^2 - Ef^2 : (FE - Ef)^2$,
or (Geom. theor. 80, or Euc. vi. 1.)
$FD \quad - Df^2 : 4CR^2$, or
(Geom. theor. 33.)
$:: 2CD \cdot 2Cf : 4CB^2$, or (Geom. theor. 33, or Euc. ii. 3.)

Whence $Cf \cdot CD : CD \cdot CT :: CD \cdot Cf : CB^2$: and (Geom. theor. 80, or Euc. vi. 9.),

$CD \cdot CT = CB^2$, or (Euc. vi. 1.)

$CD : CB :: CB : CT$ (Geom. theor. 81, or Euc. vi. 17.)

Q. E. D.

Cor. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every hyperbola described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

Cor. 2. $AD : DB :: AT : TB$, componendo et dividendo of the proposition.

Cor. 3. $AD \cdot DB = CD \cdot DT$; for each is equal to $CD^2 - DB^2$.

Cor. 4. Let another tangent be drawn to any point e, viz. eR. Then also $CB^2 = CR \cdot Cd$. Hence $CD \cdot CT = CR \cdot Cd$, or $CR : CT :: CD : Cd$.

Cor. 5. CD CT BD² : BT².
For CT CB CB : CD, by the proposition,
Or BD BT CD : CB, dividendo.
Or BD² BD² ; CD² : CB, (Geom. theor. 89, or Euc. vi. 22.)
CD : CD, by duplicate ratio.

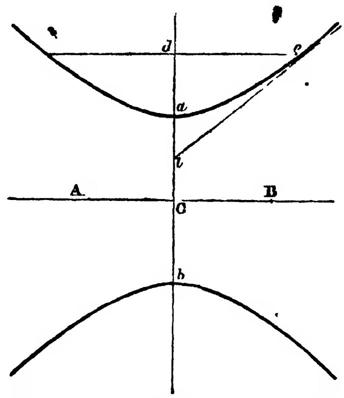
Cor. 6. If D be the focus, then TH being drawn perpendicular to the cones

AB, it is the *directrix* (def. 34.): and by this proposition the semi-transverse is a mean proportional between the distances of the focus and directrix from the centre.

Cor. 7. Let the tangent meet the conjugate axis at t, and an ordinate also be drawn to meet it at d. Then,

$$\left.\begin{array}{l} Cd : Ca :: Ca : Ct \\ da^2 : at^2 :: Cd : Ct \\ bd : da :: bt : ta \\ bd \,.\, da = Cd \,.\, dt. \end{array}\right\}$$

For these are related to the conjugate hyperbolas in precisely the same way that the corresponding parts are to the primary hyperbolas; and the same proof applies to them.



PROP. VIII. *The distance of the directrix from a point in the hyperbola has to the distance of the focus, from the same point, the same ratio wherever in the curve that point be taken.*

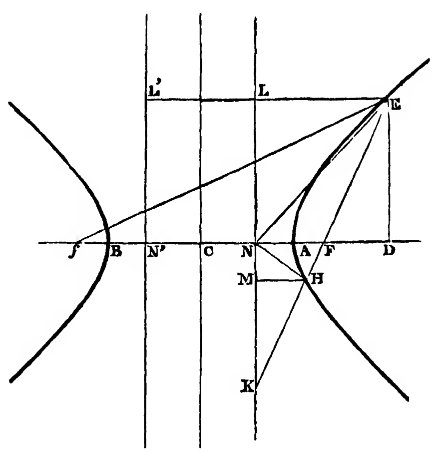
That is, FE : EL :: fE : EL′ :: CF : CA, the determining ratio.

For, CF : CA :: CA : CN, by the definition (34) and Cor. 6, prop. vii.

And CF : CA :: CI : CD, by def. of dividing point (Def. 33.)

Hence CF : CA :: CA : CN :: CI : CD :: CI − CA : CD − CN :: AI : ND (= EL) :: FE : EL (prop. v. 2.) and in the same way may it be shown of fE : EL′.

Q. E. D.



Cor. 1. If any line HE be drawn through the focus F, and HN, EN be drawn these lines will make equal angles with the axis.

For draw the ordinates ED, HP to meet the axis in D and P, and likewise the lines HM, EP parallel to the axis.

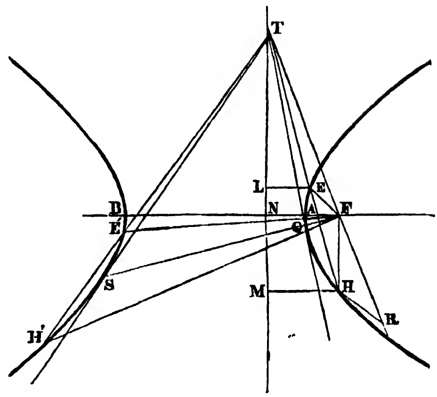
Then by the proposition HM (=NP) : EL (= DN) :: HF : FE :: HP : DE, by similar triangles. Hence the right-angled triangles NPH, NDE are similar, and the angles HNP, DNE equal.

Cor. 2. The chord HE is so divided by the focus and directrix in F and K, that EF : FH :: EK : KH, or KH : HF :: KE : EF.

For by sim. triangles KH : KE :: MH : HL, or, by the proposition, :: HF : FE.

PROP. IX. *If two lines be drawn from any point in the directrix of an hyperbola to cut the curve, one of which passes through the focus, and lines be drawn*

from that focus to the points where the other line cuts the curve, these will make equal angles with the first-named line drawn through the focus.



That is, EFT = HFR.

For draw the lines EL and MH parallel to the axis, and HR parallel to EF.

Then by parallels EF : HR :: TE : TH :: LE : MH

But by prop. viii. LE : MH :: EF : FH

Hence (Euc. v. 114.)

EF : HR :: EF : FH

Hence (Euc. v. 9.)

HR = HF, and angle HFR = HRF = EFT. Q E D.

Precisely the same mode of reasoning applies to the case when TEH cuts the opposite hyperbola, as TE′H′.

Cor. 1. If the secant TEH become a tangent TQ, the points E and H coalesce, and the angles QFR, QFT are equal, or the angle TFQ is a right angle.

In like manner, if the tangent TS be drawn (to the same or the opposite hyperbola, as the case may admit) TFS is also a right angle.

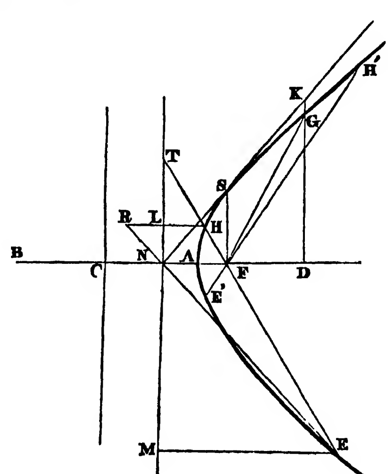
Cor. 2. If from any point in the directrix, tangents be drawn to the same or opposite hyperbolas, the chord joining the points of contact passes through the corresponding focus.

For by the last corollary, TFS, TFQ are right angles, and hence the points Q, F, S are in one right line, or QS passes through Q.

Cor. 3. The line QF bisects the angle EFH.

Cor. 4. The line HE is so divided in K by FQ that EK : KH :: ET : TH.

Prop. X. *A straight line drawn through the focus is divided in that point so that four times the rectangle contained by its segments is equal to the rectangle contained by the line itself and the parameter of the hyperbola.*



That is, 4HF . FE = P . HE.

Draw the lines HL, EM, and HN, NE as before, and produce EN to meet HL in R; and draw FS the semi-parameter.

Then (cor. 1. prop. viii.) the angles NHL, ENM are equal; and RNL = ENM. Hence RL = LH, and RH = 2HL.

By similar triangles, EH : EF :: 2HL : NF, or 2HL . EF = NF . EH.

Also, by prop. viii.

HL : HF :: NF : FS.

And by equality

2FE : 2FE :: HE : HE.

Hence, note page 102, 2HL . FE : 2HF . FE :: NF . HE : FS . HE.

But it has been proved that the

first term is equal to the third, and hence (Euc. v. 9.), the second is equal to the fourth. That is,

2HF . FE = FS . HE, or doubling

4 . HF . FE = 2FS . HE = P . HE. Q. E. D.

Cor. 1. If another line H′E′ also passes through the focus, the rectangles of the segments have the same ratio as the lines themselves : or

HF . FE : H′F . FE′ : : HE : H′E′.

Cor. 2. Draw the focal tangent NS, and produce any ordinate DG to meet it in K and join FG. Then DK = FG.

For, (prop. ix.) SF is perpendicular to NF, and therefore parallel to DK. By similar triangles, NFS, NKD; therefore,

NF : FS : : ND : DK

And by viii, NF : FS : : ND : FG. Hence DK = FG.

EXERCISES.

1. *Theorem.* The chords drawn through the focus are proportional to the squares of their parallel diameters.

2. *Problem.* Given the position of a line passing through the given focus, the position of the conjugate axis, and the ratio of the two axes, to find where it will cut the ellipse.

3. *Theorem.* The perpendicular from the centre C intercepted by the focal tangent is equal to the semi-transverse axis.

PROP. XI. *If there be any tangent meeting four perpendiculars to the axis drawn from these four points, namely, the centre, the two extremities of the axis, and the point of contact; those four perpendiculars will be proportionals.*

That is,

AG : DE : : CH : BI.

For, by prop. vii. cor. 1.

TC : AC : : AC : DC,

Therefore, dividendo,

TA : AD : : TC : AC (= CB,)

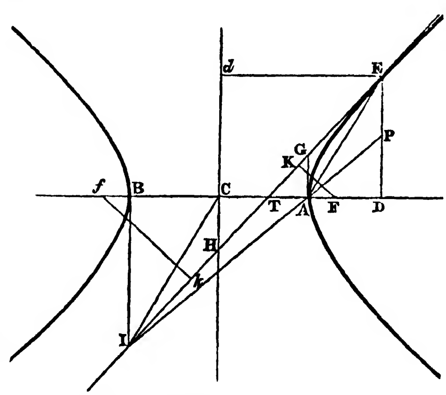
And by componendo,

TA : TD : : TC : TB,

And by sim. triangles,

AG : DE : : CH : BI.

Q. E. D.



Cor. 1. Hence TA, TD, TC, TB and TG, TE, TH, TI, are also proportionals.

For these are as AG, DE, CH, BI, by similar triangles.

Cor. 2. Draw AI to intersect DE in P; then since TA : TE : : TC : TI, the triangles TAE, TCI are similar, as well as the triangles AED, CBI, and ADP, ABI.

Hence AD : DE : : CB : BI,

And AD : DP : : AB : BI;

Therefore DE : DP : : AB : CB : : 2 : 1; *which suggests another simple practical method of drawing a tangent to an hyperbola.*

Cor. 3. The rectangle AG . BI = Ca^2, draw E*d* parallel to AB. Then C*d* = DE. Hence by this theorem and Euc. vi. 16, ED . HC = C*d* . CH = Ca^2.

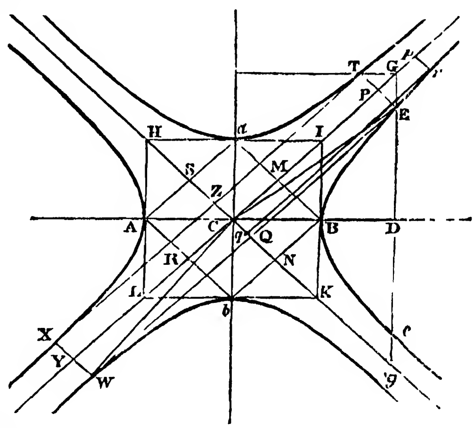
Cor. 4. Draw the perpendicular from the foci, viz. fk and FK. Then KF . kf = AG . BI. For by last cor. AG . BI = Ca^2, and by cor. 5, prop. vi., KF . kf = Ca^2. Hence, &c.

Cor. 5. Precisely similar properties obtain when the conjugate hyperbolas are substituted for the primitive ones.

SECTION II.

THE ASYMPTOTES OF THE HYPERBOLA.

Prop. A. *All the parallelograms formed by lines drawn from points in the hyperbola parallel to the asymptotes and the segments of the asymptotes, which they cut off, are equal to one another.*



Let AB, ab be the transverse and conjugate diameters, and HIKL the conjugate rectangle (def. 19): then HK and LI are the asymptotes (def. 18). Then if from E a point in the hyperbola, lines EP, EQ be drawn parallel to HK and IL, the parallelogram PEQC will be equal to CMBN.

Draw the ordinate EDe, and let it meet the asymptotes in G and g. Then, by similar triangles CBI, CDG, we have (Geom. theor. 74. Euc. vi. 22.)

$CB^2 : BI^2 :: CD^2 : DG^2$; and by prop. 2.

$CB^2 : BI^2 (= CA^2) :: AD . DB : DE^2$. Hence (Euc v. 14.)

$CD^2 : DG^2 :: AD . DB : DE^2$, and therefore, alto. et convert.

$CD^2 : CD^2 - AD . DB :: DG^2 : DE^2 - DG^2$.

But $CD^2 - AD . DB = CB^2$, and $DE^2 - DG^2 = GE . Eg$: hence

$CD^2 : CB^2 :: DG^2 : GE . Eg$, or, alternando,

$CD^2 : DG^2 :: CB^2 : GE . Eg$.

But $CD^2 : DG^2 :: CB^2 : BI^2$; hence (Euc. v. 11 and 9) $BI^2 = GE . Eg$.

Again, by similar triangles GPE, IMB and EQg, BNK,

PE : MB :: EG : IB, and

QE : BN :: Eg : BK: hence (note, page 102.)

PE . EQ : MB . BN :: GE . Eg : IB . BK.

But IB . BK = BI^2, and BI^2 = GE . Eg; hence also PE . EQ = MB . BN.

Or PE : MB :: BN : EQ. Hence (Geom. theor. 76, or Euc. vi. 16). the parallelograms PEQC and MBNC are equal. Q. E. D.

Cor. 1. If lines be drawn parallel to one asymptote of the hyperbola to meet two *conjugate* hyperbolas, they are bisected by the other asymptote. Thus, if ET be drawn parallel to HI, meeting IL in P, then EP = PT.

For, as shown in the proposition, the parallelogram TC is equal to Sa MC: but Sa MC = MCNB. Hence the parallelogram TC = CE. But they are

equal and between the same parallels, hence they stand on equal bases TP and PE, as affirmed in the enunciation of the corollary.

In the same manner it will appear, that if lines be drawn from E and T parallel to IL, they will be bisected by HI, viz. TX in Z and EW in Q.

Cor. 2. If TE be drawn parallel to the asymptote HK, and from T and E parallels to the other asymptote be drawn, meeting the hyperbolas in W and X; then WX being joined, it will be parallel to the first asymptote HK, and the figure TEWX is a parallelogram.

For by Cor. 1. $PZ = ZX$ and $EQ = QW$: hence $ZX = QW$, and they are parallel. Hence (Euc. i. 33, or Geom. theor. 24) WX is equal and parallel to ZQ, that is to TE (by Geom. theor 15, or Euc. i. 31.)

Cor. 3. The lines EC, CX are one straight line, or a diameter of the hyperbolas, as are, also, TC, CW.

For, $PE = WY = YX$, and $PC = CY$, and the alternate angles XYC, CPE equal; hence the two angles XCY, PCE are equal (Geom. theor. 2, or Euc. i. 4.) and they are vertical angles; hence again (Geom. theor. 7, conv.) the lines XC, CE are one straight line. In the same way it is shown that PC, CW are one straight line.

Cor. 4. Any parallelogram formed of lines parallel to the asymptotes, and having their vertices in the four hyperbolas, will be equal to the parallelogram A*a* B*b*, or to half the rectangle AB . *ab*.

For, the several constituent parallelograms into which the whole parallelogram TEWX is divided by the asymptotes are severally equal to the four constituent parallelograms into which the parallelogram *a*B *b*A is divided by the same asymptotes: and these latter are respectively the halves of the rectangles into which the rectangle HIKL is divided by the axes AB *ab*, of the figure.

Cor. 5. If any line G*g* be drawn parallel to one of the axes as to *ab* of an hyperbola, the rectangle of the segments GE . E*g*, (or G*e* . *eg*) intercepted between the curve and the asymptotes is always of the same magnitude, viz. the square of the semi-axis to which it is parallel, that is $GE . Eg = Ge . eg = Ca^2$.

This was shown in the demonstration of the proposition itself, for GE . E*g*, and the same mode of proof applies to G*e* . *eg*.

Cor. 6. The segments intercepted between the curve and the asymptote are equal, viz. $GE = ge$ and $Ge = Eg$.

For by props. 2 and 3, we have $DE = De$; and by the isosceles triangle GC*g*, the angle GC*g* of which is bisected by CD, we have $DG = Dg$. Hence the conclusion is manifest.

Cor. 7. The curve and its asymptotes never meet.

For, by the demonstration of the proposition we have $BI^2 = GE . Eg = DG^2 - DE^2$, or $DG^2 = BI^2 + DE^2$. Hence DG is greater than DE, and the point G lies, therefore, out of the curve. And as the same property holds good wherever in the curve the point E is taken, the line CG lies wholly out of the curve, and consequently never meets it.

Cor. 8. The curve and its asymptote approach more nearly than any distance which can be assigned.

For, let PE be the distance assigned; and take C*p* in the asymptote greater than CP, and draw *pr* parallel to PE meeting the curve in *r*, and draw *rg* parallel to *cp*. Then $cp : CP :: PE : pr$. But C*p* is greater than CP, and hence PE is greater than *pr* (Euc. v. 14). Now as the same circumstances take place wherever in the asymptote the point P be taken, it follows that however small PE may be assigned a point *p* more distant from C than P is, will

give a distance between the curve and the asymptote still less than the distance PE.

Cor. 9. If any number of distances from C be taken on either asymptote continued proportionals, the lines drawn from them to meet the curve and be parallel to the other asymptote, will also be continued proportionals.

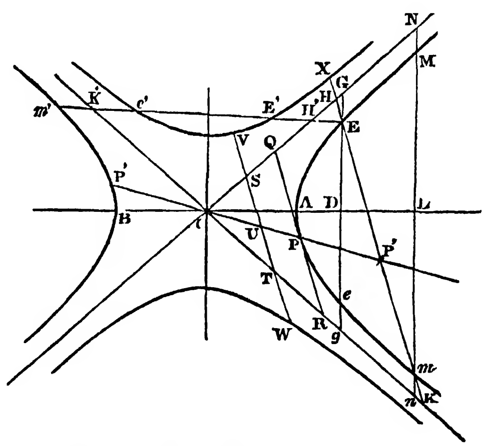
EXERCISES.

1. Given the asymptotes of an hyperbola, to find its axes and parameter when it passes through a given point.

2. Given the inclination of the asymptotes $= 42° \ 10'$ to find the absciss and ordinate of a point whose distance from the focus is 10, the ratio of the axes being 5 to 3, and distance from the centre 18.

3. Given the length of a tangent intercepted by the asymptotes, the ratio of the intercepted asymptotes cut off by it, and the length of a perpendicular from the centre upon that tangent, to find the inclination of the asymptotes, both by construction and calculation.

Prop. B. *If any straight line be drawn to meet an hyberbola, or opposite hyperbolas, the segments intercepted between the curve and its asymptotes are equal.*



Let the line HK cut the asymptotes in H and K and the curve in E and m: then HE $= m$K and Hm $=$ EK.

For, draw the lines GEg, nmN perpendicular to the axis, meeting it in D and L. Then by parallels, the triangle EgK is similar to mnK, and HEG to HmN. Hence (Theor. 86, or Euc. vi. 5.)

$nm : m\text{K} :: g\text{E} : \text{EK}$, and $m\text{N} : m\text{H} :: \text{GE} : \text{EH}$.

Hence (note page 102).

$$nm \,.\, m\text{N} : \text{K}m \,.\, m\text{H} :: \text{GE} \,.\, \text{E}g : \text{KE} \,.\, \text{EH}$$

But (prop A, cor. 4.) $\text{GE} \,.\, \text{E}g = nm \,.\, m\text{N}$: and hence (Euc. v. 9)

$$\text{K}m \,.\, m\text{H} = \text{KE} \,.\, \text{EH}, \text{ or } \text{K}m\,(m\text{E} + \text{EH}) = \text{EH}\,(\text{E}m + m\text{K}), \text{ or}$$

$$\text{K}m \,.\, m\text{E} + \text{K}m \,.\, \text{EH} = \text{EH} \,.\, \text{E}m + \text{EH} \,.\, m\text{K}, \text{ whence}$$

$$\text{K}m \,.\, m\text{E} = \text{HE} \,.\, \text{E}m, \text{ or HE} = m\text{K}.$$

And adding Em to both, Hm $=$ EK.

Similarly, $\text{EH}' = m'\text{K}'$ and $\text{EK}' = m\text{H}'$. Q. E. D.

Cor. 1. If the line HK move parallel to its present position, till it touch the hyperbola as at P, then QP $=$ PR, or the intercepted tangent is bisected at the point of contact.

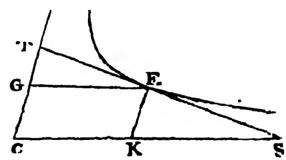
Cor. 2. If any line as, HK be drawn parallel to a tangent QR, both being intercepted by the asymptotes, the rectangle of its segments made by the curve will be equal to the square of half the intercepted tangent: that is $\text{HE} \,.\, \text{EK}$ (or $\text{H}m \,.\, m\text{K}) = \text{PQ}^2$.

Cor. 3. Any straight line terminated by the opposite hyperbolas is bisected by the diameter drawn to the point of contact of a tangent which is parallel to that line.

For, let VW be parallel to the tangent QR, and CV be joined, cutting VW in U. Then since QR is parallel to ST, and is bisected in P (cor. 1.), we have also SU = UT. Also, since through the conjugate hyperbolas the line VW is drawn meeting the asymptotes in W and S, VS = TW, by this proposition. Hence VU = UW.

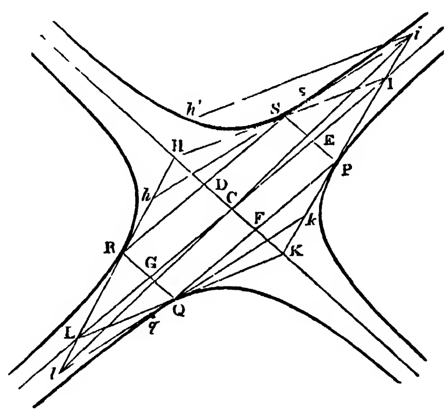
Cor. 4. All parallel chords are bisected by the same diameter, and that diameter is conjugate to them. (See def. 22.)

Cor. 5. Every inscribed triangle formed by a tangent and the intercepted points of the asymptotes is a constant quantity: viz. double the inscribed parallelogram. Thus, the triangle TCS is double of the parallelogram GEKC.



Prop. C. *If a parallelogram be described to touch the four hyperbolas, and have one of its angular points in an asymptote, it will have all its angular points upon the two asymptotes, and will be a conjugate parallelogram (def.* 19) *: and no parallelogram not having its angular points so situated can be constructed that shall be a conjugate one.*

Let any point I be taken in the asymptote CI. from which let IK, IH be drawn touching the hyperbolas in P and S, and draw HR, KQ tangents to the other two hyperbolas at Q and R: then their point of intersection L will be in the first asymptote, and the figure HIKL will be a conjugate parallelogram. And no parallelogram whose angular points are not in the asymptotes is a conjugate one.



Because IH, IK are bisected at the points of contact S and P, the line SP is bisected in E at its intersection with the asymptote (Prop. B. cor. 1.): also (Euc. vi. 2, or Geom theor. 82), SP is parallel to HK. In the same manner RQ is parallel to HK; and hence also to SP. Similarly, RS and PQ are parallel to the other asymptote IL, and the figure SPQR is a parallelogram.

So in RC, CP, SC, SQ: and because SPQR is a parallelogram inscribed between the four hyperbolas, SP = RQ and RS = QP, and (Prop. B cor. 3.) the diagonals RP and QS pass through the centre, and are diameters of the curve. But RP bisects the sides HL and IK, and SQ bisects the sides HI, KL, and hence these diameters are parallel to the sides of the parallelogram HIKL, touching the four hyperbolas: that is, each diameter is parallel to the tangents drawn to the curves from the extremities of the other diameter, or they are conjugate diameters (def. 22) and the figure HIKL is a conjugate parallelogram (def. 19.) Q. E. 1mo D.

Secondly, No parallelogram whose angular points are situated otherwise than in the asymptotes can be a conjugate parallelogram.

For, let the parallel tangents *ki*, *ih*, be drawn touching the opposite hyperbolas at P and R, and *ih*, *kl* touching their conjugate hyperbolas at *s* and *q*: and let the angular points *h*, *i*, *k*, *l*, not be in the asymptotes. Then it is to be proved that *hi* and *kl* are *not* parallel to RP.

Let *ik* and *hl* cut the asymptotes in I and L, and draw the tangents IH, LK from I and L to the conjugate hyperbolas: these will be parallel to RP by the preceding demonstration. Join S*i*. Then, since *i* is a point not in the tangent to the curve at S, and *i* lies without the curve, S*i* cuts the curve in some point. Hence the tangent from *i* to the curve lies between S*i* and *i*C, and consequently it will intersect SI in some point. It is hence not parallel to SI, and consequently not parallel to RP. Hence, the parallelogram *hikl* not having its sides parallel to the diameters drawn to the alternate points, its contact is not a conjugate parallelogram. Q. E. 2[do] D.

Cor. 1. If one diameter of a parallelogram be conjugate to another diameter, the second is conjugate to the first.

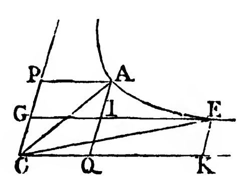
Cor. 2. All conjugate parallelograms described to touch the four hyperbolas are equal to one another, and to the rectangle contained by the two axes.

For, the parallelogram HIKL is composed of the four equal parallelograms CSIP, CPKQ, CQLR, and CRHS, which are respectively double of the four equal triangles SCP, CCQ, QCR, RCS. But these last four compose the parallelogram PQRS, which (by prop. A. cor. 4.) is equal to half the rectangle of the axes: and hence the parallelogram HIKL is equal to the rectangle contained by the axes, and therefore, always of the same magnitude.

Cor. 3. If the line CP be produced to cut any chord parallel to it in P′, that chord is bisected in P′, and P′E, P*m* are ordinates to the diameter CP. Fig. Pr. B.

Prop. D. *The three following spaces between the asymptotes and the curve, are equal; namely, the sector or trilinear space contained by an arc of the curve and two radii, or lines drawn from its extremities to the centre; and each of the two quadrilaterals, contained by the said arc, and two lines drawn from its extremities parallel to one asymptote, and the intercepted part of the other asymptote.*

That is,
The sector CAE = PAEG = QAEK,
all standing on the same arc AE.

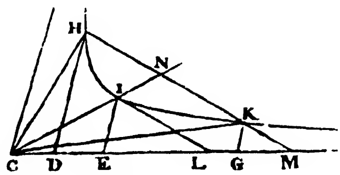


For, by prop. A, CPAQ = CGEK;
take away the common space CGIQ, and
there remains the parallelogram PI equal to the parallelogram IK.
To each add the trilineal space IAE; then
the quadrilateral space PAEG is equal to the quadrilateral space QAEK.

Again, from the quadrilateral CAEK
take the equal triangles, CAQ, CEK,
and there remains the sector CAE equal to QAEK.
Therefore CAE = QAEK = PAEG. Q. E. D.

Prop. E. *If from the point of contact of any tangent, and the two intersections of the curve with a line parallel to the tangent, three parallel lines be*

drawn in any direction, and terminated by either asymptote; those three lines shalt be in continued proportion.



That is, if HKM and the tangent IL be parallel, then are the parallels to the other asymptote DH, EI, GK in continued proportion.
For, by the parallels, HD, IE, GK, and IL, HM,

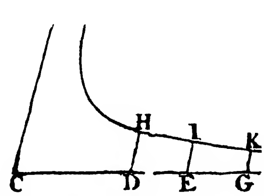
	EI IL :: DH : HM,
and,	EI IL :: GK : KM;
therefore	EI^2 IL^2 :: DH . GK : HM . MK;
but by (prop. B. cor. 2.)	HM . MK = IL^2;
and therefore	DH . GK = EI^2,
or, (Euc. vi. 17.)	DH : EI :: EI : GK. Q. E. D.

Cor. 1. Draw the semi-diameters CH, CIN, CK; then shall the sector CHI = the sector CIK.
For, since HK and all its parallels are bisected by CIN,
therefore the triangle
CNH tri. CNK, and the segment INH = seg. INK;
consequently the sector CIH = sector CIK.

Cor. 2. If the geometrical proportionals DH, EI, GK be parallel to the other asymptote, the spaces DHIE, EIKG will be equal; for they are equal to the equal sectors CHI, CIK.

So that by taking any geometrical proportionals CD, CE, CG, &c. and drawing DH, EI, GK, &c. parallel to the other asymptote, as also the radii CH, CI, CK; then the sectors CHI, CIK, &c. or the spaces DHIE, EIKG, &c. will be all equal among themselves. Or, which is the same thing, the sectors CHI, CHK, &c. or the spaces DHIE, DHKG, &c. will be in arithmetical progression. From which it follows that these sectors, or spaces, will have the same ratios as the logarithms of the lines or bases CD, CE, CG, &c.; namely, CHI or DHIE
the log. of the ratio of CD to CE, or of CE to CG, &c.;
or again of EI to DH, or of GK to EI, &c;
and CHK or DHKG the log. of the ratio of CD to CG, &c.
or of GK to DH, &c.

SCHOLIUM.



In the annexed figure if CD = 1, and CE, CG, &c. be any numbers, the hyperbolic spaces HDEI, IEGK, &c. are analogous to the *logarithms* of those numbers. For, whilst the numbers CD, CE, CG, &c. proceed in geometrical progression, the correspondent spaces proceed in arithmetical progression; and therefore, from the nature of logarithms are respectively proportional to the logarithms of those numbers. If the angle C were a right angle, and CD=DH = 1; then if CE were = 10, the space DEIH would be 2·30258509, &c.; if

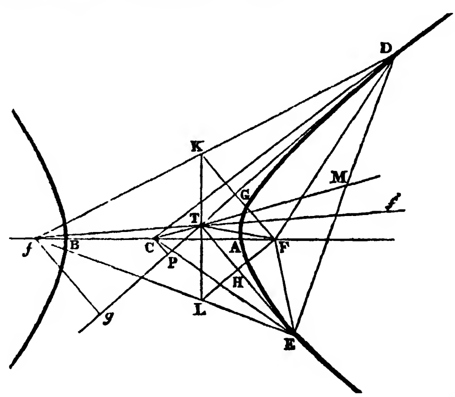
CG were $= 100$, then the space DGKH would be 4·60517018 . these being the Napierean logarithms to 10 and 100 respectively. Intermediate areas corresponding to intermediate abscissæ would be the appropriate logarithms. These are usually called *Hyperbolic* logarithms; but the term is improper: for by drawing other hyperbolic curves between HIK and its asymptotes, other systems of logarithms would be obtained. Or, by changing the angle between the asymptotes, the same thing may be effected. Thus, when the angle C is a right angle, or has its sine $= 1$, the hyperbolic spaces indicate the Napierean logarithms; but when the angle is $25^\circ\ 44'\ 27\frac{1}{4}''$, whose sine is $= \cdot 43429448$, &c. the modulus to the common, or Briggs's, logarithms, the spaces DEIH, &c. measure those logarithms. In both cases, if spaces to the right of DH are regarded as *positive*, those to the left will be *negative;* whence it follows that the logarithms of numbers less than 1 are negative also

SECTION III.

ON THE PROPERTIES OF THE HYPERBOLA DEPENDING UPON OBLIQUE DIAMETERS.

PROP. XII. *If lines be drawn from the points of contact of any two tangents to an hyberbola to the focus, and also a third line from the intersection of those tangents to the focus. the angle formed by the two former is bisected by the latter; the lines drawn from the intersection of the tangents make equal angles with the tangents; and the line drawn from the intersection of the tangents to the centre of the hyperbola bisects the chord which joins the points of contact.*

Let AB be the transverse axis; F, f, the foci; C the centre; and DT, ET the tangents from D and E meeting in T. Join FE, FD, fE, fD, CE, CD, DF, DE, TF, TC, Tf, and produce fF to f', and TC to meet DE in M, and draw fg, CP perpendicular to DT. Then



1. EFT $=$ DFT, and
   EfT $=$ DfT.
2. ETF $=$ DTf' and
   ETf' $=$ DTF
3. EM $=$ MD.

1. Draw FG, FH perpendicular to the tangents, meeting fD, fE in K and L, and join TK, TL. Then (prop. vi.) FDG $=$ KDG, and the angles at K are right angles by construction, and GD common: hence DK $=$ DF, and fK $= f$D $-$ DE $=$ AB (prop. v.) Similarly fL $=$ AB: and therefore fK $= f$L.

Again, FG $=$ GK, and GT. common to the two right angled triangles FTG and KTG. Hence TK $=$ TF. Similarly TL $=$ TF. Hence TK $=$ TL.

Hence in the two triangles TKf, TLf, TK $=$ TL, Kf $=$ Lf, and Tf common, and, therefore, the angles TfK, TfL are equal: that is DfT $=$ EfT.

In exactly the same way it may be proved that DFT $=$ EFT. Q. E. 1mo D.

2. FTf' $= \frac{1}{2}$(FTK $-$ FTL) $=$ FTD $-$ FTE. Hence

FTE $=$ FTD $-$ FTf' $= f'$TD.

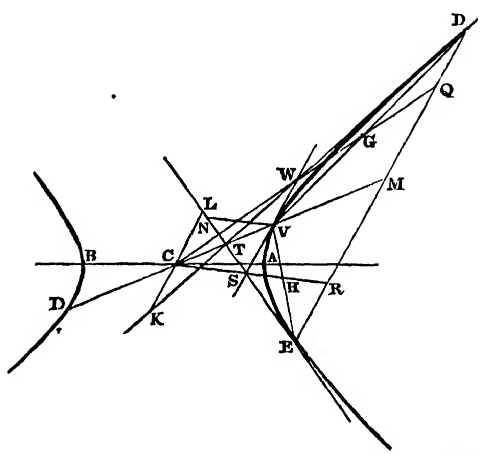
Also, adding FTf' to both, FTD $= f'$TE. Q. E. 2do D.

3. The triangle fKT $= f$DT — FDT (or KDT) $= \frac{1}{2}$DT . FG — $\frac{1}{2}$DT (fg — FG) = DT . CP = 2QCD.

But fKT has been proved equal to fLT, and therefore CDT = CLT.

Hence the triangles CDT and CLT standing on the same base CT, and on opposite sides of it, the line joining these vertices is bisected by the base: or DM = ME. Q. E. 3^{tio} D.

Prop. XIII. *Every ordinal chord (def. 23.) is bisected by its conjugate diameter: the tangents to every ordinal chord meet in that conjugate diameter produced: and that semi-diameter itself is a mean proportional between the segments estimated from the centre, cut off by the chord and tangent.*



Let WS be a tangent, and VC a semi-diameter from V: then DE parallel to WS is the ordinal chord. Then DE is bisected in M: the tangents at D and E intersect in CV; and

CM : CV :: CV : CT.

Let the tangents at D and E meet the tangent at V in W and S: join SC, WC cutting the chords DV and EV in H and G, and the chord DE in R and Q. Draw, also, LK parallel to DE through C.

First. By def. 23 and construction, the three lines WS, DE and KL are parallel: hence the triangle VWG is similar to GDQ and SVH to REH. But by the last proposition VG = GD and VH to HE. Hence the same pairs of triangles are wholly equal, and VW = DQ and SV to ER.

Again, by similar triangles LCS, RES and CWK, QWD we have

CL : RE :: CS : SH, or, by parallels
:: CW : WG, or, by the latter pair of triangles,
:: CK : QD.

But by parallels DE and LK, LC = CK (cor. prop. 12.) and hence RE = QD. But RE = SV and QD = VW: therefore SV = VW.

Also, by similar triangles, SCV, RCM and VCW, MCQ

SV : RM :: VC : CM
:: VW : MQ.

But SV = VW, and hence RM = MQ; and therefore EM = MD.

Q. E. 1^{mo} D.

Second. It was proved in proposition xii. that C, M, T are in one straight line, and in the preceding part of the present one, that C, M, V, are in one straight line: hence, both T and M are in CV, and the four points C, M, T, V, are in one straight line. That is, the tangents at D and E meet in the same point in the diameter conjugate to DE, produced. Q. E. 2^{do} D.

Third. Draw VN parallel to CS. Because SH is parallel to NV, and EH = HV (Euc. vi. 2, or Geom. theor. 82.), ES = SN.

Then, by similar triangles TNV, TSC and MET, VST, we have

$$CV : CT :: SN\ (= SE) : ST$$
$$:: MV \quad : VT$$

or, *alternando et componendo*,

$$CV + VM : CV :: CT + TV : CT: \text{ that is } CM : CV :: CV : CT.$$

Q. E. 3^{io} D.

Cor. 1. The diameter DV (made by producing VC to meet the opposite hyperbola in D) is harmonically * divided in M and T. For, *componendo et dividendo*,

$$CM + CV : CM - CV :: CT + TV : CT - CV,$$
$$\text{that is } DM : MV :: DT : TV.$$

Cor. 2. The rectangle $DM . MV = CM . MT$.

For $CM^2 - CV^2 = VM . MD$ (Theor. 31, or Euc. ii. 5. cor.) and

$$CM^2 - CV^2 = CM^2 - CT . CM = CM\ (CM - CT) = CM . MT.$$
$$\text{Hence } VM . MD = CM . MT.$$

Cor. 3. $DM . MV : DT . TV :: CM^2 : CV^2$. For (cor. 1, and theor. 89, or Euc. vi. 22.)

$CM^2 : CV^2 :: CV^2 : CT^2$. Hence, *dividendo et alternando*,

$CM^2 - CV^2 : CT^2 - CV^2 :: CM^2 : CV^2$; that is (Euc. ii 6.)

$DM . MV : DT . TV :: CM^2 : CV^2$.

Cor. 4. Every *ordinate* to a diameter is parallel to the tangent at its vertex.

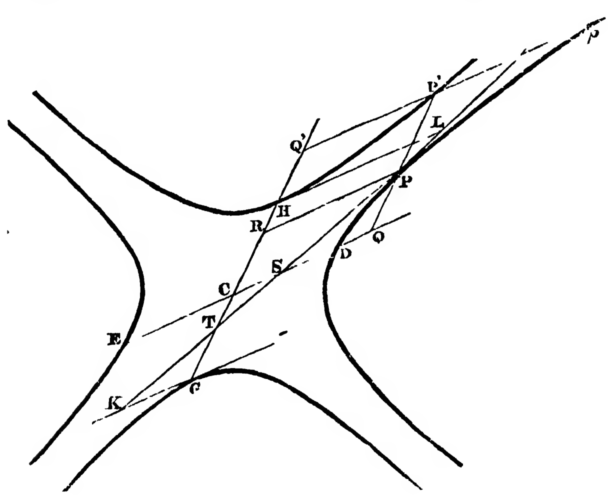
Cor. 5. All ordinates to the same diameter are parallel to the tangent at the vertex.

Cor. 6. Any straight line which bisects two parallel chords is a diameter, and is conjugate to those chords.

EXERCISE.

Given an hyperbola, to find its axes, parameter, centre, and foci: and to find the conjugate diameters, one of which makes a given angle with a given line.

Prop. XIV. *As the square of any diameter is to the square of its conjugate, so is the rectangle of the abscisses to the square of the ordinate.*



* See note on prop. xiii. Ellipse.

Let DE, GH be two conjugate diameters, and PQ an ordinate to the former, P being any point in the hyperbola: then

$CD^2 : CH^2 :: EQ . QD : QP^2$ and

$CH^2 : CD^2 :: HR . RG : RP^2$.

Draw the tangent at P to meet the conjugate diameters in S and T.

Then (prop. B.) $CQ : CD :: CD : CS$. Hence (Geom. theor. 81, cor. 2. or Euc. vi. 20, cor 2).

$CQ^2 : CD^2 :: CQ : CS$, or dividendo

$CQ^2 - CD^2 : CD :: QS : SC$, or by parallels,

$:: PQ : CT$, or since $PQ (= CR) : CH :: CH : CT$

$:: PQ^2 : CH^2$, or quadrupling

$:: ED^2 : GH^2$.

In the same way may the other case be proved. Q. E. D.

Cor. 1. The rectangles of the abscisses of any diameter are as the squares of the corresponding ordinates; for each ratio is the same as that of the squares ofthe conjugate diameters.

Cor. 2. Any diameter is to its parameter as the rectangle of the abscisses is to the square of the ordinate. This is proved as in cor. 1. prop. i.

Cor. 3. Draw the tangents GK, HL at the extremities of the diameter GH to meet the tangent at P in K and L; then $HL : PR :: CS : GK$. (See dem. of prop. ix.)

Cor. 4. If *any* tangent LK be drawn to intersect two parallel tangents GK, HL, it will cut off two segments HL, GK, whose rectangle is equal to the square of the semi-diameter CD, parallel to them.

Cor. 5. The tangent HL is harmonically divided in P and T.

Cor. 6. As the square of any diameter is to the square of its conjugate, so is the sum of the squares of that semi-conjugate and of the distance of any ordinate to it from the centre, to the square of that ordinate.

For by the proposition

$CH^2 : CD^2 :: DP^2 : CQ^2 - CD^2$,

$:: CR^2 : RP^2 - CD^2$; or *alternando et componendo.*

$:: CR^2 + CH^2 : RP^2$.

Cor. 7. In the same way it may be proved that in the conjugate hyperbola, $CD^2 : CH^2 :: CQ^2 + CD^2 : QP'^2$.

Cor. 8. By reasoning similar to that in prop. iii. cors. 1 and 2, we have

$QP^2 : QP'^2 :: CQ^2 - CD^2 : CQ^2 + CD^2$, and

$Q'P'^2 : Q'p^2 :: CQ'^2 - CH^2 : CQ'^2 + CH^2$.

EXERCISES.

1. *Theorem.* The chords which join the extremities of conjugate diameters are parallel to one another.

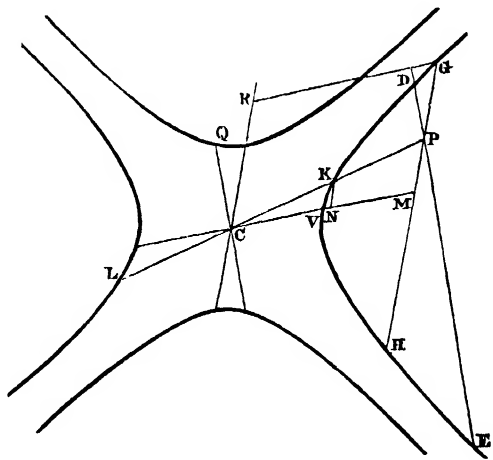
2. *Theorem.* The lines which bisect these chords are the asymptotes.

3. *Problem.* Given the ratio of two conjugate diameters of a given system of hyperbolas, to find their positions; and ascertain whether this ratio has any limit.

4. Let two pairs of hyperbolas be described within the same asymptotes, and be cut by the same chord: and let F, G be its intersections with the larger and E, D its intersections with the smaller (which lies within the other): then FE= DG and FD = EG; and the rectangle contained by the segments of the chord,

FE . EG or FD . DG is equal to the square of half the tangential chord parallel to it.

PROP. XV. *If any two straight lines which intersect one another and the same or opposite hyperbolas, be drawn parallel to two given diameters, the rectangle under the segments of the one shall be to the rectangle of the segments of the other, as the square of the diameter to which it is parallel is to the square of the other.*



Let the two lines PDE, PGH meet in P and cut the hyperbola in D, E, and in G, H; and let the semi-diameters CQ, CR be parallel to them: then $DP . PE : HP . PG :: CQ^2 : CR^2$.

Draw CV to bisect GH in M, and draw KN parallel to GH; and through P draw the diameter KL. Then CV is conjugate to CR.

Hence by the last proposition, cor. 6.

$CR^2 + KN^2 : CR^2 + GM^2 \quad CN^2 : CM^2$, or *dividendo*,

$GM^2 - KN^2 : CR^2 + KN^2 \quad CM^2 - CN^2 \quad CN^2$; or by similar triangles,

$PM^2 - KN^2 \quad KN^2$; or *altern. et divid.*

$GM^2 - PM^2 : CR^2 \quad PM - KN^2 \quad KN^2$, or by similar triangles,

$P^2 - CK^2 \quad CK^2$.

But GH and KL are bisected in M and C respectively; and hence (Euc. ii. 6.) we have

$$GP . PH : CR^2 :: KP . PL : CK^2.$$

In like manner $DP . PE : CQ^2 :: KP . PL : CK^2$, and (Euc. v. 11.)

$$GP . PH : DP . PE :: CR^2 : CQ^2.$$

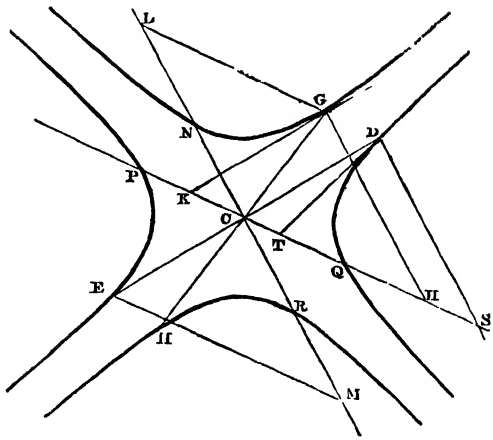
Q. E. D.

Cor. If one of the lines be a tangent, the rectangle of its segments becomes the square of the tangent; and if both be tangents, then those tangents have the same ratio as their parallel diameters.

[The examination of all the cases that can arise is left for the student's exercise.]

PROP. XVI. *If, from the extremities of two conjugate diameters of an hyperbola, ordinates be drawn to any third diameter, the rectangle of the abscisses made by one of the ordinates is equal to the square of the distance from the centre (estimated on that diameter) of the other ordinate.*

Let DE and GH be the conjugate diameters, and from the extremities D and G to any other diameter PQ draw the ordinates GH and DS. Then the rectangle contained by QH, HP is equal to the square of CS.



Draw the tangents at D and G to meet the diameter in T and K. Then by the similar triangles KGH, CDS and GHC, DPT,

$$KH \quad CS \quad KG \quad DC$$
$$KG \quad DC \quad KC \quad CT; \text{ hence (Euc. v. 11.)}$$
$$KH \quad CS \quad KC \quad CT.$$

Again, (prop. xiii.) $KC \,.\, CH = CQ^2 = CS \,.\, CT$, and (Geom. theor. 89 or Euc. vi. 16.)

$$KC : CT :: CS : CH; \text{ and (Euc. v. 11.)}$$
$$CS : CH :: KH : CS, \text{ or } CS^2 = CH \,.\, HK.$$

But (xiii. cor. 2) $CH \,.\, HK = QH \,.\, HP$. Hence $CS^2 = QH \,.\, HP$.
and similarly $CH^2 = QS \,.\, SP$. Q. E. D.

Cor. 1. $KC \,.\, CH = TC \,.\, CS$.

Cor. 2. $CH \,.\, HK = CS^2$, and $CS \,.\, ST = CH^2$. Both of which have been established in the foregoing demonstration.

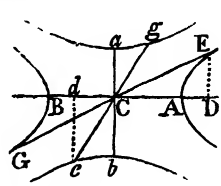
Cor. 3. $CS^2 - CH^2 = CQ^2$, and $CL^2 - CM^2 = CP^2$.
For $CS^2 = CH \,.\, HK$ by cor. 2. From these take CH^2; then $CS^2 - CH^2 = CH \,.\, HK - CH^2 = CH\,(KH - HC) = CH \,.\, CK = CQ^2$, and similarly for the other.

Cor. 4. $QC : CN :: CS : HG$, or $QC \,.\, HG = CS \,.\, CN$: for by last proposition
$$QC^2 : CN^2 :: QH \,.\, HT\ (= CS^2) : GH^2, \text{ or}$$
$$QC : CN :: CS : GH.$$

Prop. XVII. *The difference of the squares of any two conjugate diameters is always the same: and the areas of the conjugate parallelograms is always the same.*

That is, if EG and *eg* be conjugate diameters, and AB, *ab* the axes, we shall always have

$$EG^2 - eg^2 = AB^2 - ab^2.$$



Draw ED, *ed* ordinates each from one extremity of the conjugate diameters to the transverse axis. Then by the last proposition

$$Cd^2 = BD \,.\, DA = CD^2 - CA^2, \text{ or}$$
$$CA^2 = CD^2 - Cd^2, \text{ and similarly}$$
$$Ca^2 = de^2 - DE^2: \text{ hence}$$
$$CA^2 - Ca^2 = CD^2 + DE^2 - Cd^2 - de^2 = CE^2 - Ce^2, \text{ or quadrupling,}$$
$$AB^2 - ab^2 = GE^2 - ge^2. \qquad \text{Q. E. D.}$$

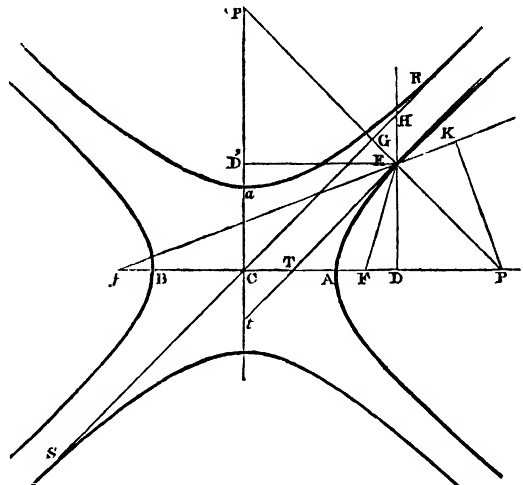
Scholium. The proposition corresponding to the second part of xvii. on the Ellipse has been more simply demonstrated in prop. C. cor 2: but it may be

given as an exercise for the student to demonstrate it in a manner analogous to that employed in the proof of that property on the Ellipse.

PROP. XVIII. *If from any point in the curve there be drawn an ordinate, and a perpendicular to the curve, or to the tangent at that point: then the*

Dist. on the trans. between the centre and ordinate CD :
Will be to the dist. PD $::$
As square of trans. axis :
To square of the conjugate.

That is,
$CA^2 : Ca^2 :: DC : DP$.
For, by theor. 2, $CA^2 : Ca^2 :: AD \cdot DB : DE^2$,
But, by rt. angled ΔS, the rect. $TD \cdot DP = DE^2$;
and, by cor. 2, prop. 16, $CD \cdot DT = AD \cdot DB$;
therefore $CA^2 : Ca^2 :: TD \cdot DC : TD \cdot DP$,
or (Euc. vi. 1) $CA^2 : Ca^2 :: CD : DP$.
Q. E. D.



Cor. 1. $AC^2 + Ca^2 : AC^2 :: PC : PD$; that is, $CF^2 : CA^2 :: PC : PD$, F being the focus.

Cor. 2. $AB : P :: DC : DP$, by def. 25, and Euc. vi. 22, cor. 2.

Cor. 3. In the same manner if we draw ED′ parallel to AB and produce EP to meet Ca in P′, we have

$$P : AB :: D'P' : CD'.$$

Cor. 4. $DP \cdot D'P' = DC \cdot CD$ from cors. 3 and 4.

Cor. 5. Draw a diameter parallel to the tangent TEt, meeting the normal in G. Then $EG \cdot EP = Ca^2$.

For, produce ED to meet CG in H. Then the angles at D and G are right angles, and hence the triangles EDP, EGH are similar, and

$$ED\ (= CD') : EP :: EG : EH\ (= Ct),\ \text{that is}$$
$$EP \cdot EG = CD' \cdot Ct = Ca^2.$$

Cor. 6. Draw PK perpendicular to Ef: then EK is half the parameter.

Let CG meet Ef in L: then the triangles EGF, EPK are similar; and $EG : EL :: EK : EP$, or $LE \cdot EK = PE \cdot EG = Ca^2$.

But EP bisects the angle exterior to the angle FEf (since it is perpendicular to the tangent which bisects the angle FEf itself) and hence $EL = \frac{1}{2}(Ef - EF) = AC$ (prop. v.) We hence have

$$AC \cdot EK = Ca^2,\ \text{or}\ AC : Ca :: Ca : EK,\ \text{or}\ EK = \tfrac{1}{2}P.$$

EXERCISES.

1. *Problem.* From a given point in either axis to draw a normal to the hyperbola.

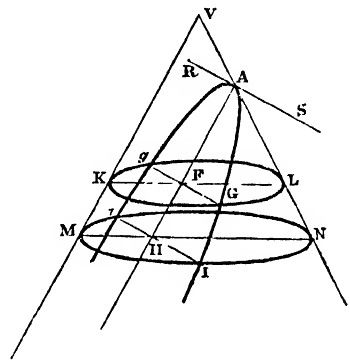
2. *Theorem.* The rectangle under fE, FE is equal to the square of the semi-diameter CR parallel to the tangent at E.

THE PARABOLA.

SECTION I.

PROPERTIES RELATIVE TO THE AXIS AND LINES CONNECTED WITH IT.

Prop. I. *In the parabola the abscisses are to one another in the same ratio as the squares of their corresponding ordinates.*



Let a plane parallel to the tangent plane of the cone cut the cone in the figure iAI: then (definition 6.) the section is a parabola. Let also the sections of the cone perpendicular to the axis be made cutting the parabolic section in Gg and Ii; and the transverse plane be drawn cutting it in AH; and, again, the tangent plane through its intersection A of the transverse and parabolic planes with the cone, in RS. Denote the intersections of Gg and Ii with the line AH by F and H.

Then, by reasoning as in the Ellipse and Hyperbola [which the student is required to apply to the present case] it will appear that AH is the axis of the parabola (def. 11.) gFG and iHI are ordinates both to the axis of the parabola and to the diameters of the circles KL, MN.

It then only remains to prove that $AF : AH :: FG^2 : HI^2$.

Since KL is parallel to MN, $AF : AH$ $FL : HN$ (Euc. vi. 2, or Geom. th. 82.)
$:: FL \,.\, KF : HN \,.\, HM$ (Euc. i. 34, and vi. 2, or Geom. 80 and 81.)
$:: FG^2$ HI^2 (Euc. vi. 2, and iii. 35, or Geom. 41 and 86. Q. E. D.

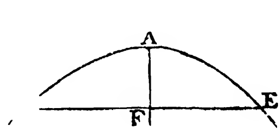
Cor. 1. If a third proportional P, to *any* absciss AF and its ordinate FG, be taken, this is a constant magnitude wherever in the same parabola the point be taken.

For, by the theorem $FG^2 : HI^2 :: AF : AH$, or (Euc. vi. 1, or Geom. 79).
$:: P \,.\, AF : P \,.\, AH$.

But by hypothesis, $AF : FG :: FG : P$, or $P \,.\, AF = FG^2$; hence also $P \,.\, AH = HI^2$, or $AH : HI :: HI : P$.

It follows, therefore, that the third proportional to AF and FG is equal to the third proportional to AH and HI; or the third proportional to any absciss and its ordinate is a constant magnitude. The propriety, then, of the definition of the parameter is hence evident. See def. 25.

Cor. 2. The distance of the vertex from the focus is equal to one-fourth of the parameter, or to half the ordinate at the focus.



That is, $AF = \frac{1}{2} FE = \frac{1}{4} P$, or $P = 4AF$, where F is the focus.

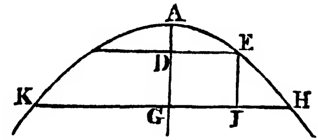
For by the proposition and cor. 1.,
$AF : FE :: FE : P$.

But by the definition of the focus
$FE = \frac{1}{2}P$; hence also $AF = \frac{1}{2} FE = \frac{1}{4}P$.

Cor. 3. As the parameter of the axis is to the sum of any two ordinates, so is the difference of those ordinates to the difference of their abscisses: that is,

$$P : GH + DE : GH - DE : DG.$$

For by the cor. 1. $P \cdot AG = GH^2$ and $P \cdot AD = DE^2$; hence $GH^2 - DE^2 = P\,(AG - AD) = P \cdot GD$. Consequently (Euc. vi. 16, or Geom. 81), and Euc. ii. 5, or Geom. 33.)



$$P : GH + DE :: GH - DE : GD.$$

Cor. 4. Produce GH to meet the curve again in K, then $KI = GH + DE$, and $IH = GH - DE$. Hence $KI \cdot IH = P \cdot GD$, or $P : KI :: IH : GD$.

Cor. 5. Since $P \cdot EI = KI \cdot IH$ and $P \cdot AG = GH^2$, therefore

$$KI \cdot IH : GH^2 : P \cdot EI : P \cdot AG$$
$$: EI \quad : AG,$$

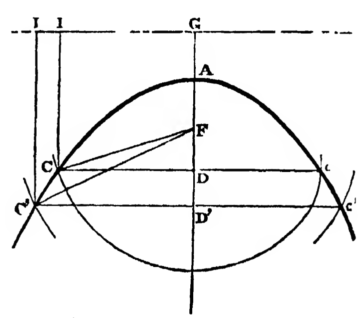
or the portions of any diameter intercepted by a given chord at right angles to the axis, are as the rectangles contained by the segments of that chord into which the diameters divide it.

Cor. 6. The parabola is capable of infinite extension, and the axis does not again meet the curve For the plane of the parabola can never meet the edge VM being parallel to it. Its two branches can never, therefore, meet in the line AH.

[*Scholium.* It will be seen in a future page that all these properties are true of any chord whatever conjugate to a given diameter.]

Cor. 7. The centre, which in analogy with the other two sections it is supposed to have, is therefore infinitely distant, and diameters from any point, are, therefore parallel.

Prop. II. *In the parabola the determining ratio is a ratio of equality; or the line drawn to the focus from any point in the curve is equal to the perpendicular drawn to the directrix.*



That is, if we make $AG = AF$ (which for the point A fulfils the condition) and draw GI perpendicular to AD, and CI, from a point C, parallel to DA, and join CF; then $CI = CF$.

For draw CD the ordinate at C. Then CDF is a right angle, and $CF^2 = FD^2 + DC^2$

$$= 4AF \cdot AD + DF^2 \text{ (prop. i. cor. 2.)}$$
$$= 2GF\,(AF + FD) + DF^2 = 2GF \cdot AF + 2GF \cdot FD + DF^2,$$
$$= \;GF^2 + 2GF \cdot FD + FD^2.$$

Hence (Euc. ii. 4, or Geom. theor. 31.)

$$CF = GF + FD = GD = CI. \qquad \text{Q. E. D.}$$

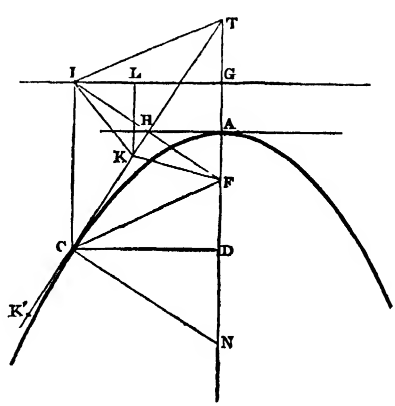
The same demonstration, *mutatis mutandis*, applies when D is between A and F: but the steps are left for the student to supply.

Cor. 1. Hence any number of points in the parabola whose parameter is given may be easily constructed.

For, taking AF = AG = $\frac{1}{4}$ parameter, and drawing lines through any points in the axis, AD, such as DC: then with F as centre and distances equal to DG, D′G, &c. describing circles to cut DC, D′C, and in C, *c*, C′, *c*′, &c. the points C, C′, *c*, *c*′, &c. will be in the curve. For CI = DG, &c. A sufficient number, therefore, of such points being constructed, the curve may be approximately traced through them, sufficiently accurate for all the purposes of constructive practice.

Cor. 2. The tangent at the vertex of the parabola is perpendicular to the axis.

Prop. III. *If a line be drawn to bisect the angle made by two lines, one of which is drawn from a point in the parabola to the focus, and the other from the same point perpendicular to the directrix, that line will be a tangent to the parabola at that point.*



Let C be the point in the parabola, F the focus, GI the directrix, perpendicular to which CI is drawn, then the line CT drawn to bisect the angle ICF will be a tangent at C.

For since C is a point in the curve, CT meets the curve: and if it be not a tangent, it will meet the curve again in some point, K. Then also K is a point in the parabola. Join KF, and draw KL perpendicular to the directrix, and join IK.

Then K being a point in the parabola, KF = KL (prop. ii): and since IC = CF, and by hypothesis ICK = KCF, and KC common to the triangles ICK, KCF, the base IK = KF (Euc. i. 4, or Geom. theor. 7.) But KF = KL, and therefore IK = KL. Whence the angles KIL, KLI are also equal.

But KLI is a right angle by construction, and therefore two angles of the triangle IKL are equal to two right angles: which is impossible (Euc. i. 17, or Geom. theor 17 cor) Hence the point K is not in the parabola.

In the same way it may be shown that no other point, K′ in the line CT is in the parabola: and therefore that CT does not cut the parabola, or it is a tangent to the parabola at C. Q. E. D.

Cor. 1. Let the tangent at C meet the axis at T. Then CF = FT.

For, ICT = TCF, and ICT = CTF (Euc. i. 29, or Geom. theor. 12): hence CTF = TCF, and CF = FT.

Cor. 2. Draw the ordinate CD: then DA = AT. For (by cor. 1.) AF + AT = FT = FC = FI = AG + AD = AF + AD. Hence AT = AD.

Cor. 3. D aw the normal CN; then FN = FT = FC. For draw FH perpendicular to CT. Then since FC = FT, and FH common to the triangles FHC, FHT, and the angles at H equal, CH = HT. Also the lines HF, CN being both perpendicular to CT (def. 28, and constr.) they are parallel. Hence TH : HC :: TF : FN; but TH = HC, therefore TF = FN. And (prop. ii.) FT = FC. Hence FN = FT = FC.

Cor. 4. T, C, N are in a circle of which F is the centre.

Cor. 5. The sub-normal is always of the same magnitude, viz. half the parameter.

For TCN is a right angle (def. 28); hence DT (= 2AD) : DC :: CD : DN (Euc. vi. 8, cor. 1, or Geom. 87.) and AD : DC :: DC : P (def. 25.)

Hence DN $= \frac{1}{2}$P = 2AF.

Cor. 6. The tangent at the vertex passes through H, and is a mean proportional between AF and AD.

For, join HA. Then, since TC and DT are bisected in H and A (cors. 2 and 3.) they are divided proportionally, and hence HA is parallel to CD, and hence also perpendicular to FT. It is hence the tangent at A (prop. ii. cor. 2). Then the conclusion follows at once (from Euc. vi. 8, cor. 1, or Geom. theor. 87), that TA (= AD) : AH :: AH : AF.

Cor. 7. FH is a mean proportional between FA and FT, or between FA and AC.

Cor. 8 Join IT, and IH: then ITFC is a rhombus, and IKF is one straight line bisected in H.

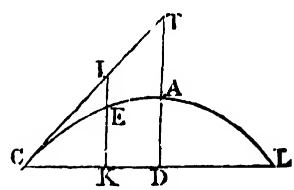
For CI = CF = FT. Hence, CI being equal and parallel to FT, the lines IT, FC are parallel and equal: and the four lines, IC, CF, FT, TI are equal.

Also the diagonals of a rhombus bisect one another at right angles; hence, since H is the middle of the diagonal CT, the line FH which bisects it at right angles is also a diameter.

Prop. IV. *If there be any tangent and a double ordinate drawn from the point of contact to the axis, and also any other line parallel to the axis, intersecting the tangent, the curve, and double ordinate; then shall this second line be divided by the curve into segments, having the same ratio as the segments into which it divides the double ordinate.*

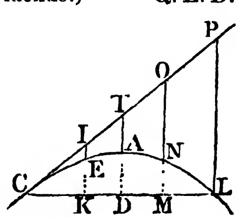
That is,

IE : EK :: CK : KL



For, by similar triangles	CK : KI :: CD : DT $=$ 2DA^2	(prop. iii. cor. 2.)
And	P : CL :: CD : 2DA,	(prop. i. cor. 1.)
Hence,	P : CK :: CL : KI,	(*altern.* and Euc. v. 11)
But	P : CK :: KL : KE,	(prop. i. cor. 4.)
Hence,	CL : LK :: KI : KE,	(Euc. v. 11. and alto.)
And	CK : KL :: IE : EK,	(dividendo.) Q. E. D.

Cor. 1. If several such lines be drawn as in the annexed figure, then the portions of them intercepted between the tangent and the curve shall have the same ratios as the squares of the portions of the tangents intercepted by them estimated respectively from the point of contact.



That is,

IE : TA : ON : PL :: CI^2 : CT^2 : CO^2 CP^2 &c.

For, by the proposition,

IE : EK :: CK : KL, or (by Euc. vi. 1.)

P . IE : P . EK :: CK^2 : CK . KL, or alternando.

P . IE : CK^2 :: P . EK : CK . KL.

But (prop. i. cor. 4.) P . EK = CK . KL; hence, P . EI = CK^2.

Similarly P . AT = AD^2, P . ON = CM^2, &c.

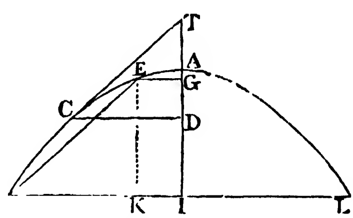
Hence, EI : AT :: CK^2 : AD^2 (Euc. vi. 1, or Geom. 79.)
:: CI^2 : CT^2 by similar triangles.

The same being true of any other lines, ON, PL, &c. the property is established.

Cor. 2. As this property is common to every position of the tangent, if the lines IE, TA, ON, &c. be appended on the points I, T, O, &c. and moveable about them, and of such lengths as that their extremities, E, A, N, &c. be in the curve of a parabola in some one position of the tangent; then making the tangent revolve about the point C, it appears that the extremities E, A, N, &c. will always fall in the curve of *some* parabola, in every position of the tangent.

Prop. V. *If a line be drawn parallel to any tangent, and cut the curve in two points; then, if two ordinates be drawn to the intersections, and a third to the point of contact, these three ordinates will be in arithmetical progression, or the sum of the extremes will be equal to double the mean.*

That is,
EG + HI = 2CD.

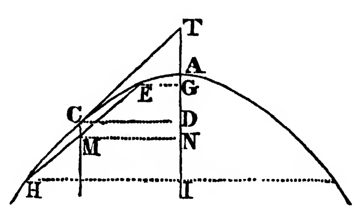


For, draw EK parallel to the axis, and produce HI to L.
Then, by sim. triangles, EK : HK :: TD or 2AD : CD;
but, by prop. i. cor. 1, EK : HK :: KL : P, the parameter.
therefore by equality, 2AD : KL :: CD : P.
But, by the defin. 2AD : 2CD :: CD : P;
therefore also the 2d terms are equal, or KL = 2CD,
that is, ... EG + HI = 2CD. Q. E. D.

Cor. 1. When the point E is on the other side of AI; then HI — GE = 2CD.

Cor. 2. HI — CD = CD — EG.

Cor. 3. Every chord HE parallel to the tangent CT is bisected by the diameter CM drawn through the point of contact.



Let CM meet HI and EG (produced) in V and S, and MN be parallel to the ordinates.

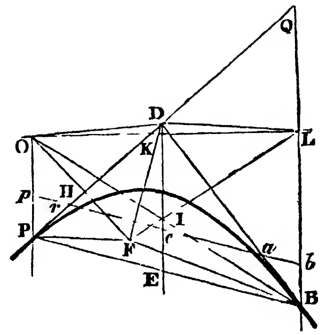
Then, by cor. 2., HV = SE, MSE = MVH, and SME = HMV; hence, also HM = ME. (Euc. i. 26.)

In the same manner it may be shewn that any other line parallel to CT is bisected by CM.

Prop. VI. *If two tangents to a parabola intersect each other, the diameter, which passes through the point of intersection, bisects the chord which joins the points of contact.*

Let the tangents at P and B intersect in D; the diameter which passes through D, bisects PB in E.

Let the diameters at P and B meet the directrix in O and L; join PF, BF,



OF, FL, and OB, meeting DE in I.

Because OH = HF, and the angles at H right angles, (prop. iii.) the triangles ODH and DHF are equal in all respects; therefore OD = DF. In the same way it may be shown that DL = DF; therefore OD = DL, and the two right-angled triangles OKD and LKD are equal in all respects; wherefore OK = KL, and by (Euc. vi. 2, or Geom. 82.) and equality of ratios,

OK : KL ∷ OI : IB ∷ PE : EB,
therefore, PE = EB.

Cor. 1. If a straight line be drawn parallel to PB, meeting the tangents and the two diameters drawn through P and B in r, a, and p, b, the intercepted parts pr and ab are equal.

For, since PE = EB, by similar triangles $re = ea$, and $pe = eb$; therefore, taking equals from equals, $pr = ab$.

Cor. 2. If the tangent PD be produced to meet the diameter in Q, PQ is bisected in D;

For, PE : EB ∷ PD : DQ (Euc. vi. 2, or Geom. 82.)
Therefore PD = DQ, and PQ is bisected in D.

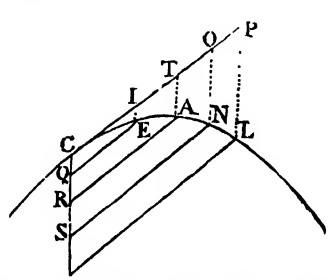
Cor. 3. A straight line, drawn from the intersection of two tangents to the focus, bisects the angle contained by two straight lines, drawn from the focus to the points of contact.

Cor. 4. If through the intersection of two tangents to a parabola, a diameter be drawn, and a straight line to the focus, they will make equal angles with the tangents.

SECTION II.

ON THE PARABOLA REFERRED TO OTHER DIAMETERS, DIFFERENT FROM THE AXIS.

Prop. VII. *The abscisses of any diameter have to one another the same ratio which the squares of the corresponding ordinates have.*



That is,

CQ : CR ∷ QE^2 : RA^2, &c.

For draw the tangent CP, and from the extremities of the lines QE, RA, &c. which are parallel to CP, the lines EI, AT, &c. to meet it.

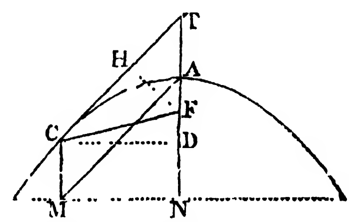
Now it has been shown (prop. v. cor. 3.) that CS bisects all the chords parallel to the tangent CP, and hence the lines QE, RA, &c. are ordinates. Also by construction QI, RT, &c. are parallelograms; hence CQ = IE. CR = TA, &c., and QE = CI, RA = CT, &c.

But, (prop. iv. cor. 1.) IE : TA :: CI² : CT², and hence
CQ : CR :: QE² : RA².

The same demonstration applies to the case where the ordinates are on the other side of the diameter CS. Hence the abscisses of any diameter are as the squares of the ordinates. Q. E. D.

Schol. Hence, as the abscisses of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the abscisses; so all the other properties of the axis and its ordinates and abscisses, before demonstrated, will likewise hold good for any diameter and its ordinates and abscisses. And also those of the parameters, understanding the parameter of any diameter, as a third proportional to any absciss and its ordinate. (Def 25.) Some of the most material of these are demonstrated in the following propositions.

Prop. VIII. *The parameter of any diameter is equal to four times the line drawn from the focus to that vertex of the diameter.*



That is, $4FC = p$, the param. of the diam. CM.

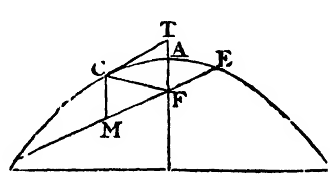
For, let A be the vertex of the parabola, and draw the ordinate MA parallel to the tangent CT: also CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

Then the abscisses AD, CM or AT, being equal, by prop. iii. cor. 2., the parameters will be as the squares of the ordinates CD, MA or CT, by the definition;

	$P . AD : p . CM :: CD^2 : CT^2$
that is,	$P : p :: CD^2 : CT^2$,
But, by sim. tri.	$FH^2 : FT^2 :: CD^2 : CT^2$,
therefore	$P : p :: FH^2 : FT^2$.
But, by cor. 6, prop. iii.	$FH^2 = FA . FT$;
therefore	$P : p :: FA . FT : FT^2$;
or, Euc. vi. 1, or Geom. 79,	$P : p :: FA : FT = FC$.
But, by prop. i. cor. 2,	$P = 4FA$,
and therefore	$p = 4FT$ or $4FC$. Q. E. D.

Corol. Hence the parameter p of the diameter CM is equal to $4FA + 4AD$, or to $P + 4AD$, that is, the parameter of the axis added to $4AD$.

Prop. IX. *If an ordinate to any diameter pass through the focus, it will be equal to half its parameter; and its absciss equal to one-fourth of the same parameter.*



That is, $CM = \frac{1}{4}p$,
and $ME = \frac{1}{2}p$.

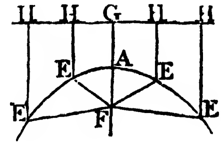
For, join FC, and draw the tangent CT.

By the parallels,	$CM = FT$;
and, by prop. iii. cor. 1,	$FC = FT$;
also, by prop. vii.	$FC = \frac{1}{4}p$;
therefore	$CM = \frac{1}{4}p$.

Again, by the defin. CM or $\frac{1}{4}p : ME :: ME : p$.
and consequently $ME = \frac{1}{2}p = 2CM$. Q. E. D.

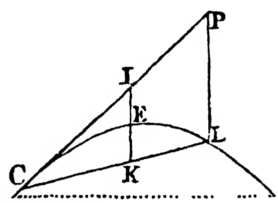
Cor. 1. Hence, of any diameter, the double ordinate which passes through the focus, is equal to the parameter, or to quadruple its absciss.

Cor. 2. Hence, and from prop. iv., cor. 1, prop. iii. cor. 1, and prop. vii., it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or $\frac{1}{4}$ of the parameter of the diameter to the point F.



Prop. X. *If there be a tangent, and any line drawn from the point of contact and meeting the curve in some other point, as also another line parallel to the axis, and limited by the first line and the tangent · then shall the curve divide the second line in the same ratio, as the second line divides the first line.*

That is,
IE : EK :: CK : KL.

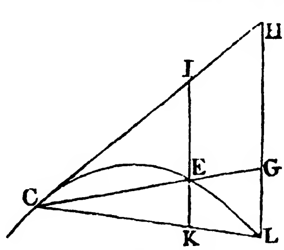


For, draw LP parallel to IK, or to the axis.

Then, by prop. iv.	$IE : PL :: CI^2 : CP^2$,
or, by sim. tri.	$IE : PL :: CK^2 : CL^2$
Also, by sim. tri.	$IK : PL :: CK : CL$,
or Euc. vi. 1, or Geom. 79,	$IK : PL :: CK^2 : CK . CL$;
therefore by equality,	$IE : IK :: CK . CL : CL^2$;
or, as above,	$IE : IK :: CK : CL$;
And, dividendo,	$IE : EK :: CK : KL$. Q. E. D.

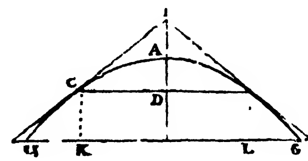
Corol. When $CK = KL$, then $IE = EK =$

Prop. XI. *If from any point of the curve there be drawn a tangent, and also two right lines to cut the curve; and diameters be drawn through the points of intersection E and L, meeting those two right lines in two other points G and K: then will the line KG joining these last two points be parallel to the tangent.*



For, by prop. x.	$CK : KL :: EI : EK$;
and *componendo*,	$CK : CL :: EI : KI$;
and by the parallels	$CK : CL :: IE : IK :: GH : LH$.
But, by sim. tri.	$CK : CL :: KI : LH$,
theref. by equality	$KI : LH :: GH : LH$
consequently	$KI = GH$,
and therefore	KG is parallel and equal to IH. Q. E. D.

Prop. XII. *If an ordinate be drawn to the point of contact of any tangent, and another ordinate be drawn and produced to meet the tangent: then, as the difference of the ordinates is to the difference added to the external part, so is double the first ordinate to the sum of the ordinates.*



That is,
KH : KI :: KL : KG;
where the tangent CI is cut in I by the double ordinate GH.

For, by cor. 1, prop. 1,	P :	DC ::	DC : DA,	
and	P :	2DC ::	DC : DT or 2DA.	
But, by sim. triangles,	KI :	KC ::	DC : DT;	
therefore, by equality,	P :	2DC ::	KI : KC,	
or,	P :	KI ::	KL : KC.	
Again, by prop 1. cor. 3,	P :	KH ::	KG : KC;	
therefore by equality,	KH :	KI ::	KL : KG.	Q. E. D.

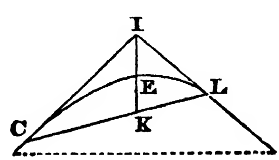
Cor. 1. Hence, by composition and division,

it is, KH : KI :: GK : GI,
and HI : HK :: HK : KL,
also HI : IK :: IK : IG;

that is, IK is a mean proportional between IG and IH.

Cor. 2. And from this last property a tangent can easily be drawn to the curve from any given point I. Thus, draw IHG perpendicular to the axis and take IK a mean proportional between IH, IG; then draw KC parallel to the axis, and C will be the point of contact, through which and the given point I the tangent IC is to be drawn.

Cor. 3. If a tangent cut any diameter produced, and if an ordinate to that diameter be drawn from the point of contact; then the distance in the diameter produced, between the vertex and the intersection of the tangent, will be equal to the absciss of that ordinate.



That is, IE = EK.
For, by the prop. IE : EK :: CK : KL.
But, by prop. v. cor. 3., CK = KL,
and therefore, also, IE = EK.

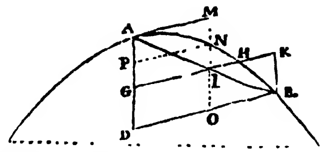
Cor. 4. The two tangents CI, LI, at the extremities of any double ordinate CL, meet in the same point of the diameter of that double ordinate produced. And the diameter drawn through the intersection of two tangents, bisects the line connecting the points of contact.

Cor. 5. Hence we have another method of drawing a tangent from any given point I without the curve. Namely, from I draw the diameter IK, in which take EK = EI, and through K draw CL parallel to the tangent at E; then C and L are the points to which the tangents must be drawn from I.

Prop. XIII. *If a line be drawn from the vertex of any diameter, to cut the curve in some other point, and an ordinate of that diameter be drawn to that point, as also another ordinate any where cutting the line, both produced if necessary:*

The three will be continual proportionals, namely, the two ordinates and the part of the latter limited by the said line drawn from the vertex.

That is, DE, GH, GI are continual proportionals, or DE : GH :: GH : GI.



For, by prop. vii.	$DE^2 : GH^2 :: AD : AG$;	
and, by sim. tri.	$DE : GI :: AD : AG$;	
therefore by equality,	$DE : GI :: DE^2 : GH^2$,	
that is, by duplicate ratios,	$DE : GH :: GH : GI$.	Q. E. D.

Cor. 1. Or their equals, GK, GH, GI, are proportionals; where EK is parallel to the diameter AD.

Cor. 2. Hence it is $DE : AG :: p : GI$, where p is the parameter,
or $AG : GI :: DE : p$.
For, by the defin. $AG : GH :: GH : p$.

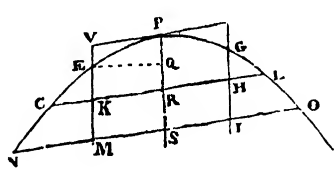
Cor. 3. Hence also the three MN, MI, MO, are proportionals, where M parallel to the diameter, and AM parallel to the ordinates.

For, by prop. vii.	MN, MI, MO,
or their equals	AP, AG, AD,
are as the squares of	PN, GH, DE,
or of their equals	GI, GH, GK,
which are proportionals by cor. 1.	

Prop. XIV. *If a diameter cut any parallel lines terminated by the curve; the segments of the diameter will be as the rectangle of the segments of those lines.*

That is, $EK : EM :: CK . KL : NM . MO$.
Or, EK is as the rectangle CK . KL.

For, draw the diameter PS to which the parallels CL, NO are ordinates, and the ordinate EQ parallel to them.



Then CK is the difference, and KL the sum of the ordinates EQ, CR; also NM the difference, and MO the sum of the ordinates EQ, NS. And the differences of the abscisses, are QR, QS, or EK, EM.

Then by cor. prop. vii. $QR : QS :: CK . KL : NM . MO$,
that is $EK : EM :: CK . KL : NM . MO$.

Cor. 1. The rect. CL . KL = rect. EK and the param. of PS.
For the rect. CK . KL = rect. QR and the param. of PS.

Cor. 2. If any line CL be cut by two diameters. EK, GH; the rectangles of the parts of the line, are as the segments of the diameters.

For EK is as the rectangle CK . KL,
and GH is as the rectangle CH . HL;
therefore $EK : GH :: CK . KL : CH . HL$.

Cor. 3. If two parallels, CL, NO, be cut by two diameters, EM, GI; the rectangles of the parts of the parallels will be as the segments of the respective diameters.

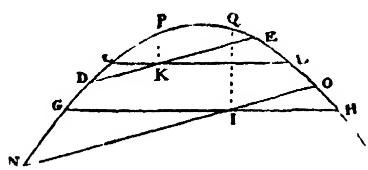
For EK : EM :: CK . KL : NM . MO,
and EK : GH :: CK . KL : CH . HL,
theref. by equal. EM : GH :: NM . MO : CH . HL.

Cor. 4. When the parallels come into the position of the tangent at P, their two extremities, or points in the curve, unite in the point of contact P; and the rectangle of the parts becomes the square of the tangent, and the same properties still follow:

So that, EV : PV :: PV : p the param.
GW : PW :: PW : p,
EV : GW :: PV^2 : PW^2,
EV : GH :: PV^2 : CH . HL.

Prop. XV. *If two parallels intersect any other two parallels; the rectangles of the segments will be respectively proportional.*

That is, CK . KL : DK . KE :: GI . IH : NI . IO



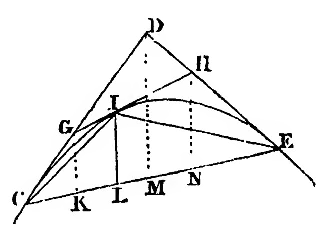
For, by cor. 3, prop. xiv. PK : QI :: CK . KL : GI . IH;
and by the same PK : QI :: DK . KE : NI . IO;
theref. by equal CK . KL : DK . KE :: GI . IH : NI . IO.

Corol. When one of the pairs of intersecting lines comes into the position of their parallel tangents, meeting and limiting each other, the rectangles of their segments become the squares of their respective tangents. So that the constant ratio of the rectangles, is that of the square of their parallel tangents, namely,

CK . KL : DK . KE :: tang^2. parallel to CL : tang^2. parallel to DE.

Prop. XVI. *If there be three tangents intersecting each other; their segments will be in the same proportion.*

That is,
GI : IH :: CG : GD :: DH : HE.



For, through the points G, I, D, H, draw the diameters GK, IL, DM, HN; as also the lines CI, EI, which are double ordinates to the diameters GK, HN, by cor. 4, prop. xii.; therefore the diameters GK, DM, HN, bisect the lines CL, CE, LE;

hence KM $=$ CM $-$ CK $= \frac{1}{2}$CE $- \frac{1}{2}$CL $= \frac{1}{2}$LE $=$ LN or NE,
and MN $=$ ME $-$ NE $= \frac{1}{2}$CE $- \frac{1}{2}$LE $= \frac{1}{2}$CL $=$ CK or KL.

But, by parallels, GI : IH :: KL : LN,
and CG : GD :: CK : KM,
also DH : HE :: MN : NE.
But the third terms KL, CK, MN are all equal;
as also the 4th terms LN, KM, NE.

Therefore the first and second terms, in all the analogies, are proportional, namely, GI : IH :: CG : GD :: DH : HE. Q. E. D.

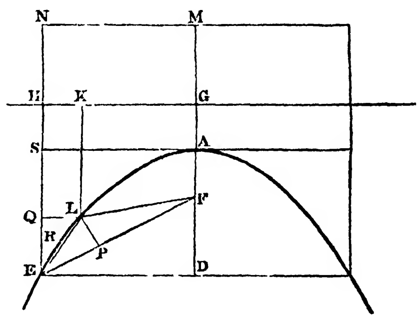
EXERCISES.

1. If any diameter intersect two parallel chords, the segments of the diameter intercepted from its vertex will have the same ratio as the rectangles of the segments into which it divides those two lines.

2. If any circle cut a parabola in four points, and from those points ordinates be drawn to the axis, the sum of the ordinates on one side of the axis is equal to the sum of those on the other.

PROP. XVII. *The area or space of a parabola is two-thirds of its circumscribing rectangle.*

Let E, R and L be any three points in the curve of the parabola, of which R is between E and L, F the focus, A the vertex, GH the directrix, and AS the tangent at the vertex. Draw the ordinate ED, and make GM = FD, and complete the rectangle EDMN.



Draw also FL, LK, FE, EH, and from L draw the perpendiculars LP, LQ to the lines EF, EH.

Then EH = EF and KL = LF. But the points E and F being in the curve of the parabola, the line EL falls within the parabola, and therefore the tangent to the curve at E falls between EL and EH. But the tangent bisects the angle HEF, (prop. iii.); and hence the angle QEL is greater than the angle LEF, and therefore, also, LQ is greater than LP. (Euc. i. 24.)

But as the point L approaches the point E, the difference of the angles QEL, PEL diminishes continually, and hence the ratio of QL to LP approaches to a ratio of equality. During this approach of L to E, the mixed figure ERLKH approaches to a rectangle, whose base is EH and altitude is LQ, whilst the parabolic sector FLRE approaches to a triangle, whose base is EF and whose altitude is LP. That is, the two figures approach to a rectangle and a triangle respectively, whose bases are equal and whose altitudes are equal.

If, now, we conceive the line EH to move parallel to its present position, having one extremity in the directrix, and so changing its length as to have the other extremity always in the curve, whilst the line EF simultaneously revolves round F, and having its other extremity coincident with the extremity of EH, which moves along the curve; then at two *consecutive points*, in which the point E is found during the motion, the secant EL will be virtually coincident with the tangent at either of them, and all the preceding conditions will be fulfilled—namely, that EK will be a rectangle, and FLE a triangle of equal bases and altitudes. Hence, (Geom. 26, or Euc. i. 41.) the rectangle EK is double the triangle FLE.

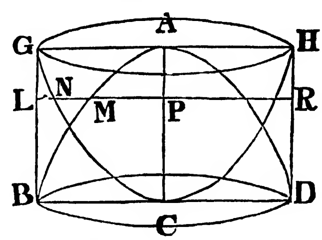
The line HF will therefore describe a series of rectangles, and EF a series of triangles, each of the former being double of the corresponding one of the latter. Hence (Geom. 72, or Euc. v. 12.), the whole external area EHGAE is double of the whole internal area EAF.

Moreover, GM = FD and MN = DE (constr.), and hence the rectangle GN is double the triangle FED. Whence the whole space NEAM is double the space AED, or, which is the same thing, the area of the parabola is one-third

of the rectangle ND. But AG = AF (prop. ii.), and GM = FD (constr.), and hence AM = AD, and the rectangle SD is half the rectangle ND. Consequently the parabolic area EAD (being one-third of the rectangle ND) is two-thirds of the rectangle SD, or of the rectangle which circumscribes the parabola. Q. E. D. *

Cor. If two parabolas have a common vertex and axis, and be both cut by a right ordinate, the areas of the curves are to one another in the sub-duplicate ratios of their parameters. So also are the sectors contained between the curve and lines from the focus to the four points of intersection.

Prop. XVIII. *The solid content of a paraboloid (or solid generated by the rotation of a parabola about its axis), is equal to half its circumscribing cylinder.*



Let GHBD be a cylinder, in which two equal paraboloids are inscribed; one BAD having its base BCD equal to the lower extremity of the cylinder; the other GCH inverted with respect to the former, but of equal base and altitude. Let the plane LR parallel to each end of the cylinder, cut all the three solids, while a vertical plane may be supposed to cut them so as to define the parabolas shown in the figure.

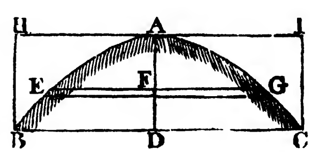
Then, in the semi-parabola ACB, $PM^2 = P \cdot AP$,
also, in the semi-parabola ACG, $PN^2 = P \cdot CP$;
consequently by addition, $PM^2 + PN^2 = P \cdot (AP + CP) = P \cdot AC$.
$= CB^2 = PL^2$.

That is, since circles are as the squares of their radii, the circular section of the cylinder, is equal to the sum of the corresponding sections of the two paraboloids.

And as the same property obtains for all sections parallel to BD, it, therefore, holds for the two paraboloids which are made up of these sections. In other words, the cylinder is equal to the two paraboloids taken together: wherefore, since the two paraboloids, having equal bases and equal altitudes, are equal to one another, it follows that each paraboloid is half of its circumscribing cylinder. Q. E. D.

Cor. If π denote as usual the perimeter of a circle to the diameter 1, the area of the base of the cylinder is $\frac{\pi}{4} \cdot BD^2 = \pi \cdot BC^2, = \pi \cdot P \cdot AC$, and the paraboloid is $\frac{\pi}{2} \cdot P \cdot AC^2$.

Prop. XIX. *The solidity of the frustum BEGC of the paraboloid, is equal to a cylinder whose height is DF, and its base half the sum of the two circular bases, EG, BC.*



By cor. prop. xviii. we have

$\frac{\pi}{2} \cdot P \cdot AD^2 =$ paraboloid BAC and

$\frac{\pi}{2} \cdot P \cdot AF^2 =$ paraboloid EAG; hence

* This demonstration is a modification of one given by Lieut. Drummond of the Royal Engineers, when he was a gentleman cadet at the Royal Military Academy.

$$\text{Frustum BEGC} = \frac{\pi}{2} \cdot P \cdot (AD^2 - AF^2)$$
$$= \frac{\pi}{2} \cdot P\,(AD - AF)\,(AD + AF)$$
$$= \frac{\pi}{2} \cdot P \cdot FD\,(AD + AF)$$
$$= \frac{\pi}{2} \cdot FD\,(P \cdot AD + P \cdot AF)$$
$$= \frac{\pi}{2} \cdot FD\,(BD^2 + EF^2)$$
$$= \frac{1}{2} \cdot FD\,(\pi \cdot BD^2 + \pi \cdot EF^2)$$
$$= \frac{1}{2} \cdot FD\,(\text{circle on BC} + \text{circle on EG}).$$

Q. E. D.

PROBLEMS, &c. FOR EXERCISE IN CONIC SECTIONS.

1. Demonstrate that if a cylinder be cut obliquely the section will be an ellipse.

2. Show how to draw a tangent to an ellipse whose foci are F, *f*, from a given point P, situated on or without the curve.

3. Show how to draw a tangent and a normal to a given parabola from a given point P, either in, or without the curve.

4. The diameters of an ellipse are 16 and 12. Required the parameter and the area, and the magnitude of the equal conjugate diameters.

5. The base and altitude of a parabola are 12 and 9. Required the parameter, and the ordinates corresponding to the abscissæ 2, 3, and 4.

6. In the actual formation of arches, the voussoirs or arch-stones are so cut as to have their faces always *perpendicular* to the respective points of the curve upon which they stand. By what constructions may this be effected for the parabola and the ellipse?

7. Construct accurately on paper, a parabola whose base shall be 12 and altitude 9.

8. A cone, the diameter of whose base is 10 inches, and whose altitude is 12, is cut obliquely by a plane, which enters at 3 inches from the vertex on one slant slide, and comes out at 3 inches from the base on the opposite slant side. Required the dimensions of the section?

9. Suppose the same cone to be cut by a plane parallel to one of the slant sides, entering the other slant side at 4 inches from the vertex, what will be the dimensions of the section?

10. Let any straight line EFR be drawn through F, the focus, of a parabola, and terminated by the curve in E and R; then it is to be demonstrated that EF . FR = ER . ¼ parameter, and that any two such chords have the same ratio as the rectangles of the segments into which they are divided at the focus.

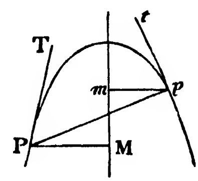
11. All circles are similar figures: all parabolas are similar figures: and hyperbolas between the same asymptotes are similar figures.

12. Similar ellipses or similar hyperbolas, have their axes in the same ratio: and if two similar figures of either kind have either their centres or a focus, coin-

cident in position, and their transverse axes also coincident in direction, all lines drawn through the common centre or common focus, will be divided by the curves in the same ratio.

13. To describe a polygon in a conic section similar to a given polygon in a given and similar conic section.

14. Let TP, tp be tangents to a parabola whose axis is VM at the extremities P and p of the right ordinates PM, pm: then



1. tan. TPM : tan. tpm :: PM : pm, and
2. tan. TPM − tan. tpm = 2 tan. pPM.

Required demonstrations of these two useful theorems in practical gunnery.

15. If the opposite sides of a hexagon inscribed in a conic section be produced till they meet, the three points of intersection will be in the same straight line: and if the opposite angles of a hexagon circumscribed about a conic section be joined by the diagonals, these three diagonals will intersect in the same point.

16. If any point in the plane of a given conic section be given, then a line can be found such that all lines drawn through the point will be harmonically divided by the curve and that line: and, conversely, if the line be given, the point can be found.

17. If the opposite sides of a quadrilateral inscribed in a conic section be produced to meet, and likewise the opposite sides of quadrilateral whose sides touch the curve at the angles of the inscribed one be also produced to meet: then the four points of intersection are in the same straight line; and the four diagonals of the two quadrilaterals intersect in the same point.

18. If about two points as poles two given angles revolve, so that one leg of one angle intersect one leg of the other angle in a straight line, then the other legs will always intersect in a conic section.

19. If three lines revolve about these given points so that two of their intersections are always in two given lines, then the third point of intersection will always be in a conic section.

20. If tangents be drawn to touch two of three similar, and similarly situated conic sections, then the three points of mutual intersection of those tangents will be in the same straight line.

ON GEODESIC OPERATIONS, AND GEOMORPHY, OR THE FIGURE OF THE EARTH.

SECTION I.

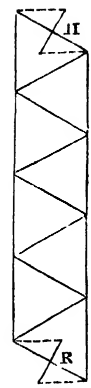
General Account of this Kind of Surveying.

Art 1. In the treatise on Land Surveying in the first volume of this Course of Mathematics, the directions were restricted to the necessary operations for surveying fields, farms, lordships, or at most counties; these being the only operations in which the generality of persons, who practise this kind of measurement, are likely to be engaged: but there are especial occasions when it is

requisite to apply the principles of plane and spherical geometry, and the practices of surveying, to much more extensive portions of the earth's surface; and when of course much care and judgment are called into exercise, both with regard to the direction of the practical operations, and the management of the computations. The extensive processes which we are now about to consider, and which are characterised by the terms *Geodesic Operations* and *Trigonometrical Snrveying*, are usually undertaken for the accomplishment of one of these three objects. 1. The finding the difference of longitude, between two moderately distant and noted meridians; as the meridians of the observatories at Greenwich and Oxford, or of those at Greenwich and Paris. 2. The accurate determination of the geographical positions of the principal places, whether on the coast or inland, in an island or kingdom; with a view to give greater accuracy to maps, and to accommodate the navigator with the actual position, as to latitude and longitude, of the principal promontories, havens, and ports. These have, till lately, been desiderata, even in this country: the position of some important points, as the Lizard, not being known within seven minutes of a degree; and, until the publication of the Board of Ordnance maps, the best county maps being so erroneous, as in some cases to exhibit *blunders of three miles in distances of less than twenty*. 3. The measurement of a degree in various situations; and thence the de rmination of the figure and magnitude of the earth.

When objects so important as these are to be attained, it is manifest that, in order to ensure the desirable degree of correctness in the results, the instruments employed, the operations performed, and the computations required, must each have the greatest possible degree of accuracy. Of these, the first depend on the artist; the second on the surveyor, or engineer, who conducts them; and the latter on the theorist and calculator: they are these last which will chiefly engage our attention in the present chapter.

2. In the determination of distances of many miles, whether for the survey of a kingdom, or for the measurement of a degree, the whole line intervening between two extreme points is not *absolutely measured;* for this, on account of the inequalities of the earth's surface, would be always very difficult, and often impossible. But, a line of a few miles in length is very carefully measured on some plain, heath, or marsh, which is so nearly level as to facilitate the measurement of an actually horizontal line; and this line being assumed as the base of the operations, a variety of hills and elevated spots are selected, at which signals can be placed, suitably distant and visible one from another: the straight lines joining these points constitute a double series of triangles, of which the assumed base forms the first side; the angles of these, that is, the angles made at each station or signal staff, by two other signal staffs, are carefully measured by a theodolite, which is carried successively from one station to another. In such a series of triangles, care being always taken that one side is common to two of them, all the angles are known from the observations at the several stations; and a side of one of them being given, namely, that of the base measured, the sides of all the rest, as well as the distance from the first angle of the first triangle, to any part of the last triangle, may be found by the rules of trigonometry. And so, again, the bearing of any one of the sides, with respect to the meridian, being determined by observation, the bearings of any of the rest, with respect to the same meridian, will be known by computation. In these operations, it is always advisable, when circumstances will admit of it, to measure another base (called a base of verification) at or near the ulterior extremity of the series: for the length of this base, *computed* as one of the sides of the chain of triangles, compared with its length determined by *actual admeasurement*, will be a test of the accuracy of all the operations made in the series between the two bases.



3. Now, in every series of triangles, where each angle is to be ascertained with the same instrument, they should, as nearly as circumstances will permit, be equilateral. For, if it were possible to choose the stations in such manner, that each angle should be exactly 60 degrees; then, the half number of triangles in the series, multiplied into the length of one side of either triangle, would, as in the annexed figure, give at once the total distance; and then also, not only the sides of the scale or ladder, constituted by this series of triangles, would be perfectly parallel, but the diagonal steps, marking the progress from one extremity to the other, would be alternately parallel throughout the whole length. Here, too, the first side might be found by a base crossing it perpendicularly of about half its length, as at H; and the last side verified by another such base, R, at the opposite extremity. If the respective sides of the series of triangles were 12 or 18 miles, these bases might advantageously be between 6 and 7, or between 9 and 10 miles respectively; according to circumstances. It may also be remarked (and the reason of it will be seen in the next section), that whenever only two angles of a triangle can be actually observed, each of them should be as nearly as possible 45°, or the sum of them about 90°; for the less the third or computed angle differs from 90°, the less probability there will be of any considerable error. See prob. 1, sect. 2, of this chapter.

4. The student may obtain a general notion of the method employed in measuring an arc of the meridian, from the following brief sketch and introductory illustrations.

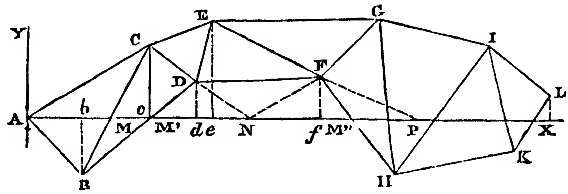
The earth, it is well known, is nearly spherical. It may be either an ellipsoid of revolution, that is, a body formed by the rotation of an ellipse, the ratio of whose axes is nearly that of equality, on one of those axes; or it may approach nearly to the form of such an ellipsoid, or spheroid, while its deviations from that form, though small relatively, may still be sufficiently great in themselves, to prevent its being called a spheroid with much more propriety than it is called a sphere. One of the methods made use of to determine this point, is by means of extensive Geodesic operations.

The earth, however, be its exact form what it may, is a planet, which not only revolves in an orbit, but turns upon an axis. Now, if we conceive a plane to pass through the axis of rotation of the earth, and through the zenith of any place on its surface, this plane, if prolonged to the limits of the apparent celestial sphere, would there trace the circumference of a great circle, which would be the *meridian* of that place. All the points of the earth's surface, which have their zenith in that circumference, will be under the same celestial meridian, and will form the corresponding *terrestrial meridian*. If the earth be an irregular spheroid, this meridian will be a curve of double curvature; but if the earth be a solid of revolution, the terrestrial meridian will be a plane curve.

5. If the earth were a sphere, then every point upon a terrestrial meridian would be at an equal distance from the centre, and of consequence every degree upon that meridian would be of equal length. But if the earth be an ellipsoid of revolution slightly flattened at its poles, and protuberant at the equator; then, as will be soon shown, the degrees of the terrestrial meridian, in receding from the equator towards the poles, will be increased in the duplicate ratio of the right sine of the latitude; and the ratio of the earth's axes, as well as their actual magnitude, may be ascertained by comparing the lengths of a degree on the meridian in different latitudes. Hence appears the great importance of measuring a degree.

6. Now, instead of actually tracing a meridian on the surface of the earth,—a measure which is prevented by the interposition of mountains, woods, rivers, and seas,—a construction is employed which furnishes the same result. It consists in this.

Let ABCDEF, &c. be a series of triangles, carried on, as nearly as may be, in the direction of the meridian, according to the observations in art. 3. These



triangles are really *spherical* or *spheroidal* triangles; but as their curvature is extremely small, they are treated the same as rectilinear triangles, either by reducing them to the *chords* of the respective terrestrial arcs AC, AB, BC, &c. or by deducting *a third* of the excess, of the sum of the three angles of each triangle above two right angles, from each angle of that triangle, and working with the remainders, and the three sides, as the dimensions of a plane triangle; the proper reductions to the centre of the station, to the horizon, and to the level of the sea, having been previously made. These computations being made throughout the series, the sides of the successive triangles are contemplated as arcs of the terrestrial spheroid. Suppose that we know, by observation, and the computations which will be explained in this chapter, the *azimuth*, or the inclination of the side AC to the first portion AM of the measured meridian, and that we find, by trigonometry, the point M where that curve will cut the side BC. The points A, B, C, being in the same horizontal plane, the line AM will also be in that plane: but, because of the curvature of the earth, the prolongation MM′, of that line, will be found *above* the plane of the second horizontal triangle BCD: if, therefore, without changing the angle CMM′, the line MM′ be brought down to coincide with the plane of this second triangle, by being turned about BC as an axis, the point M′ will describe an arc of a circle, which will be so very small, that it may be regarded as a right line perpendicular to the plane BCD: whence it follows, that the operation is reduced to bending down the side MM′ in the plane of the meridian, and calculating the distance AMM′, to find the position of the point M′. By bending down thus in imagination, one after another, the parts of the meridian on the corresponding horizontal angles we may obtain, by the aid of the computation, the direction and the length of such meridian, from one extremity of the series of triangles, to the other.

A line traced in the manner we have now been describing, or deduced from trigonometrical measures, by the means we have indicated, is called a *geodetic* or *geodesic line:* it has the property of being the shortest which can be drawn between its two extremities on the surface of the earth; and it is therefore the proper itinerary measure of the distance between those two points. Speaking rigorously, this curve differs a *little* from the terrestrial meridian, when the earth is not a solid of revolution: yet, in the real state of things, the difference between the two curves is so extremely minute, that it may safely be disregarded.

7. If now we conceive a circle perpendicular to the celestial meridian, and passing through the vertical of the place of the observer, it will represent the prime vertical of that place. The series of all the points of the earth's surface which have their zenith in the circumference of this circle will form the *perpen-*

dicular to the meridian, which may be traced in like manner as the meridian it self.

In the sphere the perpendiculars to the meridian are great circles which all intersect mutually, on the equator, in two points diametrically opposite: but in the ellipsoid of revolution, and *à fortiori* in the irregular spheroid, these concurring perpendiculars are curves of double curvature. Whatever be the nature of the terrestrial spheroid, the parallels to the equator are curves of which all the points are at the same latitude: on an ellipsoid of revolution, these curves are plane and circular.

8. The situation of a place is determined, when we know either the individual perpendicular to the meridian, or the individual parallel to the equator, on which it is found, and its position on such perpendicular, or on such parallel. Therefore, when all the triangles, which constitute such a series as we have spoken of, have been computed, according to the principles just sketched, the respective positions of their angular points, either by means of their longitudes and latitudes, or of their distances from the first meridian, and from the perpendicular to it, may be determined. The following is the method of computing these distances.

Suppose that the triangles ABC, BCD, &c. (see the fig. to art. 6), make part of a chain of triangles, of which the sides are arcs of great circles of a sphere, whose radius is the distance from the level or surface of the sea to the centre of the earth; and that we know by observation the angle CAX, which measures the *azimuth* of the side AC, or its inclination to the meridian AX. Then, having found the excess E, of the three angles of the triangle ACc (Cc being perpendicular to the meridian) above two right angles, by reason of a theorem which will be demonstrated in prob. 8 of this chapter, subtract a third of this excess from each angle of the triangle, and thus by means of the following proportions find Ac and Cc.

$$\sin.\ (90^\circ - \tfrac{1}{3}E) : \cos.\ (CAc - \tfrac{1}{3}E) :: AC : Ac;$$
$$\sin.\ (90^\circ - \tfrac{1}{3}E) : \sin.\ (CAc - \tfrac{1}{3}E) :: AC : Cc.$$

The azimuth of AB is known immediately, because BAX = CAB — CAX; and if the spherical excess proper to the triangle ABM′ be computed, we shall have

$$AM'B = 180^\circ - M'AB - ABM' + E.$$

To determine the sides AM′, BM′, a third of E must be deducted from each of the angles of the triangle ABM′; and then these proportions will obtain: viz,

$$\sin.\ (180^\circ - M'AB - ABM' + \tfrac{2}{3}E) : \sin.\ (ABM' - \tfrac{1}{3}E) :: AB : AM',$$
$$\sin.\ (180^\circ - M'AB - ABM' + \tfrac{2}{3}E) : \sin.\ (M'AB - \tfrac{1}{3}E) :: AB : BM'.$$

In each of the right-angled triangles A*b*B, M′*d*D, are known two angles and the hypothenuse, which is all that is necessary to determine the sides A*b*, *b*B, and M′*d*, *d*D. Therefore the distances of the points B, D, from the meridian and from the perpendicular, are known.

9. Proceeding in the same manner with the triangle ACN, or M′DN, to obtain AN and DN, the prolongation of CD; and then with the triangle DNF to find the side NF and the angles DNF, DFN, it will be easy to calculate the rectangular co-ordinates of the point F.

The distance *f*F and the angles DEN, NF*f*, being thus known, we shall have (th. 6, cor. 3. Geom.)

$$fFP = 180^\circ - EFD - DFN - NFf.$$

So that, in the right-angled triangle *f*FP, two angles and one side are known; and therefore the appropriate spherical excess may be computed, and thence the angle FP*f* and the sides *f*P, FP. Resolving next the right-angled triangle *e*EP, we shall in like manner obtain the position of the point E, with respect to the

meridian AX, and to its perpendicular AY; that is to say, the distances E*e*, and A*e* = AP − *e*P. And thus may the computist proceed through the whole of the series. It is requisite, however, previous to these calculations, to draw, by any suitable scale, the chain of triangles observed, in order to see whether any of the subsidiary triangles ACN, NFP, &c. formed to facilitate the computation of the distances from the meridian, and from the perpendicular to it, are too obtuse or too acute.

Such, in few words, is the method to be followed, when we have principally in view the finding the length of the portion of the meridian comprised between any two points, as A and X. It is obvious that, as in the course of the computations, the azimuths of a great number of the sides of triangles in the series are determined; it will be easy therefore to check and verify the work in its progress, by comparing the azimuths found by observation, with those resulting from the calculations. The amplitude of the whole arc of the meridian measured, is found by ascertaining the *latitude* at each of its extremities; that is, commonly by finding the differences of the zenith distances of some known fixed star, at both those extremities.

10. Some mathematicians, employed in this kind of operations, have adopted different means from the above. They draw, through the summits of all the triangles, parallels to the meridian and to its perpendicular · by these means, the sides of the triangles become the hypothenuses of right-angled triangles, which they compute in order, proceeding from some known azimuth, and without regarding the spherical excess, considering all the triangles of the chain as described on a plain surface. This method, however, is manifestly defective in point of accuracy.

Others have computed the sides and angles of all the triangles, by the rules of spherical trigonometry. Others, again, reduce the observed angles to angles of the chords of the respective arches; and calculate by plane trigonometry, from such reduced angles and their chords. Either of these two methods is equally correct with that by means of the spherical excess: so that the principal reason for preferring one of these to the other must be derived from its greater facility. As to the methods in which the several triangles are contemplated as spheroidal, they are abstruse and difficult, and may, happily, be safely disregarded: for M. Legendre has demonstrated in *Mémoires de la Classe des Sciences Physiques et Mathématiques de l'Institut*, 1806, p. 130, that the difference between spherical and spheroidal angles is less than *one-sixtieth* of a second, in the greatest of the triangles which occurred in the late measurement of an arc of a meridian between the parallels of Dunkirk and Barcelona.

11. Trigonometrical surveys for the purpose of measuring a degree of a meridian in different latitudes, and thence inferring the figure of the earth, have been undertaken by different philosophers, under the patronage of different governments. As by M. Maupertuis, Clairaut, &c. in Lapland, 1736; by M. Bouguer and Condamine, at the equator, 1736—1743; by Cassini, in lat. 45°, 1739—40; by Boscovich and Lemaire, lat. 43°, 1752; by Beccaria, lat. 44°44, 1768; by Mason and Dixon in America, 1764—8; by Colonel Lambton, and Colonels Hodgson and Everest in the East Indies, 1803, &c.; by Mechain, Delambre, &c. France, 1790—1805; by Swanberg, Ofverbom, &c. in Lapland, 1802; and by General Roy, Colonel Williams, Mr. Dalby, General Mudge, and Colonel Colby, in Britain, from 1784 to the present time. The three last-mentioned of these surveys are doubtless the most accurate and important.

The trigonometrical survey in England was first commenced, in conjunction with similar operations in France, in order to determine the difference of longi-

tude between the meridians of the Greenwich and Paris observatories: for this purpose, three of the French Academicians, MM. Cassini, Mechain, and Legendre, met General Roy and Sir Charles Blagden, at Dover, to adjust their plans of operation. In the course of the survey, however, the English philosophers, selected from the Royal Artillery and Engineer officers, expanded their views, and pursued their operations, under the patronage and at the expense of the Honourable Board of Ordnance, in order to perfect the geography of England, and to determine the lengths of as many degrees on the meridian as fell within the compass of their labours.

12. It is not our province to enter into the history of these surveys: but it may be interesting and instructive to speak a little of the instruments employed, and of the extreme accuracy of some of the results obtained by them.

These instruments are, besides the signals, those for measuring distances, and those for measuring angles. The French philosophers used for the former purpose, in their measurement to determine the length of the *metre*, rulers of platina and of copper, forming metallic thermometers. The Swedish mathematicians, Swanberg and Ofverbom, employed iron bars, covered towards each extremity with plates of silver. General Roy commenced his measurement of the base at Hounslow-Heath with *deal* rods, each of 20 feet in length. Though they, however, were made of the best seasoned timber, were perfectly straight, and were secured from bending in the most effectual manner; yet the changes in their lengths, occasioned by the variable moisture and dryness of the air, were so great, as to take away all confidence in the results deduced from them. Afterwards, in consequence of having found by experiments, that a solid bar of glass is more dilatable than a tube of the same matter, glass tubes were substituted for the deal rods. They were each 20 feet long, inclosed in wooden frames, so as to allow only of expansion or contraction in length, from heat or cold, according to a law ascertained by experiments. The base measured with these was found to be 27404·08 feet, or about 5·19 miles. Several years afterwards the same base was measured by General Mudge, with a steel-chain of 100 feet long, constructed by Ramsden, and jointed somewhat like a watch-chain. This chain was always stretched to the same tension, supported on troughs laid horizontally, and allowances were made for changes in its length by reason of variations of temperature, at the rate of ·0075 of an inch for each degree of heat from 62° of Fahrenheit: the result of the measurement by this chain was found not to differ more than $2\frac{3}{4}$ inches, from General Roy's determination by means of the glass tubes: a minute difference in a distance of more than 5 miles; which, considering that the measurements were effected by different persons, and with different instruments, is a remarkable confirmation of the accuracy of both operations. And further, as steel chains can be used with more facility and convenience than glass rods, this remeasurement determines the question of the comparative fitness of these two kinds of instruments. Still greater improvements, however, in the construction of apparatus for the measurement of a base, have been lately introduced into the survey, by its scientific and indefatigable conductor, Colonel Colby.

13. For the determination of angles, the French and Swedish philosophers employed *repeating circles* of Borda's construction: instruments which are extremely portable, and with which, though they are not above 14 inches in diameter, the observers can take angles to within 1″ or 2″ of the truth. But this kind of instrument, however great its ingenuity in theory, has the accuracy of its observations necessarily limited by the imperfections of the *small* telescope which must be attached to it. Generals Roy and Mudge made use of a very

excellent theodolite constructed by Ramsden, which, having both an altitude and an azimuth circle, combines the powers of a theodolite, a quadrant, and a transit instrument, and is capable of measuring horizontal angles to fractions of a second. This instrument, besides, has a telescope of a much higher magnifying power than had ever before been applied to observations purely terrestrial; and this is one of the superiorities in its construction, to which is to be ascribed the extreme accuracy in the results of this trigonometrical survey.

Another circumstance which has augmented the accuracy of the English measures, arises from the mode of fixing and using this theodolite. In the method pursued by the Continental mathematicians, a reduction is necessary to the plane of the horizon, and another to bring the observed angles to the true angles at the centres of the signals: these reductions, of course, require formulæ of computation, the actual employment of which *may* lead to error. But, in the trigonometrical survey of England, great care has always been taken to place the centre of the theodolite exactly in the vertical line, previously or subsequently occupied by the centre of the signal: the theodolite is also placed in a perfectly horizontal position. Indeed, as was observed by Professor Playfair, "In no other survey has the work in the field been conducted so much with a view to save that in the closet, and at the same time to avoid all those causes of error, however minute, that are not essentially involved in the nature of the problem. The French mathematicians trust to the *correction* of those errors; the English endeavour to *cut them off* entirely; and it can hardly be doubted that the latter, though perhaps the slower and more expensive, is by far the safest proceeding."

14. With a view to facilitate the observation of distant stations, many contrivances have been adopted; among which those recently (1826) invented by Lieutenant Drummond, R. E., deserve peculiar notice: of these, one is applicable by day, the other by night. The first, which consists in employing the reflection of the sun from a plane mirror as a point of observation, was first suggested by Professor Gauss; and the result of the first trials made in the survey of Hanover proved very successful. Recourse was had to this method on some occasions that occurred in the Trigonometrical Survey of England, where, from peculiar local circumstances, much difficulty was experienced in discerning the usual signals.

Even as a temporary expedient, and under a rude form, viz. that of placing tin plates at the station to be observed in such a manner that the sun's reflection should be thrown towards the observer at a particular time, the most essential service was derived from its use; and the consequence was, the invention of a more perfect instrument, of which a description is given, accompanied with a drawing, in the Philosophical Transactions for 1830.

The second method consists in the exhibition of a very brilliant light at night. At the commencement of the Survey of England, General Roy had recourse, on several occasions, and especially in carrying his triangles across the Channel, to the use of Bengal and white lights; for these, parabolic reflectors illuminated by Argand lamps were afterwards substituted as more convenient; but from want of power they appear in their turn to have gradually fallen into disuse. With a view to remedy this defect, a series of experiments was undertaken by Lieut. Drummond, the result of which was the production of a very intense light, varying between 60 and 90 times that of the brightest part of the flame of an Argand lamp.

This brilliant light is obtained from a small ball of lime about 3-8ths of an

inch diameter, placed in the focus of the reflector, and exposed to a very intense heat by means of a simple apparatus, of which a description is given, in his account. A jet of oxygen gas directed through the flame of alcohol is employed as the source of heat. Zirconia, magnesia, and oxide of zinc were also tried; but the light emanating from them was much inferior to that from lime. Besides being easily procured, the lime admits of being turned in the lathe, so that any number of the small focal balls may be readily obtained, uniform in size, and perfect in figure. The chemical agency of this light is remarkable, causing the combination of chlorine and hydrogen, and blackening chloride of silver. Its application to the very important purpose of illuminating light-houses is also suggested, especially in those situations where the lights are the first that are made by vessels arriving from distant voyages.

Both the methods now described, for accelerating geodesic operations, were resorted to with much success during the season of 1825 in Ireland; and on one occasion, where every attempt to discern a distant station had failed, the observations were effected by their means, the heliostat being seen during the day, when the outline of the hill ceased to be visible, and the light at night being seen with the naked eye, and appearing much brighter and larger at the distance of 66 miles, than a parabolic reflector, of equal size, illuminated by an Argand lamp, and placed nearly in the same direction, as an object of reference, at the distance of 15 miles.

15. In proof of the great correctness of the English survey, we shall state a very few particulars, besides what is already mentioned in art. 12.

General Roy, who first measured the base on Hounslow Heath, measured another on the flat ground of Romney Marsh in Kent, near the southern extremity of the first series of triangles, and at the distance of more than 60 miles from the first base. The length of this base of verification, as actually measured, compared with that resulting from the computation through the whole series of triangles, differed only by 28 inches.

General Mudge measured another base of verification on Salisbury Plain. Its length was 36574·4 feet, or more than 7 miles; the measurement did not differ more than *one inch* from the computation carried through the series of triangles from Hounslow Heath to Salisbury Plain. A most remarkable proof of the accuracy with which all the angles, as well as the two bases, were measured!

The distance between Beachy Head in Sussex, and Dunnose in the Isle of Wight, as deduced from a mean of four series of triangles, is 339397 feet, or more than $64\frac{1}{4}$ miles. The extremes of the four determinations do not differ more than 7 feet, which is less than $1\frac{2}{3}$ inches in a mile Instances of this kind frequently occur in the English survey. But we have not room to specify more. We must now proceed to discuss the most important problems connected with this subject; and refer those who are desirous to consider it more minutely to General Mudge and Colonel Colby's "Account of the Trigonometrical Survey;" Mechain and Delambre, ' Base du Système Métrique Décimal;" Swanberg, "Exposition des Opérations faites en Lapponie;" Puissant's works entitled "Geodesie," and "Traité de Topographie, d'Arpentage," &c. and Francœur's "Traité de la Figure de la Terre."

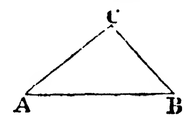
SECTION II.

Problems connected with the detail of Operations in Extensive Trigonometrical Surveys.

PROBLEM I.

It is required to determine the most advantageous conditions of triangles.

1. In any rectilinear triangle ABC, it is, from the proportionality of sides to the sines of their opposite angles, AB : BC :: sin. C : sin. A, and consequently AB . sin. A = BC . sin. C. Let AB be the base, which is supposed to be measured without perceptible error, and which therefore is assumed as constant; then finding the extremely small variation* or fluxion of the equation on this hypothesis, it is AB . cos. A . $\dot{A}$ = sin. C . $\dot{BC}$ + BC . cos. C . $\dot{C}$. Here, since we are ignorant of the magnitude of the errors or variations expressed by $\dot{A}$ and $\dot{C}$, suppose them to be equal (a probable supposition, as they are both taken by the same instrument), and each denoted by v: then will $\dot{BC} = v \times \dfrac{AB \cos. A - BC \cos. C}{\sin. C}$;



or, substituting $\dfrac{BC}{\sin. A}$ for its equal $\dfrac{AB}{\sin. C}$, the equation will become

$$\dot{BC} = v \times \left(BC \cdot \frac{\cos. A}{\sin. A} - BC \cdot \frac{\cos. C}{\sin. C}\right);$$

or finally, $\dot{BC} = v$. BC (cot. A — cot. C).

This equation (in the use of which it must be recollected that v taken in seconds should be divided by R″, that is, by the length of the radius expressed in seconds) gives the error $\dot{BC}$ in the estimation of BC occasioned by the errors in the angles A and C. Hence, that these errors, supposing them to be equal, may have no influence on the determination of BC, we must have A = C, for in that case the second member of the equation will vanish.

2. But, as the two errors, denoted by $\dot{A}$, and $\dot{C}$, which we have supposed to be of the same kind, or in the same direction, may be committed in different directions, when the equation will be $\dot{BC} = \pm v$. BC (cot. A $\mp$ cot. C); we must inquire what magnitude the angles A and C ought to have, so that the sum of their cotangents shall have the least value possible; for in this state it is manifest that $\dot{BC}$ will have its least value. But, by the formulæ in chap. 3, we have

$$\cot. A + \cot. C = \frac{\sin. (A + C)}{\sin. A \sin. C} = \frac{\sin. (A + C)}{\frac{1}{2}\cos. (A \backsim C) - \frac{1}{2}\cos (A + C)} =$$
$$\frac{2 \sin. B}{\cos. (A \backsim C) + \cos. B}.$$

Consequently, $\dot{BC} = \pm v \cdot BC \cdot \dfrac{2 \sin. B}{\cos. (A \backsim C) + \cos. B}$.

And hence, whatever be the magnitude of the angle B, the error in the value of BC will be the least when cos. (A $\backsim$ C) is the greatest possible, which is, when A = C.

* Some of these investigations cannot be thoroughly comprehended without a knowledge of the fluxional or of the differential calculus.

We may therefore infer, for a general rule, that *the most advantageous state of a triangle, when we would determine one side only, is when the base is equal to the side sought.*

3. Since, by this rule, the base should be equal to the side sought, it is evident that *when we would determine two sides, the most advantageous condition of a triangle is that it be equilateral.*

4. It rarely happens, however, that a base can be commodiously measured which is as long as the sides sought. Supposing, therefore, that the length of the base is limited, but that its direction at least may be chosen at pleasure, we proceed to inquire what that direction should be, in the case where one only of the other two sides of the triangle is to be determined.

Let it be imagined, as before, that AB is the base of the triangle ABC, and BC the side required. It is proposed to find the least value of cot. A $\mp$ cot. C, when we cannot have A = C.

Now, in the case where the negative sign obtains, we have

$$\text{cot. A} - \text{cot. C} = \frac{\text{AB} - \text{BC . cos. B}}{\text{BC . sin. B}} - \frac{\text{BC} - \text{AB . cos. B}}{\text{AB . sin. B}} = \frac{\text{AB}^2 - \text{BC}^2}{\text{AB . BC. sin. B}}.$$

This equation again manifestly indicates the equality of AB and BC in circumstances where it is possible: but if AB and BC, are constant, it is evident, from the form of the denominator of the last fraction, that the fraction itself will be the least, or cot. A − cot. C the least, when sin B is a maximum, that is, when B = 90°.

5. When the positive sign obtains, we have cot. A + cot. C =

$$\text{cot. A} + \frac{\sqrt{(\text{BC}^2 - \text{AB}^2\,\text{sin.}^2\,\text{A})}}{\text{AB sin. A}} = \text{cot. A} + \sqrt{\left(\frac{\text{BC}^2}{\text{AB}^2\,\text{sin.}^2\,\text{A}} - 1\right)}.$$

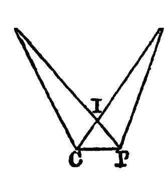
Here, the least value of the expression under the radical sign, is obviously when A = 90°. And in that case the first term, cot. A, would disappear. Therefore the least value of cot. A + cot. C, obtains when A = 90°; conformably to the rule given by M. Bouguer (*Fig. de la Terre*, p. 88). But we have already seen that in the case of cot. A − cot. C, we must have B = 90°: whence we conclude, since the conditions A = 90°, B = 90°, cannot obtain simultaneously, that a medium result would give A = B.

If we apply to the side AC the same reasoning as to BC, similar results will be obtained: therefore in general, *when the base cannot be equal to one or to both the sides required, the most advantageous condition of the triangle is, that the base be the longest possible, and that the two angles at the base be equal.* These equal angles, however, should never, if possible, be less than 23 degrees.

PROBLEM II.

To deduce, from angles measured out of one of the stations, but near it, the true angles at the station.

When the centre of the instrument cannot be placed in the vertical line occupied by the axis of a signal, the angles observed must undergo a reduction, according to circumstances



1. Let C be the centre of the station, P the place of the centre of the instrument, or the summit of the observed angle APB: it is required to find C, the measure of ACB, supposing there to be known APB = P, BPC = p, CP = d, BC = L, AC = R.

Since the exterior angle of a triangle is equal to the sum of the two interior opposite angles (theor. 16 Geom.) we have, with respect

to the triangle IAP, AIB = P + IAP; and with regard to the triangle BIC, AIB = C + CBP. Making these two values of AIB equal, and transposing IAP, there results C = P + IAP − CBP.

But the triangles CAP, CBP, give

$$\sin.\ \text{CAP} = \sin.\ \text{IAP} = \frac{\text{CP}}{\text{AC}} \sin.\ \text{APC} = \frac{d\ .\ \sin.\ (\text{P} + p)}{\text{R}};$$

$$\sin.\ \text{CBP} = \frac{\text{CP}}{\text{BC}}\ .\ \sin.\ \text{BPC} = \frac{d\ .\ \sin.\ p}{\text{L}}.$$

And, as the angles CAP, CBP, are, by the hypothesis of the problem, always very small, their sines may be substituted for their arcs or measures: therefore

$$\text{C} - \text{P} = \frac{d\ .\ \sin.\ (\text{P} + p)}{\text{R}} - \frac{d\ .\ \sin.\ p}{\text{L}}.$$

Or, to have the reduction in seconds,

$$\text{C} - \text{P} = \frac{d}{\sin.\ 1''}\ .\ \left(\frac{\sin.\ (\text{P} + p)}{\text{R}} - \frac{\sin.\ p}{\text{L}}\right).$$

The use of this formula cannot in any case be embarrassing, provided the signs of sin. p, and sin. (P + p) be attended to. Thus, the first term of the correction will be positive, if the angle (P + p) is comprised between 0 and 180°; and it will become negative, if that angle surpass 180°. The contrary will obtain in the same circumstances with regard to the second term, which answers to the angle of direction p. The letter R denotes the distance of the object A to the right, L the distance of the object B situated to the left, and p the angle at the place of observation, between the centre of the station and the object to the left.

2. An approximate reduction to the centre may indeed be obtained by a single term; but it is not quite so correct as the form above. For, by reducing the two fractions in the second member of the last equation but one to a common denominator, the correction becomes

$$\text{C} - \text{P} = \frac{d . \text{L} . \sin. (\text{P} + p) - d . \text{R} . \sin. p}{\text{LR}}.$$

But the triangle ABC gives $\text{L} = \dfrac{\text{R}\ .\ \sin.\ \text{A}}{\sin.\ \text{B}} = \dfrac{\text{R}\ .\ \sin.\ \text{A}}{\sin.\ (\text{A} + \text{C})}$.

And because P is always very nearly equal to C, the sine of A + P will differ extremely little from sin. (A + C), and may therefore be substituted for it, making $\text{L} = \dfrac{\text{R} \sin.\ \text{A}}{\sin.\ (\text{A} + \text{P})}$.

Hence we manifestly have

$$\text{C} - \text{P} = \frac{d\ .\ \sin.\ \text{A}\ .\ \sin. (\text{P} + p) - d\ .\ \sin. p\ .\ \sin.\ (\text{A} + \text{P}}{\text{R}\ .\ \sin.\ \text{A}};$$

Which, by taking the expanded expressions for sin. (P + p), and sin. (A + P), and reducing to seconds, gives

$$\text{C} - \text{P} = \frac{d}{\sin.\ 1''}\ .\ \frac{\sin.\ \text{P}\ .\ \sin.\ \text{A} - p)}{\text{R}\ .\ \sin.\ \text{A}}.$$

3. When either of the distances R, L, becomes infinite, with respect to d, the corresponding term in the expression art. 1 of this problem, vanishes, and we have accordingly

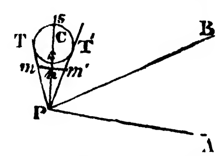
$$\text{C} - \text{P} = -\frac{d\ .\ \sin. p}{\text{L}\ .\ \sin.\ 1''}, \text{ or } \text{C} - \text{P} = \frac{d\ .\ \sin.\ (\text{P} + p)}{\text{R}\ .\ \sin.\ 1''}.$$

The first of these will apply when the object A is a heavenly body, the second when B is one. When both A and B are such, then C − P = 0.

But without supposing either A or B infinite, we may have C − P = 0, or C = P in innumerable instances: that is, in all cases in which the centre P of

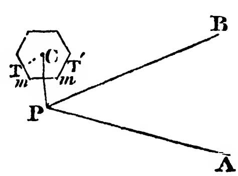
the instrument is placed in the circumference of the circle that passes through the three points A, B, C; or when the angle BPC is equal to the angle BAC, or to $180° - BAC$. Whence, though C should be inaccessible, the angle ACB may commonly be obtained by observation, without any computation. It may further be observed, that when P falls in the circumference of the circle passing through the three points A, B, C, the angles A, B, C, may be determined solely by measuring the angles APB and BPC. For, the opposite angles ABC, APC, of the quadrilateral inscribed in a circle, are (theor. 54 Geom.) $= 180°$. Consequently, $ABC = 180° - APC$, and $BAC = 180° - (ABC + ACB) = 180° - (ABC + APB)$.

4. If one of the objects, viewed from a further station, be a vane or staff in the centre of a steeple, it will frequently happen that such object, when the observer comes near it, is both invisible and inaccessible. Still there are various methods of finding the exact angle at C. Suppose, for example, the signal-staff be in the centre of a circular tower, and that the angle APB was taken at P near its base. Let the tangents PT, PT′, be marked, and on them two equal and arbitrary distances Pm, Pm', be measured. Bisect mm' at the point n; and, placing there a signal staff, measure the angle nPB, which (since Pn prolonged obviously passes through C the centre) will be the angle p of the preceding investigation. Also, the distance Ps added to the radius Cs of the tower, will give $PC = d$ in the former investigation.



If the circumference of the tower cannot be measured, and the radius thence inferred, proceed thus: Measure the angles BPT, BPT′, then will $BPC = \frac{1}{2}(BPT + BPT') = p$; and $CPT = BPT - BPC$: Measure PT, then $PC = PT \,.\, \sec.\, CPT = d$. With the values of p and d, thus obtained, proceed as before.

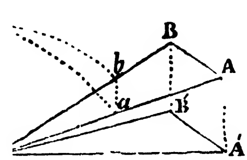
5. If the base of the tower be polygonal and *regular*, as most commonly happens: assume P in the point of intersection of two of the sides prolonged, and $BPC = \frac{1}{2}(BPT + BPT')$ as before, PT = the distance from P to the middle of one of the sides whose prolongation passes through P; and hence PC is found, as above. If the figure be a regular hexagon, then the triangle Pmm is equilateral, and $PC = m'm \sqrt{3}$.



PROBLEM III.

To reduce angles measured in a plane inclined to the horizon, to the corresponding angles in the horizontal plane.

Let BCA be an angle measured in a plane inclined to the horizon, and let B′CA′ be the corresponding angle in the horizontal plane. Let d and d' be the zenith distances, or the complements of the angles of elevation ACA′, BCB′. Then from z the zenith of the observer, or of the angle C, draw the arcs za, zb, of vertical circles, measuring the zenith distances d, d', and draw the arc ab of another great circle to measure the angle C. It follows from this construction, that the angle z, of the spherical triangle zab, is equal to the horizontal angle A′C′B; and that, to find it, the three sides $za = d$, $zb = d'$, $ab = C$, are given. Call the sum of these s; then the resulting formula of prob. 2, ch. iv., applied to the present instance, becomes



$$\sin.\ \tfrac{1}{2}z = \sin.\ \tfrac{1}{2}C = \sqrt{\frac{\sin.\ \frac{1}{2}(s-d)\ .\ \sin.\ \frac{1}{2}(s-d')}{\sin.\ d\ .\ \sin.\ d'}}.$$

If h and h' represent the angles of altitude ACA′, BCB′, the preceding expression will become

$$\sin.\ \tfrac{1}{2}C = \sqrt{\frac{\sin.\ \frac{1}{2}(C + h - h')\ .\ \sin.\ \frac{1}{2}(C + h' - h)}{\cos.\ h\ .\ \cos.\ h'}}.$$

Or, in logarithms,

$$\log.\ \sin.\ \tfrac{1}{2}C = \tfrac{1}{2}[20 + \log.\ \sin.\ \tfrac{1}{2}(C + h - h') + \log.\ \sin.\ \tfrac{1}{2}(C + h' - h) - \log.\ \cos.\ h - \log.\ \cos.\ h'.]$$

Cor. 1. If $h = h'$, then is $\sin.\ \frac{1}{2}C = \frac{\sin.\ \frac{1}{2}ACB}{\cos.\ h}$; and $\log.\ \sin.\ \frac{1}{2}A'CB' = 10 + \log.\ \sin.\ \frac{1}{2}ACB - \log.\ \cos.\ h$.

Cor. 2. If the angles h and h' be very small, and nearly equal; then, since the cosines of small angles vary extremely slowly, we may, without sensible error, take $\log.\ \sin.\ \frac{1}{2}A'CB' = 10 + \log.\ \sin.\ \frac{1}{2}ACB - \log.\ \cos.\ \frac{1}{2}(h + h')$.

Cor. 3. In this case the correction $x = A'CB' - ACB$, may be found by the expression

$$x = \sin.\ 1''\ [\tan.\ \tfrac{1}{2}C\ \left(90^\circ - \frac{d + d'}{2}\right)^2 - \cot.\ \tfrac{1}{2}C\left(\frac{d - d'}{2}\right)^2].$$

And in this formula, as well as the first given for $\sin.\ \frac{1}{2}C$, d and d' may be either one or both greater or less than a quadrant; that is, the equations will obtain whether ACA′ and BCB′ be each an elevation or a depression.

Scholium. By means of this problem, if the altitude of a hill be found barometrically, according to the method described in this volume, or geometrically, according to some of those described in heights and distances, as that announced in the following problem; then, finding the angles formed at the place of observation, by any objects in the country below, and their respective angles of depression, their horizontal angles, and thence their distances, may be found, and their relative places fixed in a map of the country; taking care to have a sufficient number of angles between intersecting lines, to verify the operations.

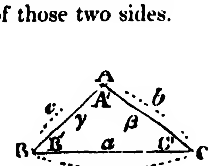
PROBLEM IV.

Given the angles of elevation of any distant object, taken at three places in a horizontal right line, which does not pass through the point directly below the object; and the respective distances between the stations; to find the height of the object, and its distance from either station.

[See a solution under Heights and Distances in vol. i.]

PROBLEM V.

In any spherical triangle, knowing two sides and the included angle; it is required to find the angle comprehended by the chords of those two sides.



Let the angles of the spherical triangle be A, B, C, the corresponding angles included by the chords A′, B′, C′; the spherical sides opposite the former a, b, c, the chords respectively opposite the latter, α, β, γ; then, there are given b, c, and A, to find A′.

Here, from prob. 1, equa. i., page 56, we have

$\cos.\ a = \sin.\ b\ .\ \sin.\ c\ .\ \cos.\ A + \cos.\ b\ .\ \cos.\ c$.

But $\cos.\ c = \cos.\ (\frac{1}{2}c + \frac{1}{2}c) = \cos.^2\ \frac{1}{2}c - \sin.\ \frac{1}{2}c$ by equa. v., ch. iii.) $= (1 - \sin.^2\ \frac{1}{2}c) - \sin.^2\ \frac{1}{2}c = 1 - 2\sin.^2\ \frac{1}{2}c$. And in like manner $\cos.\ a = 1 -$

$2\sin.^2 \frac{1}{2}a$, and $\cos. b = 1 - 2\sin.^2 \frac{1}{2}b$. Therefore the preceding equation becomes

$$1 - 2\sin.^2 \tfrac{1}{2}a = 4\sin. \tfrac{1}{2}b \,.\, \cos. \tfrac{1}{2}b \,.\, \sin. \tfrac{1}{2}c \,.\, \cos. \tfrac{1}{2}c \,.\, \cos. A + (1 - 2\sin.^2 \tfrac{1}{2}b).(1 - 2\sin.^2 \tfrac{1}{2}c).$$

But $\sin. \frac{1}{2}a = \frac{1}{2}\alpha$, $\sin. \frac{1}{2}b = \frac{1}{2}\beta$, $\sin. \frac{1}{2}c = \frac{1}{2}\gamma$: which values substituted in the equation, we obtain, after a little reduction,

$$\frac{\beta^2 + \gamma^2 - \alpha^2}{2} = \beta\gamma \,.\, \cos. \tfrac{1}{2}b \,.\, \cos. \tfrac{1}{2}c \,.\, \cos. A + \tfrac{1}{4}\beta^2\gamma^2.$$

Now, (equa. ii., page 18), $\cos. A' = \dfrac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$. Therefore, by substitution,

$$\beta\gamma \,.\, \cos. A' = \beta\gamma \,.\, \cos. \tfrac{1}{2}b \,.\, \cos. \tfrac{1}{2}c \,.\, \cos. A + \tfrac{1}{4}\beta^2\gamma^2;$$

whence, dividing by $\beta\gamma$, there results

$$\cos. A' = \cos. \tfrac{1}{2}b \,.\, \cos. \tfrac{1}{2}c \,.\, \cos. A + \tfrac{1}{2}\beta \,.\, \tfrac{1}{2}\gamma;$$

or, lastly, by restoring the values of $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$, we have

$$\cos. A' = \cos. \tfrac{1}{2}b \,.\, \cos. \tfrac{1}{2}c \,.\, \cos. A + \sin. \tfrac{1}{2}b \,.\, \sin. \tfrac{1}{2}c \quad . \; . \quad (\text{I.})$$

Cor. 1. It follows evidently from this formula, that when the spherical angle is right or obtuse, it is always *greater* than the corresponding rectilinear angle.

Cor. 2. The spherical angle, if acute, will be *less* than the corresponding rectilinear angle, when we have $\cos. A$ greater than $\dfrac{\sin. \frac{1}{2}b \,.\, \sin. \frac{1}{2}c}{1 - \cos. \frac{1}{2}b \,.\, \cos. \frac{1}{2}c}$.

PROBLEM VI.

Knowing two sides and the included angle of a rectilinear triangle, it is required to find the spherical angle of the two arcs of which those two sides are the chords.

Here β, γ, and the angle A' are given, to find A. Now, since in all cases, $\cos. = \sqrt{(1 - \sin.^2)}$, we have

$$\cos. \tfrac{1}{2}b \,.\, \cos. \tfrac{1}{2}c = \sqrt{[(1 - \sin.^2 \tfrac{1}{2}b) \,.\, (1 - \sin.^2 \tfrac{1}{2}c)]};$$

we have also, as above, $\sin. \frac{1}{2}b = \frac{1}{2}\beta$, and $\sin. \frac{1}{2}c = \frac{1}{2}\gamma$. Substituting these values in the equation i. of the preceding problem, there will result, by reduction,

$$\cos. A = \frac{\cos. A' - \frac{1}{4}\beta\gamma}{\sqrt{(1 - \frac{1}{2}\beta) \,.\, (1 + \frac{1}{2}\beta) \,.\, (1 - \frac{1}{2}\gamma) \,.\, (1 + \frac{1}{2}\gamma)}} \quad . \; . \; . \; . \; . \quad (\text{II.})$$

To compute by this formula, the values of the sides β, γ, must be reduced to the corresponding values of the chords of a circle whose radius is unity. This is easily effected by dividing the values of the sides given in feet, or toises, &c. by such a power of 10, that neither of the sides shall exceed 2, the value of the greatest chord, when radius is equal to unity.

From this investigation, and that of the preceding problem, the following corollaries may be drawn.

Cor. 1. If $c = b$, and of consequence $\gamma = \beta$, then will $\cos. A' = \cos. A \,.\, \cos.^2 \frac{1}{2}c + \sin.^2 \frac{1}{2}c$; and thence $1 - 2\sin.^2 \frac{1}{2}A' = (1 - 2\sin.^2 \frac{1}{2}A)\cos.^2 \frac{1}{2}c + (1 - \cos.^2 \frac{1}{2}c)$: from which may be deduced

$$\sin. \tfrac{1}{2}A' = \sin. \tfrac{1}{2}A \,.\, \cos. \tfrac{1}{2}c \quad . \; . \; . \; . \quad (\text{III.})$$

Cor. 2. Also, since $\cos. \frac{1}{2}c = \sqrt{(1 - \sin.^2 \frac{1}{2}c)} = \sqrt{(1 - \frac{1}{4}\gamma^2)}$, equa. ii., will in this case, reduce to

$$\sin. \tfrac{1}{2}A = \frac{\sin. \frac{1}{2}A'}{\sqrt{(1 - \frac{1}{2}\gamma) \,.\, (1 + \frac{1}{2}\gamma)}} \quad . \; . \; . \; . \; . \quad (\text{IV.})$$

Cor. 3. From the equation iii., it appears that the vertical angle of an isosceles spherical triangle is always *greater* than the corresponding angle of the chords.

Cor. 4. If $A = 90°$, the formulæ i., ii., give

$$\cos. A' = \sin. \tfrac{1}{2}b \,.\, \sin. \tfrac{1}{2}c = \tfrac{1}{4}\beta\gamma \quad . \; . \; . \; . \; . \quad (\text{V.})$$

These five formulæ are strict and rigorous, whatever be the magnitude of the triangle. But if the triangles be small, the arcs may be put instead of the sines in equa. v. then

Cor. 5. As cos. $A' =$ sin. $(90^\circ - A') =$ in this case, $90^\circ - A'$; the small excess of the spherical right angle over the corresponding rectilinear angle, will, supposing the arcs b, c, taken in seconds, be given in seconds by the following expression,

$$90^\circ - A' = \frac{\frac{1}{4}bc}{R''} = \frac{bc}{4R''} \quad \ldots\ldots \quad (\text{VI.})$$

The error in this formula will not amount to a second, when $b + c$ is less than 10°, or than 700 miles measured on the earth's surface.

Cor. 6. If the hypothenuse does not exceed $1\frac{1}{2}^\circ$, we may substitute a sin. C instead of c, and a cos C instead of b; this will give $bc = a^2$. sin. C . cos. C $= \frac{1}{2}a^2$. sin. $2(90^\circ - B) = \frac{1}{2}a^2$. sin. 2B : whence

$$(90^\circ - A') = \frac{a^2 \,.\, \text{sin. } 2C}{8R''} = \frac{a^2 \,.\, \text{sin. } 2B}{8R''} \quad \ldots \quad (\text{VII.})$$

If $a = 1\frac{1}{2}^\circ$, and $B = C = 45^\circ$ nearly; then will $90^\circ - A' = 17''{\cdot}7$.

Cor. 7. Retaining the same hypothesis of $A = 90^\circ$, and $a =$ or $< 1\frac{1}{2}^\circ$, we have $B - B' = \dfrac{b^2 \,.\, \text{cot. B}}{8R''} = \dfrac{bc}{8R''} \quad \ldots \quad (\text{VIII.})$

$$\text{Also } C - C' = \frac{bc}{8R''} \quad \ldots\ldots\ldots \quad (\text{IX.})$$

Cor. 8. Comparing formulæ viii., ix., with vi., we have $B - B' = C - C' = \frac{1}{2}(90^\circ - A')$. Whence it appears that the sum of the two excesses of the oblique spherical angles, over the corresponding angles of the chords, in a small right-angled triangle, is equal to the excess of the right angle over the corresponding angle of the chords. So that either of the formulæ vi., vii., viii., ix., will suffice to determine the difference of each of the three angles of a small right-angled spherical triangle, from the corresponding angles of the chords. And hence *this* method may be applied to the measuring an arc of the meridian by means of a series of triangles. See arts. 8, 9, sect. 1.*

PROBLEM VII.

In a spherical triangle ABC, right angled at A, knowing the hypothenuse BC (*less than* 4°) and the angle B, it is required to find the error e committed through finding by plane trigonometry, the opposite side AC.

Referring still to the diagram of prob. 5, where we now suppose the spherical angle A to be right, we have (theor. 10, page 62) sin. $b =$ sin. a . sin. B. But it has been shown, in Plane Trigonometry, vol. i., that the sine of any arc A is equal to the sum of the following series:

$$\text{sin. } A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$$

$$\text{or, sin. } A = A - \frac{A^3}{6} + \frac{A^5}{120} - \frac{A^7}{5040} + \&c.$$

And, in the present inquiry, all the terms after the second may be neglected, because the 5th power of an arc of 4° divided by 120, gives a quotient not ex-

* On this subject some elegant investigations by Captain, now Colonel, Everest, of the Bengal Artillery, are inserted in the *Memoirs of the Astronomical Society of London*, vol. ii., page 37, &c.

ceeding 0″·01. Consequently, we may assume sin. $b = b - \frac{1}{6}b^3$, sin. $a = a - \frac{1}{6}a^3$; and thus the preceding equation will become,

$$b - \tfrac{1}{6}b^3 = \text{sin. B } (a - \tfrac{1}{6}a^3)$$

$$\text{or, } b = a \text{ . sin. B} - \tfrac{1}{6}(a^3 \text{ . sin. B} - b^3).$$

Now, if the triangle were considered as rectilinear, we should have $b = a$. sin. B; a theorem which manifestly gives the side b or AC too great by $\frac{1}{6}(a^3$. sin. B $- b^3)$. But, neglecting quantities of the fifth power, for the reason already assigned, the last equation but one gives $b^3 = a^3$. sin.^{3} B. Therefore, by substitution, $e = -\frac{1}{6}a^3$. sin. B (1 — sin.^{2} B): or, to have this error in seconds, take R″ = the radius expressed in seconds, so shall $e = - a$. sin. B . $\frac{a^2 \text{ . cos.}^2 \text{ B}}{6\text{R}'' \text{ R}''}$.

Cor. 1. If $a = 4^\circ$, and $\text{B} = 35^\circ 16'$, in which case the value of sin. B . cos.^{2} B is a maximum, we shall find $e = - 4\frac{1}{2}''$.

Cor. 2. If, with the same data, the correction be applied, to find the side c adjacent to the given angle, we should have

$$e' = a \text{ . cos. B . } \frac{a^2 \text{ . sin.}^2 \text{ B}}{3\text{R}'' \text{ R}''}.$$

So that this error is of a contrary kind from the other; the one being subtractive, the other additive.

Cor. 3. The data being the same, if we have to find the angle C, the error to be corrected will be $e'' = a^2 \text{ . } \frac{\text{sin. 2B}}{4\text{R}''}$.

As to the excess of the arc over its chord, it is easy to find it correctly from the expressions in prob. 5: but for arcs that are very small, compared with the radius, a near approximation to that excess will be found in the same measures as the radius of the earth, by taking $\frac{1}{24}$ of the quotient of the cube of the length of the arc divided by the square of the radius.

PROBLEM VIII.

It is required to investigate a theorem, by means of which, spherical triangles, whose sides are small compared with the radius, may be solved by the rules for plane trigonometry, without considering the chords of the respective arcs or sides.

Let a, b, c, be the sides, and A, B, C, the angles of a spherical triangle. on the surface of a sphere whose radius is r: then a similar triangle on the surface of a sphere whose radius = 1, will have for its sides $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$; which, for the sake of brevity, we represent by α, β, γ, respectively: then by equa. i. chap. iv., we have cos. A $= \frac{\text{cos. } \alpha - \text{cos. } \beta \text{ . cos. } \gamma}{\text{sin. } \beta \text{ . sin. } \gamma}$. Now, r being very great with respect to the sides a, b, c, we may, as in the investigation of the last problem, omit all the terms containing higher than 4th powers, in the series for the sine and cosine of an arc, given at the commencement of the Plane Trigonometry, vol. i.: so that we shall have, without perceptible error,

$$\text{cos. } \alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{2.3.4} \text{ . . . sin. } \beta = \beta - \frac{\beta^3}{2.3}.$$

And similar expressions may be employed for cos. β, cos. γ, sin. γ. Thus, the preceding equation will become

$$\text{cos. A} = \frac{\frac{1}{2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24}(\alpha^4 - \beta^4 - \gamma^4) - \frac{1}{4}\beta^2\gamma^2}{\beta\gamma(1 - \frac{1}{6}\beta^2 - \frac{1}{6}\gamma^2)}$$

Multiplying both terms of this fraction by $1 + \frac{1}{6}(\beta^2 + \gamma^2)$, to simplify the denominator, and reducing, there will result,

$$\cos. A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2\beta^2 - 2\alpha^2\gamma^2 - 2\beta^2\gamma^2}{24\beta\gamma}.$$

Here, restoring the values of α, β, γ, the second member of the equation will be entirely constituted of like combinations of the letters, and therefore the whole may be represented by

$$\cos. A = \frac{M}{2bc} + \frac{N}{24bcr^2} \quad . \; . \; . \; . \; (1.)$$

Let, now, A′ represent the angle opposite to the side a, in the rectilinear triangle whose sides are equal in length to the arcs a, b, c; and we shall have

$$\cos. A' = \frac{b^2 + c^2 - a^2}{2bc} = \frac{M}{2bc}.$$

Squaring this, and substituting for $\cos.^2 A'$ its value $1 - \sin.^2 A'$, there will result $-4b^2c^2 \sin.^2 A' = a^2 + b^2 + c^2 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = N$.
So that, equa. i. reduces to the form

$$\cos. A = \cos. A' - \frac{bc}{6r^2} \sin.^2 A'.$$

Let A $A' + x$, then, as x is necessarily very small, its second power may be rejected, and we may assume $\cos. A = \cos. A' - x \,.\, \sin. A'$: whence, substituting for cos. A this value of it, we shall have $x = \frac{bc}{6r^2} \sin. A'$.

It hence appears that x is of the second order, with respect to $\frac{b}{r}$ and $\frac{c}{r}$; and of course that the result is exact to quantities within the fourth order. Therefore, because $A = A' + x$,

$$A = A' + \frac{bc}{6r^2} \,.\, \sin. A'.$$

But, by prob. 2, rule 2, Mensuration of Planes, $\frac{1}{2}bc \sin. A'$ is the area of the rectilinear triangle, whose sides are a, b, and c.

$$\text{Therefore } A = A' + \frac{\text{area}}{3r^2};$$

$$\text{or } A' = A - \frac{\text{area}}{3r^2}.$$

$$\text{In like manner} \begin{cases} B' = B - \dfrac{\text{area}}{3r^2}. \\ C' = C - \dfrac{\text{area}}{3r^2}. \end{cases}$$

$$\text{And } A' + B' = 180° = A + B + C - \frac{\text{area}}{r^2}:$$

$$\text{or, } \frac{\text{area}}{r^2} = A + B + C - 180°.$$

Whence, since the spherical excess is a measure of the area (page 68, note), we have this theorem, viz.

A spherical triangle being proposed, of which the sides are very small, compared with the radius of the sphere: if from each of its angles one-third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, whose sides are equal in length to those of the proposed spherical triangle *.

* This curious theorem was first announced by M. Legendre, in the Memoirs of the Paris Academy, for 1787. Legendre's investigation is nearly the same as the above: a shorter

Scholium.

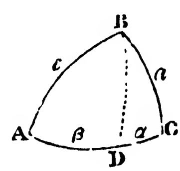
We have already given, at theor. 5, chap. iv., pp. 68—71, expressions for finding the spherical excess, in two or three cases. A few additional rules may with propriety be presented here.

1. The spherical excess E, may be found in seconds, by the expression $E = \frac{R''S}{r}$; where S is the surface of the triangle $= \frac{1}{2}bc \cdot \sin. A = \frac{1}{2}ab \cdot \sin. C = \frac{1}{2}ac \cdot \sin. B = \frac{1}{2}a^2 \cdot \frac{\sin. B \cdot \sin. C}{\sin. (B + C)}$, r is the radius of the earth, in the same measures as a, b, and c, and $R'' = 206264''\cdot8$, the seconds in an arc equal in length to the radius.

If this formula be applied logarithmically; then $\log. R'' = \log. \frac{1}{\text{arc } 1''} = 5\cdot3144251$.

2 From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant log 9·3267737, then the remainder is the logarithm of the excess above 180°, in seconds nearly *.

3. Since $S = \frac{1}{2}bc \cdot \sin. A$, we shall manifestly have $E = \frac{R''}{2r^2} bc \cdot \sin. A$.



Hence, if from the vertical angle B we demit the perpendicular BD upon the base AC, dividing it into the two segments α, β, we shall have $b = \alpha + \beta$, and thence $E = \frac{R''}{2r^2} c\,(\alpha + \beta) \sin. A = \frac{R''}{2R^2} \alpha c \cdot \sin. A + \frac{R''}{2r^2} \beta c \cdot \sin. A$. But the two right angled triangles ABD, CBD, being nearly rectilinear, give $\alpha = A \cdot \cos. C$, and $\beta = c \cdot \cos. A$; whence we have

$$E = \frac{R''}{2r^2} ac \cdot \sin. A \cdot \cos. C + \frac{R''}{2r^2} c^2 \cdot \sin. A \cdot \cos. A.$$

In like manner, the triangle ABC, which itself is so small as to differ but little from a plane triangle, gives $c \cdot \sin. A = a \cdot \sin. C$. Also, $\sin. A \cdot \cos. A = \frac{1}{2} \sin. 2A$, and $\sin. C \cdot \cos. C = \frac{1}{2}\sin. 2C$ (equa. xv. page 25). Therefore, finally,

$$E = \frac{R''}{4r^2} a^2 \cdot \sin. 2C + \frac{R''}{4r^2} c^2 \cdot \sin. 2A.$$

From this theorem a table may be formed, from which the spherical excess may be found; entering the table with each of the sides above the base and its adjacent angle, as arguments.

4. If the base b, and height h, of the triangle are given, then we have evidently $E = \frac{1}{2}bh \frac{R''}{r^2}$. Hence results the following simple logarithmic rule: Add the logarithm of the base of the triangle, taken in feet, to the logarithm of the perpendicular, taken in the same measure; deduct from the sum the

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investigation is given by Svanberg, at p. 40, of his " Exposition des Opérations faites en Lapponie ;" but it is defective in point of perspicuity.

* This is commonly called " General Roy's rule," and given by him in the Philosophical Transactions, for 1790, p. 171 ; it is, however, due to the late Mr. Isaac Dalby, who was then General Roy's assistant in the Trigonometrical Survey, and for several years the entire conductor of the mathematical department. See p. 68.

logarithm 9·6278037; the remainder will be the common logarithm of the spherical excess in seconds and decimals.

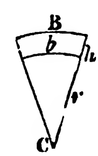
5. Lastly, when the three sides of the triangle are given in feet; add to the logarithm of half their sum, the logs. of the three differences of those sides and that half sum, divide the total of these 4 logs. by 2, and from the quotient subtract the log. 9·3267737; the remainder will be the logarithm of the spherical excess in seconds, &c. as before.

One or other of these rules will apply to all cases in which the spherical excess will be required.

PROBLEM IX.

Given the measure of a base on an elevated level; to find its measure when reduced to the level of the sea.

Let r represent the radius of the earth, or the distance from its centre to the surface of the sea, $r + h$ the radius referred to the level of the base measured, the altitude h being determined by the rule for the measurement of such altitudes by the barometer and thermometer (given farther on in this vol.); let B be the length of the base measured at the elevation h, and b that of the base referred to the level of the sea. Then because the measured base is all along reduced to the horizontal plane, the two, B and b, will be concentric and similar arcs, to the respective radii $r + h$ and r. Therefore, since similar arcs, whether of spheres or spheroids, are as their radii of curvature, we have



$$r + h : r :: \mathrm{B} : b = \frac{r\mathrm{B}}{r+h}.$$

Hence, also $\mathrm{B} - b = \mathrm{B} - \frac{r\mathrm{B}}{r+h} = \frac{\mathrm{B}h}{r+h}$; or, by actually dividing $\mathrm{B}h$ by $r + h$, we shall have

$$\mathrm{B} - b = \mathrm{B} \times \left(\frac{h}{r} - \frac{h^2}{r^2} + \frac{h^3}{r^3} - \frac{h^4}{r^4} + \&c.\right)$$

Which is an *accurate* expression for the excess of B above b.

But the mean radius of the earth being more than 21 million feet, if h the difference of level were 50 feet, the second and all succeeding terms of the series could never exceed the fraction $\frac{1}{176000000000}$; and may therefore safely be neglected; so that for all practical purposes we may assume $\mathrm{B} - b = \frac{\mathrm{B}h}{r}$. Or, in logarithms, add the logarithm of the measured base in feet, to the logarithm of its height above the level of the sea, subtract from the sum the logarithm 7·3223947, the remainder will be the logarithm of a number, which taken from the measured base, will leave the reduced base required.

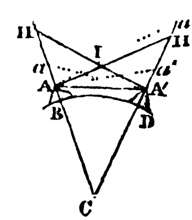
PROBLEM X.

To determine the horizontal refraction.

1. Particles of light, in passing from any object through the atmosphere, or part of it, to the eye, do not proceed in a right line; but the atmosphere being composed of an infinitude of strata (if we may so call them) whose density increases as they are posited nearer the earth, the luminous rays which pass through it are acted on as if they passed successively through media of increasing density, and are therefore inflected more and more towards the earth as the density augments. In consequence of this it is, that rays from objects, whether celestial or terrestrial, proceed in curves which are *concave* towards the earth; and thus it

happens, since the eye always refers the place of objects to the direction in which the rays reach the eye, that is, to the direction of the tangent to the curve at that point, that the apparent, or observed elevations of objects, are always *greater* than the true ones. The difference of these elevations, which is, in fact, the *effect* of refraction, is, for the sake of brevity, called *refraction:* and it is distinguished into two kinds, ***horizontal*** or ***terrestrial*** refraction, being that which affects the altitudes of hills, towers, and other objects on the earth's surface; and ***astronomical*** refraction, or that which is observed with regard to the altitudes of heavenly bodies. Refraction is found to vary with the state of the atmosphere, in regard to heat or cold, humidity or dryness, &c.: so that, determinations obtained for one state of the atmosphere, will not answer correctly for another, without modification. Tables commonly exhibit the refraction at different altitudes, for some assumed mean state.

2. With regard to the *horizontal* refraction, the following method of determining it has been successfully practised in the British Trigonometrical Survey.



Let A, A', be two elevated stations on the surface of the earth, BD the intercepted arc of the earth's surface, C the earth's centre, AH', A'H, the horizontal lines at A, A', produced to meet the opposite vertical lines CH', CH. Let *a*, *a'*, represent the apparent places of the objects A, A', then is *a'*AA' the refraction observed at A, and *a*A'A the refraction observed at A'; and half the sum of those angles will be the horizontal refraction, if we assume it equal at each station.

Now, an instrument being placed at each of the stations A, A', the reciprocal observations are made at the same instant of time, which is determined by means of signals or watches previously regulated for that purpose: that is, the observer at A takes the apparent depression of A', at the same moment that the other observer takes the apparent depression of A.

In the quadrilateral ACA'I, the two angles, A, A' are right angles, and therefore the angles I and C are together equal to two right angles: but the three angles of the triangle IAA' are together equal to two right angles; and consequently the angles A and A' are together equal to the angle C, which is measured by the arc BD. If therefore the sum of the two depressions HA'*a*, H'A*a'*, be taken from the sum of the angles HA'A, H'AA', or, which is equivalent, from the angle C (which is known, because its measure BD is known); the remainder is the sum of both refractions, or angles *a*A'A, *a'*AA'. Hence this rule, ***take the sum of the two depressions from the measure of the intercepted terrestrial arc, half the remainder is the refraction.***

3. If, by reason of the minuteness of the contained arc BD, one of the objects, instead of being depressed, appears elevated, as suppose A' to *a''*: then the sum of the angles *a''*AA' and *a*A'A will be greater than the sum IAA' + IA'A, or than C, by the angle of elevation *a''*AA'; but if from the former sum there be taken the depression HA'A, there will remain the sum of the two refractions. So that in this case the rule becomes as follows: ***take the depression from the sum of the contained arc and elevation, half the remainder is the refraction.***

4. The quantity of this terrestrial refraction is estimated by Dr. Maskelyne at one-tenth of the distance of the object observed, expressed in degrees of a great circle. So, if the distance be 10000 fathoms, its 10th part, 1000 fathoms, is the 60th part of a degree of a great circle on the earth, or 1', which therefore is the refraction in the altitude of the object at that distance.

But M. Legendre is induced, he says, by several experiments, to allow only

$\frac{1}{14}$th part of the distance for the refraction in altitude. So that, on the distance of 10000 fathoms, the 14th part of which is 714 fathoms, he allows only 44″ of terrestrial refraction, so many being contained in the 714 fathoms. See his Memoir concerning the Trigonometrical Operations, &c.

Again, M. Delambre, a late eminent French astronomer, makes the quantity the terrestrial refraction to be the 11th part of the arch of distance. But the English measurers, especially Gen. Mudge, from a multitude of exact observations, determine the quantity of the medium refraction to be the 12th part of the said distance.

The quantity of this refraction, however, is found to vary considerably, with the different states of the weather and atmosphere, from the $\frac{1}{7}$th to the $\frac{1}{18}$th of the contained arc. See *Trigonometrical Survey*, vol. i. p. 160, 355.

Scholium.

Having given the mean results of observations on the terrestrial refraction, it may not be amiss, though we cannot enter at large into the investigation, to present here a correct table of mean astronomical refractions. The table which has been most commonly given in books of astronomy is Dr. Bradley's, computed from the rule $r = 57'' \times \cot.\ (a + 3r)$, where a is the altitude, r the refraction, and $r = 2'35''$ when $a = 20°$. But it has been found by numerous observations, that the refractions thus computed are rather too *small*. Laplace, in his *Mecanique Celeste* (tome iv. page 27) deduces a formula which is strictly similar to Bradley's; for it is $r = m \times \tan.\ (z - nr)$, where z is the zenith distance, and m and n are two constant quantities to be determined from observation. The only advantage of the formula given by the French philosopher, over that given by the English astronomer, is, that Laplace and his colleagues have found more correct coefficients than Bradley had.

Now, if $R = 57°2957795$, the arc equal to the radius, if we make $m = \frac{kR}{n}$ (where k is a constant coefficient which, as well as n, is an abstract number), the preceding equation will become $\frac{nr}{R} = k \times \tan.\ (z - nr)$. Here, as the refraction r is always very small, as well as the correction nr, the trigonometrical tangent of the arc nr may be substituted for $\frac{nr}{R}$; thus we shall have $\tan.\ nr = k\ .\ \tan.\ (z - nr)$.

But $nr = \frac{1}{2}z - (\frac{1}{2}z - nr)\ .\ .\ .\ .\ z - nr = \frac{1}{2}z + (\frac{1}{2}z - nr)$;

$$\text{conseq.}\ \frac{\tan.\ nr}{\tan.\ (z - nr)} = \frac{\tan.\left(\frac{z}{2} - \frac{z - 2nr}{2}\right)}{\tan.\left(\frac{z}{2} + \frac{z - 2nr}{2}\right)} = \frac{\sin.\ z - \sin.\ (z - 2nr)}{\sin.\ z + \sin.\ (z - 2nr)} =$$

Hence, $\sin.\ (z - 2nr) = \frac{1 - k}{1 + k}\ .\ \sin.\ z.$

This formula is easy to use, when the coefficients n and $\frac{1 - k}{1 + k}$ are known: and it has been ascertained, by a mean of many observations, that these are 4 and ·99765175 respectively. Thus Laplace's equation becomes

$$\sin.\ (z - 8r) = \cdot 99765175\ \sin.\ z:$$

and from this the following table has been computed. Besides the refractions, the differences of refraction, for every 10 minutes of altitude, are given; an addition which will render the table more extensively useful in all cases where great accuracy is required.

TABLE OF REFRACTIONS.

Barometer 29 92 *inches, Fahrenheit's Thermometer* 54°.

Alt. app	Refract.	Diff. on 10′	Alt. app.	Refrac.	Diff. 10′	Alt. app.	Refrac.	Diff. 10′.	Alt. app.	Refr.	Diff. 10′.
D. M.	M. S.	S.	D. M.	M. S.	S.	D.	M. S.	S.	D.	S.	S.
0 0	33 46·3	112·0	7 0	7 24·8	9·5	14	3 49·8	2·58	56	39·3	0·25
10	31 54·3	105·0	10	7 15·3	9·0	15	3 34·3	2·28	57	37·8	0·24
20	30 9·3	97·3	20	7 6·3	8·6	16	3 20·6	2·02	58	36·4	0·24
30	28 32·1	89·8	30	6 57·7	8·1	17	3 8·5	1·82	59	35·0	0·23
40	27 2·2	83·6	40	6 49·6	7·7	18	2 57·6	1·65	60	33·6	0·22
50	25 38·6	77·4	50	6 41·9	7·5	19	2 47·7	1·48	61	32·3	0·22
1 0	24 21·2	71·6	8 0	6 34·4	7·3	20	2 38·8	1·37	62	31·0	0·21
10	23 9·6	66·2	10	6 27·1	7·1	21	2 30·6	1·24	63	29·7	0·21
20	22 3·4	61·5	20	6 20 0	6 9	22	2 23·2	1·11	64	28·4	0·20
30	21 1·9	57·1	30	6 13·1	6·7	23	2 16·5	1·05	65	27·2	0·20
40	20 4·8	53·3	40	6 6 4	6·5	24	2 10·2	0·98	66	25·9	0·20
50	19 11·5	49·3	50	5 59·9	6·3	25	2 4·3	0·90	67	24·7	0·20
2 0	18 22·2	45·9	9 0	5 53·6	6·2	26	1 58·9	0·83	68	23·5	0·20
10	17 36 3	43·1	10	5 47·4	5·9	27	1 53·9	0·78	69	22·4	0·20
20	16 53·2	39·8	20	5 41·5	5·7	28	1 49·2	0·73	70	21·2	0·20
30	16 13·4	37·4	30	5 35·8	5·5	29	1 44·8	0·70	71	20·0	0·19
40	15 36·0	35·1	40	5 30 3	5·3	30	1 40·6	0·65	72	18·9	0·18
50	15 0·9	32·8	50	5 25·0	5·2	31	1 36·7	0 60	73	17·8	0·18
3 0	14 28·1	30·8	10 0	5 19·8	5·1	32	1 33·1	0·58	74	16·7	0·18
10	13 57·3	28·8	10	5 14·7	5·0	33	1 29·6	0·56	75	15·6	0 18
20	13 28·5	27·2	20	5 9·7	4·8	34	1 26·2	0·53	76	14·5	0·17
30	13 1·3	25·7	30	5 4·9	4·6	35	1 23·1	0·50	77	13·5	0·17
40	12 35·6	24 3	40	5 0·3	4·4	36	1 20·1	0·48	78	12·4	0·17
50	12 11·3	23·0	50	4 55·9	4 2	37	1 17·2	0·47	79	11·3	0·17
4 0	11 48·3	21·7	11 0	4 51·7	4·1	38	1 14·4	0·43	80	10·3	0·17
10	11 26 6	20·5	10	4 47·6	4 0	39	1 11·8	0·42	81	9·2	0 17
20	11 6·1	19·4	20	4 43·6	4·0	40	1 9·3	0·40	82	8·2	0·17
30	10 46·7	18·4	30	4 39 6	3·9	41	1 6·9	0·38	83	7·2	0·17
40	10 28·3	17·4	40	4 35 7	3·9	42	1 4·6	0·37	84	6·1	0·17
50	10 10 9	16·6	50	4 31·8	3·8	43	1 2·4	0·35	85	5·1	0·17
5 0	9 54·3	15·9	12 0	4 28·0	3·7	44	1 0·3	0·34	86	4·1	0·17
10	9 38·4	15·0	10	4 24·3	3·6	45	0 58·2	0·33	87	3·1	0·17
20	9 23·4	14·4	20	4 20·7	3·5	46	0 56 2	0·32	88	2·0	0·17
30	9 9·0	13·7	30	4 17·2	3·4	47	0 54·3	0·31	89	1·0	0·17
40	8 55·3	13·0	40	4 13·8	3·2	48	0 52·4	0·30	90	0 0	
50	8 42·3	12·4	50	4 10·6	3·1	49	0 50·6	0·29			
6 0	8 29·9	11·8	13 0	4 7·5	3·1	50	0 48·9	0·28			
10	8 18·1	11·5	10	4 4·4	3·0	51	0 47·2	0 27			
20	8 6·6	11·0	20	4 1·4	3·0	52	0 45·5	0·26			
30	7 55·6	10 6	30	3 58·4	2·9	53	0 43·9	0·26			
40	7 45·0	10·3	40	3 55 5	2·9	54	0 42·3	0·25			
50	7 34·7	9·9	50	3 52·6	2·8	55	0 40·8	0·25			
7 0	7 24·8		14 0	3 49·8		56	0 39·3				

For refraction under different temperatures,

$$\text{Ref.} = \frac{a}{29{\cdot}6} \times \tan.\,(z - 3r) \times 57'' \times \frac{400}{350 + h}, \text{ according to Dr. Maskelyne.}$$

$$\text{Ref.} = \frac{a}{29{\cdot}6} \times \tan.\,(z - 3{\cdot}2r) \times 56{\cdot}9'' \times \frac{500}{450 + h}, \text{ according to Dr. Brinkley.}$$

Where $a =$ alt. barometer in inches, $z =$ zenith distance, $r = 57''$ tan. z, $h =$ height of Fahrenheit's thermometer, and 29·6 is assumed for the mean height of the barometer.

On the general subject of astronomical and terrestrial refractions, the reader may advantageously consult an elaborate paper by Mr. H. Atkinson of Newcastle, in *Mem. Astron. Soc.* London, vol. ii.

PROBLEM XI.

To find the angle made by a given line with the meridian.

1. The easiest method of finding the angular distance of a given line from the meridian, is to measure the greatest and the least angular distance of the vertical plane in which the star marked α in Ursa minor (commonly called the *pole star*), is from the said line: for half the sum of these two measures will manifestly be the angle required.

2. Another method is to observe when the sun is on the given line; to measure the altitude of his centre at that time, and correct it for refraction and parallax. Then, in the spherical triangle ZPS, where Z is the zenith of the place of observation, P the elevated pole, and S the centre of the sun, there are supposed given ZS the zenith distance, or co-altitude of the sun, PS the co-declination of that luminary, PZ the co-latitude of the place of observation, and ZPS the hour angle, measured at the rate of 15° to an hour, to find the angle SZP between the meridian PZ and the vertical ZS, on which the sun is at the given time. And here, as three sides and one angle are known, the required angle is readily found, by saying, as sine ZS : sine ZPS :: sine PS : sine PZS; that is, as the cosine of the sun's altitude, is to the sine of the hour angle from noon; so is the cosine of the sun's declination, to the sine of the angle made by the given vertical and the meridian.

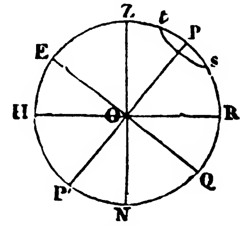
Note. Many other methods are given in books of Astronomy; but the above are sufficient for our present purpose. The first is independent of the latitude of the place; the second requires it.

PROBLEM XII.

To find the latitude of a place.

The latitude of a place may be found by observing the greatest and least altitude of a circumpolar star, and then applying to each the correction for refraction; so shall half the sum of the altitudes, thus corrected, be the altitude of the pole, or the latitude.

For, if P be the elevated pole, *st* the circle described by the star, PR = EZ the latitude: then since P*s* = P*t*, PR must be = ½ (R*t* + R*s*).



This method is obviously independent of the declination of the star: it is therefore most commonly adopted in trigonometrical surveys, in which the telescopes employed are of such power as to enable the observer to see stars in the daytime: the pole-star being here also made use of.

Numerous other methods of solving this problem likewise are given in books of Astronomy; but they need not be detailed here.

Corol. If the mean altitude of a circumpolar star be thus measured, at the two extremities of any arc of a meridian, the difference of the altitudes will be the measure of that arc: and if it be a small arc, one, for example, not exceeding a degree of the terrestrial meridian, since such small arcs differ extremely little from arcs of the circle of curvature at their middle points, we may, by a simple

proportion, infer the length of a degree whose middle point is the middle of that arc

Scholium.

Though it is not consistent with the purpose of this chapter to enter largely into the doctrine of astronomical spherical problems; yet it may be here added, for the sake of the young student, that if a = right ascension, d = declination, l = latitude, λ = longitude, p = angle of position (or, the angle at a heavenly body formed by two great circles, one passing through the pole of the equator and the other through the pole of the ecliptic), i = inclination or obliquity of the ecliptic, then the following equations, most of which are new, obtain generally, for all the stars and heavenly bodies.

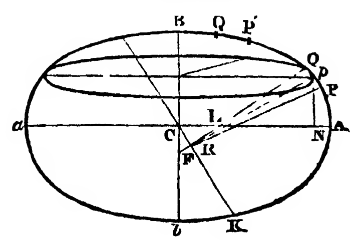
1. $\tan. a = \tan. \lambda . \cos. i - \tan. l . \sec. \lambda . \sin. i.$
2. $\sin. d = \sin. \lambda . \cos. l . \sin. i + \sin. l . \cos. i.$
3. $\tan. \lambda = \sin. i . \tan. d . \sec. a + \tan. a . \cos. i.$
4. $\sin. l = \sin. d . \cos. i - \sin. a . \cos. d . \sin. i.$
5. $\text{cotan.} p = \cos. d . \sec. a . \cot. i + \sin. d . \tan. a.$
6. $\text{cotan.} p = \cos. l . \sec. \lambda . \cot. i - \sin. l . \tan. \lambda.$
7. $\cos. a . d = \cos. l . \cos. \lambda.$
8. $\sin. p . \cos. d = \sin. i . \cos. \lambda.$
9. $\sin. p . \cos. \lambda = \sin. i . \cos. a.$
10. $\tan. a = \tan. \lambda . \cos. i.$ } when $l = 0$, as is always the case, practically,
11. $\cos. \lambda = \cos. a . \cos. d.$ } with the sun.

The investigation of these equations, which is omitted for the sake of brevity, depends on the resolution of the spherical triangle whose angles are the poles of the ecliptic and equator, and the given star, or luminary.

FIGURE OF THE EARTH.

PROBLEM XIII.

The earth being supposed a spheroid, to express the length of a small arc of a meridian at any point, in terms of the difference of latitude of its extremities.



Let APQP be the meridian passing through the extremities P and Q of a small arc of latitude: at P and Q draw normals meeting in R; then R is very nearly the centre of curvature of p, which bisects PQ. And if D be the difference of latitude of P and Q, PQR = D; and hence PQ = D $\times$ pR.

$$\text{Now } pR = \frac{CK^2}{pF} = \frac{AC^2 . BC^2}{pF^3} = \frac{a^2 c^2}{pF^3},$$

putting a and c for the semi-axes.

$$\text{But } pF = \frac{c^2}{pL}; \ \therefore pR = \frac{a^2 \cdot pL^3}{c^4}.$$

And $pLN =$ latitude of $p = l$; $\therefore$ $LN = pL \cos. l$;

$$\therefore CN = \frac{a^2}{c^2} pL \cos. l: \text{ and } Np = pL \sin. l.$$

Substituting these in the equation $\frac{CN^2}{a^2} + \frac{Np^2}{c^2} = 1$, we get

$$(pL)^2 \cdot \frac{a^2 \cos.^2 l + c^2 \sin.^2 l}{c^4} = 1;$$

$$\therefore (pL)^3 = \frac{c^6}{(a^2 \cos.^2 l + c^2 \sin.^2 l)^{\frac{3}{2}}},$$

$$\text{and } pR = \frac{a^2 c^2}{(a^2 \cos.^2 l + c^2 \sin.^2 l)^{\frac{3}{2}}},$$

$$\text{hence } PQ = D \cdot \frac{a^2 c^2}{(a^2 \cos^2 + c^2 \sin.^2 l)^{\frac{3}{2}}}.$$

If the ellipticity be small, let $a = c(1 + e)$; then

$$PQ = \frac{D \cdot c^4 (1 + 2e)}{\{c^2 (1 + 2e) \cos.^2 l + c^2 \sin.^2 l\}^{\frac{3}{2}}} = Dc(1 + 2e - 3e \cos^2 l),$$

$$\text{Or} = D \cdot c (1 - e + 3e \sin.^2 l), \text{ nearly.}$$

Suppose now, that two arcs PQ, $P'Q'$, have been measured. Suppose the latitudes of the middle to be l, l': the difference of the latitudes of P and Q to be D, that of P' and Q' to be D'. Then

$$\frac{PQ}{D} = c(1 - e + 3e \sin.^2 l),$$

$$\frac{P'Q'}{D'} = c(1 - e + 3e \sin.^2 l').$$

$$\text{Subtracting, } \frac{P'Q'}{D'} - \frac{PQ}{D} = 3ce (\sin.^2 l' - \sin.^2 l);$$

$$\therefore e = \frac{\frac{P'Q'}{D'} - \frac{PQ}{D}}{3c(\sin.^2 l' - \sin.^2 l)}.$$

As e is small, we may, without sensible error, put for c, $\frac{PQ}{D}$, or $\frac{P'Q'}{D'}$:

$$\text{And thus we have } e = \frac{1 - \frac{PQ}{Q} \cdot \frac{D'}{D}}{3(\sin.^2 l' - \sin.^2 l)}.$$

Example. By Lambton's measures in India, the arc of the meridian from lat. $8^\circ . 9' . 38''$, 4 to lat $10^\circ . 59' . 48'' . 9 = 1029100$ 5 feet.

By Svanberg's measures in Sweden, the arc of the meridian from lat. $65^\circ . 31' . 32'' . 2$ to lat. $67^\circ . 8' . 49'' . 8 = 59327{\cdot}5$ feet.

Here $PQ = 1029100{\cdot}5$; $P'Q' = 593277{\cdot}5$; $D = 10210''{\cdot}5$; $D' = 5836''{\cdot}6$; $l = 9^\circ . 34' . 44''$; $l' = 66^\circ . 20' . 10''$. Make

$$\frac{PQ}{P'Q'} \cdot \frac{D'}{D} = \cos.^2 \theta,$$

or $2 \log. \cos. \theta = 20 + \log. PQ + \log. D' - \log. P'Q' - \log. D$

$= \log. PQ + \log. D' +$ ar. com. $\log. P'Q' +$ ar. com. $\log. D$:

then the numerator $= \sin.^2 \theta$. And the denominator $= 3 \sin. (l' + l) . \sin. l' - l$.

Hence $\log. e = 2 \log. \sin. \theta - \log. 3 - \log. \sin. (l' + l) - \log. \sin. (l' - l)$.

By calculating from the data above, $e = \frac{1}{306{\cdot}3}$.

Attempts have also been made to determine the ellipticity of the earth by measuring the distance between two places on the same parallel, and determining the difference of longitude, either by observations on Jupiter's satellites, or by

observing the flash of gunpowder fired on a conspicuous place between them. The difference of longitude may again be determined by mere observation of angles, (see *Phil. Trans.* 1790).

PROBLEM XIV.

To express the distance of two places on the same parallel, in terms of their difference of longitude.

Let p, q, (preceding figure) be the places; L their difference of longitude. Then pq (which, when the arc is small, may be measured as a great circle without sensible error,) $= \mathrm{L} \times \mathrm{CN}$.

$$\text{Now CN} = \frac{a^2}{c^2}\,p\mathrm{L}\cos. l = \frac{a^2 . \cos. l}{\sqrt{(a^2\cos.^2 l + c^2 \sin.^2 l)}};$$

$$\therefore pq = \mathrm{L} . \frac{a^2\cos. l}{\sqrt{(a^2\cos.^2 l + c^2\sin^2 l)}} = \mathrm{L} . c\cos. l\,(1 + 2e - e\cos.^2 l)$$

$$= \mathrm{L} . c\cos. l\,(1 + e - e\sin.^2 l), \text{ if the ellipticity be small.}$$

If, therefore, one arc has been measured in the meridian, and another on a parallel, and if l be the latitude of the middle of the meridional arc, l' that of the parallel, we shall have these equations .

$$\frac{\mathrm{PQ}}{\mathrm{D}} = c\,(1 - e + 3e . \sin.^2 l),$$

$$\frac{pq}{\mathrm{L}} = c\cos. l'\,(1 + e + e\sin.^2 l').$$

Eliminating c, e may be found. This method, however, is not considered to be *practically* accurate.

The method which on account of its great facility is now very extensively used, is that of observing the intensity of gravity in different latitudes, by means of the pendulum. It is usual to observe the number of vibrations made in a day by the same pendulum, in the different places at which it is proposed to compare the force of gravity; and likewise the number of vibrations made at London or Paris. The comparative number of vibrations being found, the comparative force of gravity, or the comparative length of the seconds' pendulum, can be found (see the *Mechanics*): and, as the length of the seconds' pendulum has been very accurately determined at London and Paris, its length is known at all the places of observation. The French astronomers have used a method more direct, but less convenient, and probably less accurate: it is described at length in the Additions to Biot's *Astronomie Physique*, p. 138.

Let p and p' be the lengths of the seconds' pendulum in latitudes l and l', P that at the equator. Since these lengths are proportional to the intensities of gravity, we have, by the doctrine of the pendulum—

$$\left.\begin{aligned} p &= \mathrm{P}\,(1 + n\sin.^2 l) \\ p' &= \mathrm{P}\,(1 + n\sin.^2 l') \end{aligned}\right\}, \text{ where } n = \frac{5m}{2} - e: m \text{ being the ratio}$$

of the centrifugal force at the equator to the equatoreal gravity, or $m = \frac{1}{289}$.

From these equations,

$$n = \frac{\mu}{\sin.^2 l' - \sin.^2 l},$$

which may be calculated as the last example: then $e = \frac{5m}{2} - n$.

Example. At Madras, $l = 13°.4'.9''$, $p = 39\cdot0234$.
At Melville Island, $l' = 74°.47'.12''$, $p' = 39\cdot2070$.

Hence, $n = ,0053214$, and $\frac{3m}{2} = ,0086505$;

$$\therefore\ e = ,0033291 = \frac{1}{300}\ *.$$

GENERAL SCHOLIUM AND REMARKS.

1. The value e, or $\frac{a}{c} - 1, = \frac{a-c}{c}$, is called the *compression* of the terrestrial spheroid, and it manifestly becomes known when the ratio $\frac{a}{c}$ is determined. But the measurements of philosophers, however carefully conducted, furnish resulting compressions, in which the discrepancies are much greater than might be wished. General Roy has recorded several of these in the *Phil. Trans.* vol. lxxvii., and later measurers have deduced others. Thus, the degree measured at the equator by Bouguer, compared with that of France measured by Mechain and Delambre, gives for the compression $\frac{1}{334}$, also $a = 3271208$ toises, $c = 3261443$ toises, $a - c = 9765$ toises. General Roy's sixth spheroid, from the degrees at the equator and in latitude 45°, gives $\frac{1}{309{\cdot}3}$. Mr. Dalby makes $a = 3489932$ fathoms, $c = 3473656$, Gen. Mudge $a = 3491420$, $c = 3468007$, or 7935 and 7882 miles. The degree measured at Quito, compared with that measured in Lapland by Swanberg, gives compression $= \frac{1}{309{\cdot}4}$. Svanberg's observations, compared with Bouguer's, give $\frac{1}{329{\cdot}25}$. Svanberg's compared with the degree of Delambre and Mechain $\frac{1}{307{\cdot}4}$. Compared with Major Lambton's degree $\frac{1}{307{\cdot}17}$. A minimum of errors in Lapland, France, and Peru, gives $\frac{1}{323{\cdot}4}$. Laplace, from the lunar motions, finds compression $= \frac{1}{314}$. From the theory of gravity as applied to the latest observations of Burg, Maskelyne, &c. $\frac{1}{309{\cdot}05}$. From the variation of the pendulum in different latitudes $\frac{1}{335{\cdot}78}$ Dr. Robison, assuming the variation of gravity at $\frac{1}{180}$, makes the compression $\frac{1}{319}$. The results from Capt. Sabine's experiments on the pendulum in different latitudes, give $\frac{1}{300}$, † and from Capt. Foster's $\frac{1}{285{\cdot}26}$. Others give results varying from $\frac{1}{178{\cdot}4}$ to $\frac{1}{577}$: but far the greater number of observations differ but little from $\frac{1}{304}$, which the computation from the phenomena of the precession of the equinoxes and the nutation of the earth's axis, gives for the maximum limit of the compression.

* See Mr. *Airy's Mathematical Tracts*, from which the last two problems and subsequent remarks have been taken; and the whole of which the student may most advantageously consult.

† See Ivory in Phil. Mag. July 1826, and Ladies' Diary for 1835.

2. From the various results of careful admeasurements it happens, as Gen. Roy has remarked, "that philosophers are not yet agreed in opinion with regard to the exact figure of the earth; some contending that it has no regular figure, that is, not such as would be generated by the revolution of a curve around its axis. Others have supposed it to be an ellipsoid; regular, if both polar sides should have the same degree of flatness; but irregular if one should be flatter than the other. And lastly, some suppose it to be a spheroid differing from the ellipsoid, but yet such as would be formed by the revolution of a curve around its axis." According to the theory of gravity, however, the earth must of necessity have its axes approaching nearly to either the ratio of 1 to 680 or of 303 to 304; and as the former ratio obviously does not obtain, the figure of the earth *must* be such as to correspond nearly with the latter ratio.

3. Besides the method above described, others for determining the figure of the earth by measurement have been proposed. Thus, that figure might be ascertained by the measurement of a degree in two parallels of latitude; but not so accurately as by meridional arcs, 1st. Because, when the distance of the two stations, in the same parallel, is measured, the celestial arc is not that of a parallel circle, but is nearly the arc of a great circle, and always exceeds the arc that corresponds truly with the terrestrial arc. 2ndly. The interval of the meridian's passing through the two stations must be determined by a time-keeper, a very small error in the going of which will produce a very considerable error in the computation. Other methods which have been proposed, are, by comparing a degree of the meridian in any latitude, with a degree of the curve perpendicular to the meridian in the same latitude; by comparing the measures of degrees of the curves perpendicular to the meridian in different latitudes; and by comparing an arc of a meridian with an arc of the parallel of latitude that crosses it. The theorems connected with these and some other methods are investigated by Professor Playfair in the *Edinburgh Transactions*, vol. v. to which, together with the books mentioned at the end of the 1st section of this chapter, the reader is referred for much useful information on this highly interesting subject.

Having thus solved the chief problems connected with Trigonometrical Surveying, the student is now presented with the following examples by way of exercise.

Ex. 1. The angle subtended by two distant objects at a third object is $66°30'39''$; one of those objects appeared under an elevation of $25'47''$, the other under a depression of $1''$. Required the reduced horizontal angle.

Ans. $66°30'36\frac{1}{2}''$.

Ex. 2. Going along a straight and horizontal road which passed by a tower, I wished to find its height, and for this purpose measured two equal distances each of 84 feet, and at the extremities of those distances took three angles of elevation of the top of the tower, viz. $36°50'$, $21°24'$, and $14°$. What is the height of the tower? Ans. 53·96 feet.

Ex. 3. Investigate General Roy's rule for the spherical excess, given in the scholium to prob. 8.

Ex. 4. The three sides of a triangle measured on the earth's surface (and reduced to the level of the sea) are 17, 18, and 10 miles: what is the spherical excess? Ans. $1''·096$.

Ex. 5. The base and perpendicular of another triangle are 24 and 15 miles. Required the spherical excess. Ans. $2''21'''52\frac{1}{2}^{iv}$.

Ex. 6. In a triangle two sides are 18 and 23 miles, and they include an angle of 58°24′36″. What is the spherical excess? Answer 2″·31639.

Ex. 7. The length of a base measured at an elevation of 38 feet above the level of the sea is 34286 feet: required the length when reduced to that level? Ans. 34285·9379.

Ex. 8. Given the latitude of a place 48°51′N, the sun's declination 18°30′N, and the sun's apparent altitude at $10^h 11^m 26^s$ AM, 52°35′; to find the angle that the vertical on which the sun is, makes with the meridian. Ans. 45°23′2″$\frac{5}{6}$.

Ex. 9. When the sun's longitude is 29°13′43″, what is his right ascension? The obliquity of the ecliptic being 23°27′40″. Ans. 27°10′13″$\frac{1}{6}$.

Ex. 10 Required the longitude of the sun, when his right ascension and declination are 32°46′52″$\frac{1}{2}$, and 13°13′27″N respectively. See the theorems in the scholium to prob. 12.

Ex. 11. The right ascension of the star α Ursæ majoris is 162°50′34″, and the declination 62°50′N: what are the longitude and latitude? The obliquity of the ecliptic being as above.

Ex. 12. Given the measure of a degree on the meridian in N. lat. 49°3′, 60833 fathoms, and of another in N. lat. 12°32′, 60494 fathoms: to find the ratio of the earth's axes.

Ex. 13. Demonstrate that, if the earth's figure be that of an oblate spheroid, a degree of the earth's equator is the first of two mean proportionals between the last and first degrees of latitude.

Ex. 14. Demonstrate that the degrees of the terrestrial meridian, in receding from the equator towards the poles, are increased very nearly in the duplicate ratio of the sine of the latitude.

Ex. 15. If p be the measure of a degree of a great circle perpendicular to a meridian at a certain point, m that of the corresponding degree on the meridian itself, and d the length of a degree on an oblique arc, that arc making an angle a with the meridian, then is $d = \frac{pm}{p + (m - p)\sin.^2 a}$. Required a demonstration of this theorem.

THE GEOMETRY OF CO-ORDINATES.

If any figure whose genesis is known be drawn upon a plane, and through a given point in that plane lines be drawn in any given directions; and if parallel to these lines, lines be drawn from any point whatever of the figure, and terminating each in that to which the other is parallel: then there will be, in every case, a constant relation between these last-drawn lines, which can always be expressed by means of an equation *.

[For example, in the case of plane curves:—let the angle be a right one, and the given point be the vertex A of a parabola, the one line drawn through it coinciding with the axis AH of the parabola, and the other with its tangent (see

* In like manner, if any surface be given by means of its genesis, and through a given point planes be drawn parallel to three given planes; and if from any point of the surface three lines be drawn parallel to the three planes, each of which is terminated by the two planes to which the other two are parallel: then the relation between these three lines will be constantly the same, and may, in every case, be expressed by means of an equation.

fig. p. 142) R S. Then P denoting the parameter, we have by cor. 1. prop. i. the relation between AF and FG, (where AF is equal to the line drawn from G a point in the curve parallel to AH) expressed by the equation $P \cdot AF = FG^2$.

Or, in the circle, (see fig. Geom. theor. 87, vol. i.) we have the equation $AD \cdot DB = DC^2$, where AB is the diameter of the circle, and AD is equal to the line drawn from C parallel to AB, and terminated by the tangent drawn to the circle at A.

Or again, in the ellipse and hyperbola we have (theor. i. cor. i. of each) $AD \cdot DB : DE^2 :: AB^2 : ab^2$; or, as an equation,

$$DE^2 = \frac{ab^2}{AB^2} \cdot AD \cdot DB \text{; or } DE^2 = \frac{ab^2}{AB^2}(CD^2 \backsim CA^2).$$

These examples, which are very simple, and with which the student is already familiar, will be sufficient to explain the fundamental idea of this branch of Geometry; and we proceed to lay down a few definitions and first principles.]

Def. 1. The lines drawn through the given point (A in all these cases) are called the *axes of co-ordinates;* or the *axes of reference,* since to these all the parts of the figure are referred.

Def. 2. The lines drawn from the points of the curve parallel to the axes of reference, are called the *co-ordinates of those several points.*

[The segments of the axes intercepted by these lines are also called the co-ordinates, since they are equal to them in magnitude.]

Def. 3. The point in which the axes intersect (A) is called the *origin of co-ordinates,* since it is from that point that their several lengths are estimated.

Def. 4. The angle under which those axes intersect is called the *angle of co-ordinates.* When that is a right angle, the co-ordinates are said to be *rectangular,* and when that angle is oblique, the system is said to be *oblique.*

Def. 5. The equation which expresses the particular relation existing between the co-ordinates of any curve, is called the *equation of that curve.*

Def. 6. The co-ordinates are usually expressed in the equation by x and y, and the given parts of the figure by other single letters as in the "Application of Algebra to Geometry," in the first volume.

[If AF, fig. prop. ii. Parabola, p. 143, be denoted by a, the equation of the parabola referred to AD as the axes of x, will be $y^2 = 4ax$.

If AB, fig. theor. 87, Geom. p. 326, vol. i. be denoted by $2a$, then the equation of the circle referred to AB as the axes of x will be $y^2 = (2a - x)\, x$.

In a similar manner in the ellipse and hyperbola (where $AB = 2a$, and $ab = 2b$), the equations are respectively

$$y^2 = (2a - x)\, x \cdot \frac{b^2}{a^2} = \frac{b^2}{a^2}(2ax - x^2), \text{ and}$$

$$y^2 = (2a + x)\, x \cdot \frac{b^2}{a^2} = \frac{b^2}{a^2}(2ax + x^2).$$

If the origin had been assumed in a different place, the inclination of the axes of co-ordinates to each other and to the axes of the curves, the equations would have been of a *different form,* and in most cases more complex. The *degree* or *dimension* of the equation is, however, not altered by these circumstances.]

Def. 7. All curves are called *lines,* so that the straight line may be included under one common denomination with them: and they are also called in reference to their equations, the *loci* of those equations.

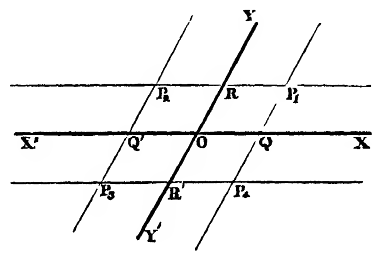
Def. 8. Curves are said to be of different orders according to the order or degree of the equation which expresses the relation between their co-ordinates.

Thus, if the equation be of the first degree (or $ax + by + c = 0$) its representative locus is called a line of the first order: if of the second degree, as $ax^2 + bxy + cy^2 + ex + fy + g = 0$, the locus is a line of the second order: and so on.

Those coefficients may be + or —, or 0; still the locus is of the same degree, so long as the *degree* of the equation is not lowered by those particular values given to the coefficients.

[It will presently be shown that the straight line is the locus of an equation of the first degree: and it is obvious that each of the conic sections (vide p. 185), is a locus of *some* equation of the second degree. It will hereafter be shown that *every* equation of the second degree expresses *some* conic section.]

PRINCIPLES.



1. Let X'X and Y'Y be the axes of reference, O, their point of intersection, the origin of co-ordinates. Then it is usual to consider the ordinates which are measured on the axis of x on the right hand as + and then (as was the case with the cosines in Trigonometry, vol. i. page 384.) those which are measured on the left as —. In a similar manner, the ordinates measured above O on OY are taken +, and consequently, those below will (like the sines in Trigonometry) be —.

2. The position of a point is given when its distances from the two co-ordinate axes (measured on lines parallel to those axes) are given, with the signs prefixed to those distances which indicate on which sides of them it is situated. Thus if $OQ = RP_{,} = q$ and $OR = P_{,}R = r$, were the *lengths* of the co-ordinates: then the point $P_{,}$ is defined by the co-ordinates q, r; P_2, by $-q, r$; P_3 by $-q, -r$; and P_4 by $-q, -r$.

[These *quantities*, for the sake of designating their being *co-ordinates* are generally designated by x and y; and to shew that they are *given* quantities, they are marked by an accent, (or if there be several sets, each set by a different number of accents) placed below them:—As $x_{,}y_{,}$, $x_{,,}y_{,,}$, &c. Sometimes also instead of $x_{,,}y_{,,}$, $x_{,,,}y_{,,,}$, &c. or $x''y''$, $x'''y'''$, they are simply denoted by x_2y_2, x_3y_3, &c. Generally, too, where these quantities are supposed to involve their own signs, the comma, as between the q and r above, is entirely dropped: but cases occur in which this cannot be done without creating ambiguity. This is especially the case when the values are given in numbers, in which case the sign must be prefixed: and then if the commas were omitted, the signification would be dubious, if not false.

CHAPTER I.

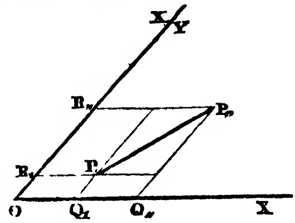
THE STRAIGHT LINE.

I. *To find the distance between two points whose co-ordinates* $x_{,}y_{,}$ *and* $x_{,,}y_{,,}$ *are given.*

From the annexed figure, where $P_{,,}yP_{,} = \pi - a$, the lines $P_{,}Q_{,}$, $P_{,}R_{,}$, and $P_{,,}Q_{,,}$, $P_{,,}R_{,,}$ being the co-ordinates of $P_{,}$ and $P_{,,}$ respectively, by Trig. vol. i. p. 404,

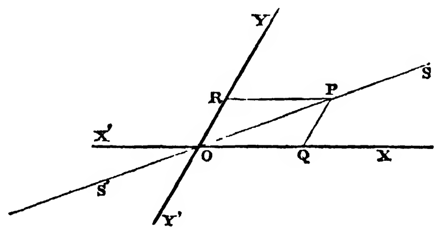
$P_{,}P_{,,}\ \pm\sqrt{(x_{,}-x_{,,})^2+2(x_{,}-x_{,,})(y_{,}-y_{,,})\cos.\alpha+(y_{,}-y_{,,})^2}$ when $\alpha=\frac{\pi}{2}$ it becomes simply $P_{,}P_{,,}=$

$\pm\sqrt{(x_{,}-x_{,,})^2+(y_{,}-y_{,,})^2}$ . . . . . . (14)



II. *To find its Equation.*

1. Let it pass through the origin of co-ordinates, as OS. Take any point P in the straight line, and draw the lines PQ, PR parallel to the axes. Denote PQ by y, and PR or OQ by x, and the angle YOX, of inclination of the axes by α, and SOX by β.



Then $\frac{QP}{QO}=\frac{y}{x}=\frac{\sin.\ POQ}{\sin.\ OPQ}=\frac{\sin.\beta}{\sin.(\alpha-\beta)}$.

Whence $y=x\frac{\sin.\beta}{\sin.\overline{\alpha-\beta}}$ . . . . (3)

or $y\sin.\overline{\alpha-\beta}-x\sin.\beta=0$ . . . . (4)

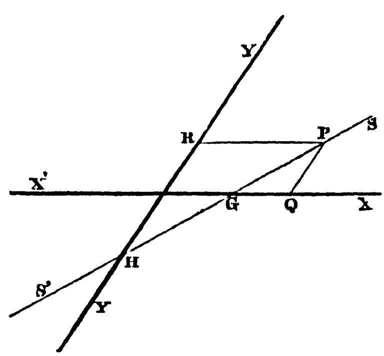
2. Let the lines not pass through the origin, but cut the axes in G and H. Let GO $=a$, OH $=b$, and the other quantities as before. Then

$\frac{QP}{QG}=\frac{y}{x-a}=\frac{\sin\beta}{\sin.\overline{\alpha-\beta}}$.

whence $x\sin.\beta-y\sin.(\alpha-\beta)$
$-a\sin.\beta=0$ . . . . (5)

or $y=\frac{(x-a)\sin.\beta}{\sin.\overline{\alpha-\beta}}=$

$x\frac{\sin.\beta}{\sin.\overline{\alpha-\beta}}-a.\frac{\sin.\beta}{\sin.\overline{\alpha-\beta}}$ . . (6)



The general form of the equation, then, as before stated, is

$$\left.\begin{array}{l} y=ax+b \\ \text{or } a_{,}x+b_{,}y+c_{,}=0 \end{array}\right\} \quad . . . . (7)$$

in which $a_{,}$, $b_{,}$, $c_{,}$ are known quantities.

3. If the angle of co-ordination be right, or $\alpha=\frac{\pi}{2}$, then these take a simpler form, as sin. $(\alpha-\beta)$ becomes cos. β: viz.

$y=x\tan.\beta$, and $y=x\tan.\beta+c$ . . . . (8)

The student should trace the mutations which arise from the variations of β.

III. *Given the equation of a straight line, and its angle of co-ordination, to construct it.* (See last figure.)

Since the equation holds good for *all values* of x, it is true when $x = 0$; and in this case, taking (5) we have $y = -\frac{c'}{b'}$:

Set off this value on the axis OY above or below as the sign + or — belongs to it. This will indicate one point in the line to be constructed.

In like manner it is true for *all* the values of x which correspond to values of y in the equation (5); and therefore when $y=0$. But in this case $x = -\frac{c_{,}}{a}$. Set off this on the axis of x, to the right or left as it is + or —. Then this gives another point of the line.

The line drawn through these two points is the locus of equation (5.)

It is obvious, however, that if we had taken any two values of x or y, and found the two corresponding values of y or x respectively, and constructed the points, then the line through these would have been that sought. The advantage of the method first indicated is—that the calculation and the construction are both less laborious than in the latter one.

IV. *To find the equation of a straight line passing through two points whose co-ordinates are given.*

[Here the object is to discover the *coefficients* of the equation.]

Denote the given co-ordinates by $x_{,}y_{,}$ and $x_{,,}y_{,,}$ respectively.

Then, the general form of the equation is given in (7) in two ways. Take the latter: and since $x_{,}y_{,}$ and $x_{,,}y_{,,}$ are *known values* of x and y in it, we have

$y_{,} = a_{,}x_{,} + b$, and $y_{,,} = a_{,}x_{,,} + b$. From which we have

$a = \frac{y_{,} - y_{,,}}{x_{,} - x_{,,}}$ and $b = \frac{x_{,}y_{,,} - x_{,,}y_{,}}{x_{,} - x_{,,}}$, and the equation of the line itself is

$$y = \frac{y_{,} - y_{,,}}{x_{,} - x_{,,}} . x + \frac{x_{,}y_{,,} - x_{,,}y_{,}}{x_{,} - x_{,,}} \quad \ldots\ldots\ldots \quad (9.)$$

It is sometimes more commodiously written in the equivalent form

$$\left.\begin{array}{l}(y - y_{,})\,(x_{,} - x_{,,}) = (x - x_{,})\,(y_{,} - y_{,,}) \\ \text{or } (y - y_{,,})(x_{,} - x_{,,}) = (x - x_{,,})\,(y_{,} - y_{,,})\end{array}\right\} \quad \ldots\ldots \quad (10.)$$

[The fraction $\frac{y_{,} - y_{,,}}{x_{,} - x_{,,}}$ obviously denotes $\frac{\sin. \beta}{\sin. a - \beta}$: and when $a = \frac{\pi}{2}$ or the angle of co-ordination is right, it is simply = tan. β.]

V. *To find the equation of a line which passes through a given point and is parallel to a given line.*

In this case, (10) may be written

$y - y_{,} = \frac{y_{,} - y_{,,}}{x_{,} - x_{,,}}\,(x - x_{,}) = 0$, in which the value of $\frac{y_{,} - y_{,,}}{x_{,} - x_{,,}}$ expressing the inclination is given in the problem. Put it equal to k. Then the equation of the line is, $(x - x_{,}) = k\,(y - y_{,})$ . . . . (11)

VI. *To find the co-ordinates of the point of intersection of two straight lines whose equations are given.*

At this point, x and y have the same values. If we denote the equations generally by $a_{,}x + b_{,}y + c_{,} = 0$, and $a_{,,}x + b_{,,}y + c_{,,} = 0$, we have as usually found in simultaneous equations.

$$\frac{b_,c_{,,} - b_{,,}c_,}{b_,a_{,,} - b_{,,}a_,}, \text{ and } y = \frac{a_,c_{,,} - a_{,,}c_,}{b_,a_{,,} - b_{,,}a_,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

VII. *To find the angle under which two lines whose equations are given intersect each other.*

Denote the lines by the equations $y = a_,x + b_,$ and $y = a_{,,}x + b_{,,}$. Then if the angle of co-ordination be α, and $\beta_,$, $\beta_{,,}$ be the inclinations to the axis of x, we have

$$a_, = \frac{\sin. \beta_,}{\sin. \overline{\alpha - \beta_,}} \text{ and } a_{,,} = \frac{\sin. \beta_{,,}}{\sin. \overline{\alpha - \beta_{,,}}}. \quad \text{From which we obtain}$$

$$\tan. \beta_, = \frac{a_, \sin. \alpha}{1 + a_, \cos. \alpha} \text{ and } \tan. \beta_{,,} = \frac{a_{,,} \sin. \alpha}{1 + a_{,,} \cos. \alpha} : \text{ and hence}$$

$$\tan. \overline{\beta_, - \beta_{,,}} = \frac{(a_, - a_{,,}) \sin. \alpha}{1 + (a_, + a_{,,}) \cos. \alpha + a_,a_{,,}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

When the angle of co-ordination is right, $\tan. \overline{\beta_, - \beta_{,,}} = \dfrac{a_, - a_{,,}}{1 + a_,a_{,,}}$ . . . (14.)

When the lines are at right angles, the condition $\tan. \overline{\beta_, - \beta_{,,}} = \tan. \dfrac{\pi}{2} = \dfrac{1}{0}$ can only be fulfilled by $1 + (a_, + a_{,,}) \cos. \alpha + a_,a_{,,} = 0$, which gives

$$a_{,,} = -\frac{1 + a_, \cos. \alpha}{a_, + \cos. \alpha}; \quad \text{or when } \alpha = \frac{\pi}{2},\ a_{,,} = -\frac{1}{a_,} \quad . \quad . \quad . \quad . \quad (15.)$$

VIII. *To find the equation of a straight line which makes a given angle with a given straight line, and passes through a given point.*

Let $x_,y_,$ be the co-ordinates of the given point; $y = ax + b$ the given line, and γ the given angle. Then the general form (10) is $y - y_, = (x - x_,) \tan. \gamma$.

But by (13), $\tan. \gamma = -\dfrac{1 + a \cos. \alpha}{a + \cos. \alpha}$; and hence

$$y - y_, = -\frac{1 + a \cos. \alpha}{a + \cos. \alpha} (x - x_,) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 16.)$$

IX. *To find the length of the perpendicular from a given point* $x_,y_,$ *upon a given line* $y = ax + b$.

By finding the co-ordinates of the point of intersection of (16) with the given line, we have, denoting those co-ordinates by $x'y'$,

$$\left.\begin{aligned} x' - x_, &= \frac{(a + \cos. \alpha)(y' - ax - b)}{1 + 2a \cos. \alpha + a^2} \\ \text{and } y' - y_, &= \frac{-(1 + a \cos. \alpha)(y' - ax - b)}{1 + 2a \cos. \alpha + a^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

Inserting these values in the expression for the length (p) of the line $x'y'$, $x_,y_,$ (see equation 1) we shall obtain

$$(x' - x_,)^2 + 2(x' - x_,)(y' + y_,) \cos. \alpha + (y' + y_,)^2 = \frac{(y_, - ax_, - b)^2 \sin.^2 \alpha}{1 + 2a \cos. \alpha + a^2}, \text{ or}$$

$$p = \frac{\pm (y_, - ax_, - b) \sin. \alpha}{\sqrt{1 + 2a \cos. \alpha + a^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

EXERCISES ON THE STRAIGHT LINE.

1. Construct the equations $-5x + 4y - 10 = 0$, $5x - 4y - 10 = 0$, $-5x - 4y - 10 = 0$, and $5x + 4y - 10 = 0$: the angle of ordination being 75°, and all the lines referred to the same co-ordinate axes.

2. Verify by scale the calculated values of y when $x = -\frac{3}{4}$, and find the inclination of each line to the axes of x and y.

3. Find the equation of the lines passing through the several pairs of points referred to rectangular co-ordinates, viz. $-5, 4$; $5, -4$; $-5, -4$; and $5, 4$: and find the distances from the origin at which they severally cut the axes.

4. Construct the lines $2y + 5 = 0$, $4x - 2 = 0$, and $x + y = 0$: and find the equation of the line which cuts from x and y, the segments a and b.

5. Two straight lines pass through the given point $x_{,}y_{,}$, make a given angle with each other, and cut from the axis of x a segment which is divided at the origin in extreme and mean ratio. What are these equations? And what is the ratio of the segments it cuts from the axis of y, and the angles of the inclination of each to the rectangular axes?

6. The base and area of a triangle are given, what is the path of its vertex? Likewise, find the locus of the vertex of an isoceles triangle whose base is given: and when the vertical angle and the inclination of the base to one of the sides are given; find the locus of a point which divides the base (or base produced) in a given ratio?

7. Show that the lines drawn from the angles A, B, C, of a triangle to bisect the opposite sides meet in one point; that those drawn to bisect the angles meet in a point; and those drawn perpendicular to the opposite sides also meet in a point.

[This will be done if it can be shown that the co-ordinates of the point of intersection of one pair of the three lines, and those of another pair, are identical, in each case.]

8. If upon the sides of a triangle as diagonals parallelograms be described whose sides are parallel to two given lines: then the other three diagonals will intersect in the same point.

9. The lines AL, BK, CM (figure 47, i. Euc.) intersect in the same point: and the same is true if instead of squares the figures described on the sides of the triangle had been any *similar* rectangles of which AB, BC, CA, were homologous sides.

10 Transform the equation of a straight line through two given points $x_{,}y_{,}$ and $x_{,,}y_{,,}$ into the form $\frac{x}{a} + \frac{y}{b} = 1$; and also into the form $\frac{\cos.}{a_{,}} + \frac{\sin.\theta}{b_{,}} = 1$.

11. From any (even $2n$) number of points, lines are drawn to meet in one point Z; and such that the sum of the squares of n of them is equal to the sum of the squares of the others: then the locus of Z is a straight line whose equation it is required to find.

12. A straight is drawn to cut the three sides of a given triangle, and is divided by them in a given ratio, and also passes through a given point; it is required to find its equation, those of the three sides of the triangle being given as well as the co-ordinates of the given point.

13. If two triangles be described on the opposite sides of a quadrilateral figure, each having its vertex in the other's base: then the lines joining the points of mutual intersection of the sides will always pass through the same point. Prove this, and find the co-ordinates of that point.

14. If three sides of a given triangle be cut by a line, and in it a fourth point be taken so that the three segments shall have amongst themselves given ratios, the locus of the fourth point is a straight line.

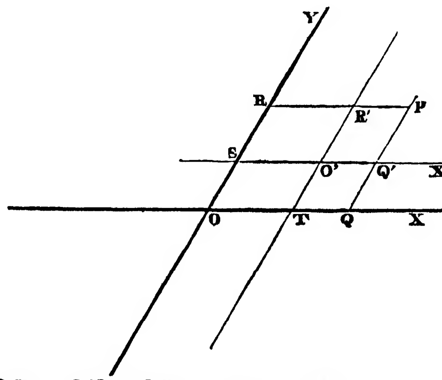
CHAPTER II.

ON THE TRANSFORMATION OF THE EQUATION OF CO-ORDINATES.

This is a method of changing the equation between the co-ordinates of any point referred to one system of axes to an equation adapted to another system, the co-ordinates of the new origin, and the inclination of the new axes to the old being also given. It comprises two separate operations, either of which, according to convenience, may be performed the first.

I. *When the origin is to be changed, but the directions of the new axes are parallel to the old.*

Let OX, OY be the old and O′X′, O′Y′ the new axes, parallel to the former, and a, b the co-ordinates of O′ the new origin. Then R′P, Q′P are the new co-ordinates of P. Denote them by $y'x'$.

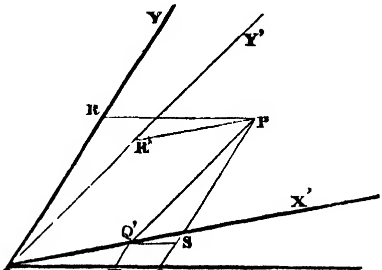


$$\left.\begin{aligned} &\text{Then, evidently, } QP = QQ' + Q'P = OT + Q'P; \text{ or, } y = b + y' \\ &\text{and} \qquad OQ = OT + TQ = SO + O'Q', \text{ or } x = a + x'. \end{aligned}\right\} \quad . \; . \; (19.)$$

which substituted in *any* equation between x and y gives the transformed equation.

II. *When the directions of the axes are to be changed, the origin remaining the same.*

Denote OQ, QP the co-ordinates of P referred to the original system of axes OX, OY by x and y, and OQ′, Q′P, those referred to the new system OX′, OY′ by x' and y'. Let $YOX = \alpha$, $X'OY' = \delta$, $X'OX = \epsilon$, and $XAY = \eta$; and draw the parallels as in the figure.



$$\left.\begin{aligned} &\text{Then } y = PQ = TQ' + SP = OQ' \cdot \frac{\sin. Q'OT}{\sin. OTQ'} + PQ' \cdot \frac{\sin. PQ'S}{\sin. PSQ'} = \frac{x' \cdot \sin. \epsilon + y' \sin. \eta}{\sin. \delta} \\ &\text{and } x = OQ = OT + Q'S = OQ' \cdot \frac{\sin. OQ'T}{\sin. OTQ'} + PQ' \cdot \frac{\sin. Q'PS}{\sin. PSQ'} = \frac{x' \sin(\alpha - \epsilon) + y' \sin(\alpha - \eta)}{\sin. \delta} \end{aligned}\right\}$$

$$. \; . \; . \; . \; . \; . \; . \; . \; (20.)$$

If the original axes be oblique, and the new ones rectangular, then $\varepsilon - \eta = \frac{\pi}{2}$, and these become

$$y = \frac{x' \sin. \varepsilon + y' \cos. \varepsilon}{\sin. \delta}, \text{ and } x = \frac{x' \sin. (\alpha - \varepsilon) - y' \cos. (\alpha - \varepsilon)}{\sin. \delta} \quad . . . (21)$$

If the original axes be rectangular and the new ones oblique, then $\alpha = \frac{\pi}{2}$, and the equations of transformation become

$$y = x' \sin. \varepsilon + y' \sin. \eta, \text{ and } x = x' \cos. \varepsilon + y' \cos. \eta \quad . . . . (22.)$$

And, lastly, if both be rectangular, we obtain

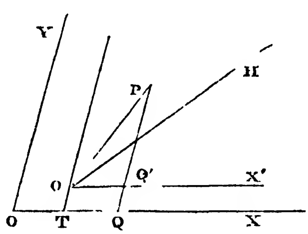
$$y = x' \sin. \varepsilon + y' \cos. \varepsilon, \text{ and } x = x' \cos. \varepsilon - y' \sin. \varepsilon \quad . . . . (23.)$$

CHAPTER II.

ON POLAR CO-ORDINATES.

Any two quantities, whether both lines, or both angles, or one a line, and the other an angle, may be employed as co-ordinates by means of which to designate the position point on a plane: but the last is that which has been, hitherto, most frequently used, the line being one of the sides containing the angle, and the other a fixed line through the angular point or origin.

Thus, let the case where the relation is between O′P and the angle POH be the method employed. This is called the "POLAR SYSTEM OF CO-ORDINATES," since O′P in the case revolves about O′ as a pole. Let O′H be the line from which the angles are estimated. O′P is denoted, usually, by r, the angle PO′H by θ, and the angle made by the line O′H (which is *given* when the conditions are complete) by κ.



Draw O′T and PQ parallel to the axis of x. Then

$$\left.\begin{aligned} y &= PQ = O'T + PQ' = b + r \cdot \frac{\sin. \overline{\theta + \kappa}}{\sin. \alpha}, \\ \text{and } x &= OQ = OT + OQ' = a + r \, \frac{\sin. (\alpha - \theta + \kappa)}{\sin. \alpha} \end{aligned}\right\} \quad . . . . . (24.)$$

These are the quantities to be substituted for x and y to effect the transformation of linear to polar co-ordinates.

If the original axes be rectangular, $\alpha = \frac{\pi}{2}$, and the equations of transformation become

$$y = b + r \sin. (\theta + \kappa) \text{ and } x = a + r \cos. (\theta + \kappa) \quad . . . . (25.)$$

If the origin be the same, we have simply, since $a = 0$, $b = 0$,

$$y = r \sin. (\theta + \kappa), \text{ and } x = r \cos. (\theta + \kappa) \quad . . . . (26.)$$

And if the axis of x also be the origin of θ, $\kappa = 0$, and they become

$$y = r \sin. \theta, \text{ and } x = r \cos. \theta \quad . . . . (27.)$$

Again, for the transformation of polar into linear co-ordinates, we have

$$\frac{x-a}{y-b} = \frac{\sin.\ \{a-(\theta+\kappa)\}}{\sin.\ (\theta+\kappa)} = \sin.\ a \cot.\ (\theta+\kappa) - \cos.\ a, \text{ or}$$

$$\tan.\ (\theta+\kappa) = \frac{(y-b)\sin.\ a}{(x-a)+(y-b)\cos.\ a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28.)$$

and $r^2 = (x-a)^2 + 2\,(x-a)\,(y-b)\cos.\ a + (y-b)^2 \quad . \quad . \quad . \quad . \quad (29.)$

Modifications similar to those above, for the particular cases of relation between the new and old co-ordinates, must be made by the student himself.

EXERCISES UPON TRANSFORMATION.

1. Transform the rectangular equation $2y + 3x - 10 = 0$ into a polar equation: when the new origin is $-2, 3$, and the inclination of the origin of θ to x is 45°. Also transform the equation $ax + by + c = 0$, where the inclination of the axes is β, into a polar one, the new origin being k, l, and the inclination of the origin of θ to x is ι.

2. Transform the following rectangular equation into another rectangular equation $ay^2 + bxy + cx^2 + dy + cx + f = 0$, the two systems of co-ordinates being parallel, and the co-ordinates of the new origin lines for x and y,

$$\text{respectively } \frac{2ae-bd}{b^2-4ac} \text{ and } \frac{2cd-be}{b^2-4ac}.$$

$$\text{Answer, } ay'^2 + bx'y' + cx'^2 + \frac{ae^2+cd^2-bde}{b^2-4ac} + f = 0.$$

3. Transform the rectangular equation last found into another having the same origin, but having the old and new axis of x inclined at an angle ϵ such that $\tan.\ 2\epsilon = \dfrac{b}{c-a}$.

Answer, The *general form* is $Ay^2 + By^2 + C = 0$: or, in full, it is

$$\left\{a+c\pm\sqrt{(a-c)^2+b^2}\right\} y''^2 + \left\{a+c\mp\sqrt{(a-c)^2+b^2}\right\} x''^2 + \frac{2(e^2+cd^2-bde)}{b^2-4ac} + 2f = 0$$

where x'', y'' are the co-ordinates of the point P referred to the new axes and the new origin.

4. Transform the rectangular equation $a^2y^2 \pm b^2x^2 = a^2b^2$ into a polar equation whose origin has for co-ordinates 0, 0, and the origin of θ upon the axis of x: also into a polar equation whose origin has for co-ordinates $\pm\sqrt{a^2 \mp b^2}$ and origin of θ upon the axis of x. Also transform the rectangular equation $y^2 = 4ax$ into a polar one having for co-ordinates of the pole, a, 0, and origin of θ upon the axis of x.

5. Transform the equation $a^2y^2 \pm b^2x^2 = a^2b^2$ into one whose origin is at the points $\pm a$, 0, and also 0, $\pm b$.

6. Find the position of two new axes passing through the same origin, such that the transformed equation shall have the same *general form* as the given one, viz. $a_iy'^2 \pm b_ix'^2 = a_i^2b_i^2$. Ans. There are innumerable positions.

7. Transform $\dfrac{a}{x} + \dfrac{b}{y} = \sin.\ a$ to a new origin whose co-ordinates are a sin. a and b sin. a.

8. The equation $ax^2 + bxy + cy^2 = k^2$ is referred to co-ordinates whose inclination is a: find the position of the new axes, inclined to one another, so that the equation shall take the form $A'x^2 + By'^2 = K^2$

CHAPTER III.

THE CONIC SECTIONS.

It has been explained in the preliminary part of this treatise, that the equation of each of the conic sections is of the second degree. By recurring to the exercises in Transformation which the student has just performed, it will also be apparent that every equation of the second degree belongs to *some* conic section*. For the *general form* of the equation of the second degree is successively reducible to the forms $ay^2 + bxy + cx^2 + f_{,} + 0$, to $a_{,}y^2 + b_{,}x^2 + f = 0$, and to $a_{,,}y^2 + b_{,}x^2 + c_{,,}x + f_{,,} = 0$, in the last of which we recognise the general form of the equations of the Conic Sections noticed at page 186, when referred to the axis of x and y, and having their origin at the vertex of the curve. It hence follows, that however complex the general rectangular equation of the second degree may be, we can by the mere transformation of co-ordinates reduce it to one of great simplicity; and thereby render it subservient to the purpose of investigating the properties of the conic sections.

The investigation of the properties of the conic sections by means of algebraic equations is often very much facilitated by the employment of polar co-ordinates, the centre or the focus being made the pole and the origin of θ at the transverse axis. The origin of θ so taken, and the focus taken as the pole, gives the formula most generally and efficiently used in Astronomical Researches.

ON THE SECANTS AND TANGENTS OF A CONIC SECTION.

I. *A straight line can meet a conic section in no more than two points.*

The equations of the line and conic section are respectively

$$hy = kx + m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad . \quad . \quad . \quad . \quad (b)$$

Insert the value of y from (a) in (b): then

$$x^2 + \frac{2akm + bhm + dkh + eh^2}{ak^2 + bkh + ch^2}x + \frac{am^2 + dhm + fh^2}{ak^2 + bkh + ch^2} = 0 \quad . \quad . \quad . \quad . \quad (c)$$

which gives two values of x; and from these, and equation (a) we obtain the two corresponding values of y. The co-ordinates of the two points of intersection may therefore be calculated: and by the theory of equations (vol. i. p. 222) these are the only values, and hence there are only two points of intersection possible.

II. *To find the condition of these two points coalescing, or of the line being a tangent to the conic section.*

In this case the two values of x in equation (c) must be equal. Complete the

---

* Those exercises, and consequently this inference from them, referred only to the case of the co-ordinates being rectangular: but the same conclusion may be deduced when the angle of co-ordination is oblique. For, if we first transform the equation of oblique co-ordination to one of right co-ordination by means of the appropriate values given in equations (21) we shall have an equation of the *same general form*, and differing only in the *values of the co-efficients*. We are hence entitled to infer that whatever be the angle of co-ordination, the general equation of the second degree expresses a conic section.

From those equations of transformation we learn generally, that whatever be the order of the proposed equation, that order is not altered by any mode of transformation of co-ordinates.

square, and make the second side equal to 0: then this condition being fulfilled, we obtain

$$(2akm + bhm + dhk + eh^2) - 4(am^2 + dhm + fh^2)(ak^2 + bhk + ch^2) = 0 \;.\; (d)$$

In this case, the two values of x are equal: but in order that the line may touch the curve it is also necessary that the two values of y should be equal. Substituting then in (b) the value of derived from (a) and proceeding as was done for x, we obtain a second condition, viz.

$$(bkm - 2chm + ehk + k^2d)^2 - 4(cm^2 - ekm + k^2f)(ak^2 + bhk + ch^2) = 0 \;.\;.\; (e)$$

These are the relations that must exist amongst the coefficients of the given equations, in case of a line touching the conic section.

As there are three quantities h, k, m to be determined, and only two equations of condition (viz. (d) and (e)) there must be another condition introduced before the tangent becomes a fixed line. It follows, therefore, that a *tangent may be drawn to any point of a conic section.*

III. *The equation of the tangent to a conic section at a given point.*

Suppose, as in the last case, two points $x_{,}y_{,}$ and $x_{,,}y_{,,}$ to be taken in the conic section. Then we shall have the two equations

$$ay_{,}^2 + bx_{,}y_{,} + cx_{,}^2 + dy_{,} + ex_{,} + f = 0 \;\;.\;.\;.\;.\;.\;.\;.\; (f)$$

$$ay_{,,}^2 + bx_{,,}y_{,,} + cx_{,,}^2 + dy_{,,} + ex_{,,} + f = 0 \;\;.\;.\;.\;.\;.\;.\;.\; (g)$$

Subtract the latter from the former equation: then

$$a(y_{,}^2 - y_{,,}^2) + b(x_{,}y_{,} - x_{,,}y_{,,}) + c(x_{,}^2 - x_{,,}^2) + d(y_{,} - y_{,,}) + e(x_{,} - x_{,,}) = 0 \;.\;.\; (h)$$

But $x_{,}y_{,} - x_{,,}y_{,,} = x_{,}(y_{,} - y_{,,}) + y_{,,}(x_{,} - x_{,,})$, $y_{,}^2 - y_{,,}^2 = (y_{,} - y_{,,})(y_{,} + y_{,,})$ and $x_{,}^2 - x_{,,}^2 = (x_{,} - x_{,,})(x_{,} + x_{,,})$; and these values being substituted in the above equation, give

$$\frac{y_{,} - y_{,,}}{x_{,} - x_{,,}} = -\frac{c(x_{,} + x_{,,}) + by_{,,} + e}{a(y_{,} + y_{,,}) + bx_{,} + d} \;.\;.\;.\;.\;.\;.\;.\;.\;.\; (k.)$$

Now the equation of the line through $x_{,}y_{,}$ and $x_{,,}y_{,,}$ is

$$y - y_{,} = \frac{y_{,} - y_{,,}}{x_{,} - x_{,,}}(x - x_{,}) \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (l)$$

and hence the equation of the *secant* $x_{,}y_{,}$, $x_{,,}y_{,,}$ is

$$y - y_{,} = -\frac{c(x_{,} + x_{,,}) + by_{,,} + e}{a(y_{,} + y_{,,}) + bx_{,} + d}(x - x_{,}) \;.\;.\;.\;.\;.\; (m.)$$

But when the points $x_{,}y_{,}$, $x_{,,}y_{,,}$ coalesce, the line (m) becomes a tangent; and since in this case $x_{,,} = x_{,}$ and $y_{,,} = y_{,}$, its equation is

$$y - y_{,} = -\frac{2cx_{,} + by_{,} + e}{2ay_{,} + bx_{,} + d}(x - x_{,}) \;.\;.\;.\;.\;.\;.\;.\; (n.)$$

Moreover, since this equation represents the line which touches the conic section in the point $x_{,}y_{,}$, we may suppose a point in it to be given, whose co-ordinates are g and h, and from (n) and (f) determine the points of contact. Hence we can always find the co-ordinates of the point of contact of a tangent to a conic section drawn through a given point.

IV. *The equation of the normal to a conic section.*

Since the normal is perpendicular to the tangent at the point in question, if that point be in the curve, the equation sought is derived immediately from the last (n) by means of the properties already deduced in conics. It is

$$y - y_{,} = \frac{2ay_{,} + bx_{,} + d}{2cx_{,} + by_{,} + e}(x - x_{,}) \;.\;.\;.\;.\;.\;.\;.\;.\;.\; (o.)$$

If, on the contrary, the point be given through which the normal is to pass, and we denote its co-ordinates by g and h: then the points of intersection with

the conic section can be found from (o) and (f), as the points of contact of the tangent were in the last case.

CHAPTER IV.

The classification of the curves represented by the general equation of the second degree.

Transform the equation to a new origin whose co-ordinates are $\frac{2ac - bd}{b^2 - 4ac}$ and $\frac{2cd - be}{b^2 - 4ac}$, as in Exercise 2, page 194. Then it becomes $ay'^2 + bx'y' + cx'^2 + \frac{ac^2 + cd^2 - bde}{b^2 - 4ac} + f = 0$, where $x'y'$ are the new co-ordinates of any point in the curve, which was before denoted by xy.

Again, transform this according to Exercise 3; and we obtain the *general form* $Ay^2 + Bx^2 + C = 0$ . . . . $(p.)$

Then, by comparison of this with the properties of the curves deduced geometrically, we learn, that if A and B have the same signs, the curve is an ellipse, and if different signs that it is the hyperbola; and that the co-ordinates of its centre are represented by the quantities $\frac{2ac - bd}{b^2 - 4ac}$ and $\frac{2cd - bc}{b^2 - 4ac}$.

Now, by recurring to the result of the transformation in Example (3) it is obvious that while $(a + c)^2$ is greater than $(a - c)^2 + b^2$, both the coefficients A and B will have the *same sign;* and the locus of the equation will be an *ellipse:* but when $(a + c)^2$ is less than $(a - c)^2 + b^2$, the locus will be an *hyperbola,* the coefficients A and B having *different signs.* These conditions are the same things as $b^2 - 4ac$ being respectively negative and positive; and the quantities a, b, c, being those given in the general equation, this question can be determined at once without any preliminary transformation.

But, besides the conditions $b^2 - 4ac$ being either negative or positive we may have $b^2 - 4ac = 0$. This would indicate that the centre of the curve is infinitely distant, and suggest that the locus is a parabola. As this case recurs, too, at the interval when the ellipse is changed into the hyperbola, its analogy to the circumstance which takes place when the plane of the section changes its position in revolving about a line the tangent to the cone itself drawn at right angles to the edge of the cone, would still further justify that conjecture. It is not, however, necessary to build this deduction on those analogies; for by again transposing the origin to the *vertex of the curve,* we shall have the equation converted into a form which will designate the sections as before, and give amongst them the equation of the parabola as depending upon $b^2 - 4ac = 0$. Unfortunately the limits of this course do not allow of further detail on this head; and, indeed, it is the less necessary, as the operation is very easy, and may be properly left as an exercise to the student himself. The general form of the resulting equation it may, however, be stated is $hy^2 + kx^2 + bx + m = 0$, where h, k, l, m are determinable functions of a, b, c, d, e, f.

Again, for the classification of the curves in reference to polar co-ordinates.

Since $x = r \cos. \theta$ and $y = r \sin. \theta$, the equation referred to the same origin and having θ measured from the axis of x, obviously become

$$ar^2 \sin.^2\theta + br^2 \sin.\theta \cos.\theta + cr^2 \cos.^2\theta + dr \sin.\theta + er \cos.\theta + f = 0,$$

and it follows at once that the same conditions as before determine the species of the curve.

The particular cases of the several curves

1. *The circle.* This is in fact the ellipse whose axes are equal; and any diameters at right angles to each other are conjugate diameters.

Its equation is $x^2 + y^2 + 2hx + 2ky + m = 0$, where the co-ordinates of the centre are $-h$, $-k$, and whose radius is $\pm\sqrt{h^2 + k^2 - m}$.

The equation of its tangent is

$$y - y_, = -\frac{2x_, + h}{2y_, + k}(x - x_,), \text{ or } (2y_, + k)(y - y_,) + (2x_, + h)(x - x_,) = 0.$$

Special hypotheses respecting the circle, as when the origin is at the centre or in the circumference, are left for the student to deduce.

In the ellipse, no other notable simplification occurs: nor in the parabola, since all parabolas are similar figures, is any to be expected.

2. In the hyperbola, when $a = -c$, the case is that of the equilateral hyperbola; or that whose asymptotes intersect in a right angle.

When b — are each equal to 0, the quantity $b^2 - 4ac$ is positive, and the figure is an hyperbola as has been already shown. The axes of reference (whether right or oblique) are in this case parallel to the asymptotes of the hyperbola. Whenever, therefore, we see from the coefficients of the given equation that it represents an hyperbola, and we desire to investigate any properties connected with the asymptotes, we should transform it into oblique co-ordinates, by means of equations (21, 22, 23, whichever of them the case may require) into another: and by making the coefficients of x'^2 and y'^2 each equal to zero, deduce the corresponding values of ϵ and η.

[We shall now add a sufficient number of examples for the student's exercise, so as to fully impress upon his mind the character of the investigations we have led him through. A few other curves are also proposed of which he is required to find the equations: but the properties of curve lines cannot be *fully* made out without the use of the fluxional or differential calculus; the subject may therefore be resumed under those heads. For the present, it will be sufficient if he find the equations of the several proposed curves, the points in which they cut the axis of co-ordinates, and the cases where they admit of indefinite extension in *the direction of either* axis. In the conic sections, however, he should find the equations and positions of the tangents and normals, the centres, vertices and foci, as well as the magnitudes of those diameters.]

EXERCISES ON THE CIRCLE.

1. Calculate the co-ordinates of the centres, and the radii of the circles designated by the following equations: $x^2 + y^2 = 100$; $2x^2 + 2y^2 - 4(x+y) - 1 = 0$; $y^2 + x^2 - 6x + 4y = 3$; $(6y - 21)y + (6x - 8)x = -14$; $y^2 + x^2 + 4y - 3x = 0$; $(x + 2)x + (y - 4)y = 0$; $y^2 + x^2 = 4y$; $y^2 + 8 = -(x + 6)x$; $y^2 - 8y + x^2 - 12x + 52 = 0$; and $(y - 4)y + (x + 2)x + 9 = 0$

2. Find the equations of the circles which severally pass through the following sets of three points: $(-2, 5;\ 4, -6;\ -2, -6;)$; $(-6, -1;\ 0, 0, 0, -1)$; $(-1, 1;\ 1, -1;\ 1, 1)$; $(0, 1;\ 1, 0;\ 0, 0)$; $(-4, -1;\ 4, 1;\ -1, 4)$ and $(\frac{1}{6}, 0;\ 1, 1;\ -\frac{1}{2}, \frac{1}{2})$.

3. Given the base and vertical angle, and, likewise, the base and ratio of the sides, to find in each case the locus of the vertex of the triangle.

4. Given the base and vertical angle to find the loci of the centres of the four circles which touch the three sides of the triangle.

5. Are the following four points in the same circle, viz. 0, 0; 0, 4; 1, 1, and 1, — 1? And is 2, 6 in the same circle with any three of these?

6. From the angular points of any polygon inscribed in a circle draw lines to a point z, such that n_1 times the square of the first, n_2 times the square of the sum of the second, &c. be equal to a given space, what is the locus of z?

7. The co-ordinates of the centre of a circle are 3, — 4, the radius is 5; and the abscisses of points in it at which tangents are drawn, are 1, — 1, and 2. Find the co-ordinates of the angles of the triangle which those tangents form, together with the co-ordinates of the centre and radius of the circle which circumscribes it.

8. Given the co-ordinates $x_{,}y_{,}$, $x_{,,}y_{,,}$, $x_{,,,}y_{,,,}$ of the angles of a triangle to find the equation of the circle which touches the sides of the triangle, and the radius of that which circumscribes it.

9. Find the same things when the three given points are designated by polar co-ordinates $r_{,}\theta_{,}$, $r_{,,}\theta_{,,}$, and $r_{,,,}\theta_{,,,}$.

10. A quadrilateral figure is inscriptible in a circle: the co-ordinates of three of its angles are 1, 2; — 1, — 2; 3, — 3: and the abscisses of the fourth angle is $2\frac{1}{2}$. What is the ordinate of that point?

11. Tangents being drawn to touch three given circles, two and two, have their points of intersection in the same straight line.

12. Equal lines are drawn from a point P to touch two given circles, what is the locus of P?

13. Find the equation of a circle which touches these given circles.

14. A circle of a given radius r *rolls* upon the inside of another whose radius is $2r$; the path traced out by any given point in the former is a diameter of the latter. [This is one of the contrivances for producing the "*parallel motion in a steam engine.*"]

EXERCISES ON LINES OF THE SECOND ORDER.

1. A line passing through the points — 1, 2 and — 2, 1 cuts both an ellipse and hyperbola whose axes are 6 and 8, and are parallel to the co-ordinate axes, the co-ordinates of the centre being $-\frac{3}{2}$ and $\frac{5}{2}$. What portion of it is intercepted by the two curves respectively?

2. The co-ordinates of the vertex of a parabola are — 2, 1; and its axis makes an angle of $-45°$ with the axis of x: what is its equation, when it passes through the origin of co-ordinates?

3. Two parabolas whose parameters are p and $p_{,}$ having a common vertex at the origin of co-ordinates, and their axes inclined to one another at an angle of 60°, intersect in a point whose co-ordinates are required.

4. Trace the curve whose equation is $3xy + 6x - 9y - 12 = 0$; and that whose equation is $4x^2 - 6xy \pm 8y^2 - 10x + 10y = 120$.

5. Given the base and the difference or sum of the angles at the base, to find the locus of the vertex, the locus of the centres of the circle which is described about the triangle and of those which touch its three sides, in both cases.

6. Given the base and sum or difference of the sides, to find the locus of the vertex, and those of the centres of the four circles of contact.

7. Given two radii-vectores $r_{,}$ and $r_{,,}$ and the angle a between them to find the polar equation of the parabola, the focus being the pole.

8. Given three radii-vectores, r_1, r_2, r_3, and the angles a_1, a_2 between them, to find the equation of the ellipse, the focus being the pole.

9. What is the locus of a point whose distances from a given point and a given line are always in a given ratio? Specify the cases that arise, and find in all cases the magnitude of the axis and position of the foci.

10. A given polygon is so moved that two of its angular points are always situated on two given lines: what are the loci of all the other angular points?

11. Parabolas have the same given vertex and touch the same given line, what is the locus of the focus? And if they have the same focus, and touch the same given line, what is the locus of the vertex?

12. What is the locus of the point of intersection of tangents to a conic section which intersect under a given angle? And what, when the tangents are always parallel to pairs of conjugate diameters?

13. Given the base of a triangle to find the locus of the vertex when the sum, difference, product or ratio, of the tangents to the angles at the base are given.

14. Two curves, referred to oblique axes whose inclination is a, are denoted by the equations

$$\frac{1}{ax} + \frac{b}{by} = \frac{1}{xy} + \frac{1}{ab} \text{ and } \frac{1}{a_{,}x} + \frac{1}{b_{,}y} = \frac{1}{xy} + \frac{1}{a_{,}b_{,}}:$$

determine their nature and whether they will intersect one another.

15. Through any given point draw two lines to intersect a conic section, and join the points of section by other straight lines; these straight lines will always intersect in two other parallel straight lines, one of which passes through the given point.

16. The opposite sides of a hexagon inscribed in a conic section being produced to meet, will have their three points of intersection in the same straight line.

17. Find the co-ordinates of the focus of a conic section $ay^2 + bxy + ex^2 + dy + cx + f = 0$.

18. If the three sides of a triangle are tangents to a parabola, the circle described about the triangle will always pass through the focus.

19. The points of contact of a quadrilateral described about a conic section being made the angular points of an inscribed one; prove that the diagonals of both quadrilaterals intersect in the same point.

20. From the angles of a circumscribed pentagon draw lines to the points of contact of the opposite sides: these intersect in one point.

21. A line moving along one side of a given triangle, and inclined to it in a given angle is cut by the other two; and a point is taken in it, so that the three distances shall always be in arithmetical, geometrical, or harmonical proportion. Show that in each case the locus of the point so taken is a conic section.

22. From a point lines are drawn making given angles with any number of given lines, and they are so related that the sums or differences of the squares of any given number of them, or sums or differences the rectangles of any number of them, taken in a given order shall have a given ratio, sum, or difference: then the locus of the point so taken is a conic section.

[This problem is remarkable as being the first to which the Geometry of co-

ordinates was applied by Descartes. It had resisted the united efforts of all preceding Geometers, including Euclid and Apollonius.]

23. If an ellipse and hyperbola have the same diameters, then if from any point in either curve lines be drawn the extremities of either axis to cut the other curve, the lines joining the other points of intersection of the lines and curve will be parallel to the other axis.

24. Of points, lines, or circles, any two being given to find the locus of the centres of the circles which touch these two.

25. The three angles of a triangle move upon the three sides of another ABC; and two sides of the triangle DEF are divided in given ratios by given lines: what is the locus of the points which divide the third side in a given ratio?

EXERCISES ON VARIOUS CURVES.

1. Perpendiculars from the centre of an ellipse or hyperbola are drawn to all the tangents: what are the equations* of the loci of the points of intersection? [This is the *lemniscata of James Bernoulli.*]

2. Given the base and rectangle of the sides of a triangle to find the locus of its vertex, and likewise that of the centre of its inscribed and circumscribed circles. [This curve is called *Cassini's Ellipse.*]

3. On the diameter AB of a circle take a point P and produce the ordinate PD to M till AP : PD :: AB : PM; then the locus of M is *Donna Agnesi's "witch,"* and its equation and general form are required.

4. Take two points P and S equi-distant from the extremities A, B of the diameter AB of a circle, and draw ST, PM perpendicular to AB: and from the point T where ST cuts the circle draw AT, cutting PM in M. The locus of M is the *Cissoïd of Diocles.* Its equation is required.

5. A line AB revolves about a given point A and cuts a given line CB in B; and from B a line BD equal to a given line is set off in the line AB; the locus of D is the *Conchoïd of Nicomedes.* Find its equation.

6. The sum of the cosines of the angles at the base is given, equal to 2 cos. β and the base itself equal to $2a$: what is the locus of its vertex? [This is the *Magnetic Curve.*]

7. BC is a straight line of given length ($2b$) having its extremities always moving on two given equal circles, whose common radius is r, and distance of their centres $2a$; to find the locus of its middle point. [This is *Watt's parallel motion* in the steam engine.]

8. The line a is given, and the ordinate of a curve is every where equal to a times the logarithm of the abscissa. How is its equation to be represented? [*The logarithmic curve.*]

9. The sum of the given line a and the ordinate is always equal to half the sum of the ordinate of a logarithmic curve and its reciprocal. Write the equation. [This is the *Catenary*, or figure of a flexible chain suspended at both ends.]

10. From a given point in the circumference of a circle draw chords; and from the other extremities of them, set off lines equal to the radius of the circle: the curve thus generated is called the *Trisectrix;* and its equation is required.

11. The *Quadratrix of Dinostratus* is thus formed: two lines move uniformly

* The student is to understand in most of these, that the polar as well as rectangular equations are required.

from the one extremity of a quadrant, the other from the other extremity parallel to the tangent at that extremity, and by their mutual intersection trace the curve. Find its equation.

12. The *Quadratrix of Tschirnhausen* is generated by two lines at right angles to one another, the one moving uniformly through the quadrantal circumference while the other moves uniformly through the radius. Its polar equation is required.

13. The *Cycloïd* is generated by a point in the plane of one circle which rolls on a straight line. When the point is in circumference of the rolling circle, it is the *common cycloïd;* when within the circumference it is the *prolate cycloïd* or *Trochoid of Newton:* and when in the plane of the circle but without the circumference, it is the *curtate cycloïd.* Find the equations of each.

14. The *Epicycloïd* is similarly described, but the circle rolls *upon* the circumference of another: and the *Hypocycloïd* when it rolls *within* the second circle. When the two circles are equal, the Epicycloïd is called the *Cardioïde.* Find these equations.

15. The *Epicycle* is generated by the uniform motion of a point in the circumference of a circle, the centre of which moves uniformly in the circumference of another circle. What is its equation?

16. The *Companion to the Cycloïd* is generated by the motion of a right line always at right angles to the diameter from its vertex and equal to the portion of the circle (estimated from the vertex) which it cuts off. Express its property algebraically.

17. If the circumference of a circle be opened out into a straight line, and the sines, cosines, tangents, &c. be set off at right angles to it, the curves are called the *figures of the sines, cosines,* &c. respectively. Find their equations.

18. If the figure of the sines have its ordinates increased or diminished in a constant ratio, their extremities mark out the *Harmonic Curve.* Write its equation. Show also that if a right cylinder be cut obliquely and made to *roll* upon a plane, the successive points of contact of the section of the cylinder with the plane will generate this curve.

19. A string being unwound from the circumference of a circle, its extremity describes a curve called the *Involute of the circle.* Its equation is also sought.

20. A line revolves about a point, and in it another point is taken at a distance which is a given multiple of some power of the arc of the circle cut off by the line. Write this condition algebraically in polar co-ordinates.

When that power is the first and the multiple unity, it is the *Spiral of Archimedes*: when the multiple is unity and the power -1, it is called the *Reciprocal Spiral, the Equiangular Spiral, or* the *Hyperbolic Spiral:* when the power is $-\frac{1}{2}$, it is the *Lituus.*

21. If the axis of a parabola be wound round the circumference of a circle, find the equation of the *Parabolic Spiral* into which the parabola is thus converted.

22. The curve whose distance from the pole is proportional to a power of a given number denoted by the angular distance of the point from the origin is called the *Logarithmic Spiral.* Define it by an equation in rectangular co-ordinates.

23. One parabola rolls upon another, their vertices being originally in contact: what are the loci of the vertex and focus of the rolling one?

24. An ellipse or hyperbola rolls upon another equal to it in the same manner, find the loci of the vertices, foci, and centre of the rolling one.

25 A leaf is doubled down so that the crease is always of the same given length, what are the loci of the vertex, middle of the crease, and centre of gravity of the displaced part? Also, when the area of the doubled part is given.

26. When the radius of the rolling circle in the genesis of the Epicycloïd and Epitrochoïd (Ex 13.) is half that of the fixed circle, the former curve becomes an ellipse and the latter a straight line.

THE DOCTRINE OF FLUXIONS.

DEFINITIONS AND PRINCIPLES.

Art. 1. In the Doctrine of Fluxions, magnitudes or quantities of all kinds are considered, not as made up of a number of small parts, but as generated by continued motion, by means of which they increase or decrease. As, a line by the motion of a point; a surface by the motion of a line; and a solid by the motion of a surface. So likewise, time may be considered as represented by a line, increasing uniformly by the motion of a point. And quantities of all kinds whatever, which are capable of increase and decrease, may in like manner be represented by geometrical magnitudes, conceived to be generated by motion. Indeed, notwithstanding all that has been advanced to the contrary, this seems the most natural, as well as the simplest, way of conducting the higher investigations; since it is impossible to conceive a geometrical magnitude to be brought into existence, or to change its magnitude, figure, or place, without motion *.

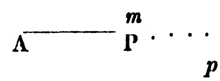
2. Any quantity thus generated, and variable, is called a *Fluent*, or a Flowing Quantity. And the rate or proportion according to which any flowing quantity increases, at any position or instant, is the *Fluxion* of the said quantity, at that position or instant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increased in a given time, with the generating celerity uniformly continued during that time.

3. The small quantities that are actually generated, produced, or described,

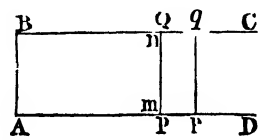
* The Editor has long been of opinion that, in point of intellectual conviction and certainty, the fluxional calculus is decidedly superior to the differential and integral calculus. To think of a mathematical quantity as becoming greater or less, broader or narrower, more or less crooked, &c., *without motion*, is a greater mystery than ever the boldest writer on fluxions attempted to explain. Whatever may be the influence of fashion in producing a change, or whatever may be the relative advantages and disadvantages of the different modes of conducting the modern analysis, he feels that there can be no logical impropriety in introducing the consideration of motion into our mathematical investigations, until there is established a new law of human thought, which will compel us to think of war without bloodshed, gardening without spades, machines without wheels and pinions, chemistry without acids and alkalies, and systems of astronomy without suns and planets. The reader who wishes to give the matter a cautious investigation may advantageously peruse "A Comparative View of the Principles of the Fluxional and Differential Calculus, by the Rev. D. M. Peacock." Those mathematicians who employ the Differential Calculus have, however, in various cases, given improved practical methods; on which account, as well as because all foreign Mathematicians, and a great proportion of British Mathematicians employ the Differential Calculus exclusively, it is desirable that the student should acquaint himself with this analysis as well as the fluxional. We shall, therefore, explain its principles in an Appendix to this volume. The books we have usually employed at the R. M. Academy for this purpose are those of *Thomson*, *Francœur*, *Garnier*, *Hind*, *J. R. Young*, and *Lubbe*.

in any small given time, and by any continued motion, either uniform or variable, are called *Increments.*

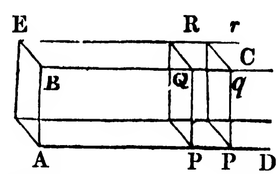
4. Hence, if the motion of increase be uniform, by which increments are generated, the increments will in that case be proportional, or equal, to the measures of the fluxions: but if the motion of increase be accelerated, the increment so generated, in a given finite time, will exceed the fluxion: and if it be a decreasing motion, the increment, so generated, will be less than the fluxion. But if the time be indefinitely small, so that the motion be considered as uniform for that instant; then these nascent increments will always be proportional, or equal, to the fluxions, and may be substituted instead of them, in any calculations.



5. To illustrate these definitions: Suppose a point *m* be conceived to move from the position A, and to generate a line AP, by a motion any how regulated; and suppose the celerity of the point *m*, at any position P, to be such as would, if from thence it should become or continue uniform, be sufficient to cause the point to describe, or pass uniformly over, the distance P*p*, in the given time allowed for the fluxion: then will the said line P*p* represent the fluxion of the fluent, or flowing line, AP, at that position.

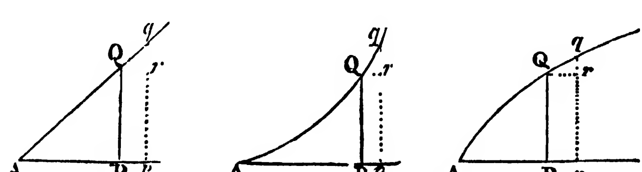


6. Again, suppose the right line *mn* to move from the position AB, continually parallel to itself, with any continued motion, so as to generate the fluent or flowing rectangle ABQP, while the point *m* describes the line AP: also, let the distance P*p* be taken, as before, to express the fluxion of the line or base AP; and complete the rectangle PQ*qp*. Then, like as P*p* is the fluxion of the line AP, so is P*q* the fluxion of the flowing parallelogram AQ; both these fluxions, or increments, being uniformly described in the same time.



7. In like manner, if the solid AERP be conceived to be generated by the plane PQR, moving from the position ABE, always parallel to itself, along the line AD; and if P*p* denote the fluxion of the line AP: Then, like as the rectangle PQ*qp*, or PQ × P*p*, denotes the fluxion of the flowing rectangle ABQP, so also shall the fluxion of the variable solid, or prism ABERQP, be denoted by the prism PQR*rqp*, or the plane PR × P*p*. And, in both these last two cases, it appears that the fluxion of the generated rectangle, or prism, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.

8. Hitherto the generating line, or plane, has been considered as of a constant and invariable magnitude; in which case the fluent, or quantity generated, is a



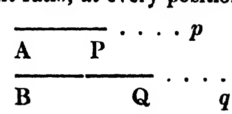
rectangle, or a prism, the former being described by the motion of a line, and the latter by the motion of a plane. So, in like manner, are other figures, whe-

ther plane or solid, conceived to be described by the motion of a Variable Magnitude, whether it be a line or a plane. Thus, let a variable line PQ be carried by a parallel motion along AP; or while a point P is carried along, and describes the line AP, suppose another point Q to be carried by a motion perpendicular to the former, and to describe the line PQ: let pq be another position of PQ, indefinitely near to the former; and draw Qr parallel to AP. Now in this case there are several fluents, or flowing quantities, with their respective fluxions; namely, the line or fluent AP, the fluxion of which is Pp or Qr; the line or fluent PQ, the fluxion of which is rq; the curve or oblique line AQ, described by the oblique motion of the point Q, the fluxion of which is Qq; and lastly, the surface APQ, described by the variable line PQ, the fluxion of which is the rectangle PQrp, or PQ $\times$ Pp. In the same manner may any solid be conceived to be described, by the motion of a variable plane parallel to itself, substituting the variable plane for the variable line; in which case the fluxion of the solid, at any position, is represented by the variable plane, at that position, drawn into the fluxion of the line along which it is carried.

9. Hence then it follows in general, that the fluxion of any figure, whether plane or solid, at any position, is equal to the section of it, at that position, drawn into the fluxion of the axis, or line along which the variable section is supposed to be perpendicularly carried; that is, the fluxion of the figure AQP, is equal to the plane PQ $\times$ Pp, when that figure is a solid, or to the ordinate PQ $\times$ Pp, when the figure is a surface.

10. It also follows from the same premises, that in any curve, or oblique line AQ, whose absciss is AP, and ordinate is PQ, the fluxions of these three form a small right-angled plane triangle Qqr; for Qr = Pp is the fluxion of the absciss AP, qr the fluxion of the ordinate PQ, and Qq the fluxion of the curve or right line AQ. And consequently that, in any curve, the square of the fluxion of the curve, is equal to the sum of the squares of the fluxions of the absciss and ordinate, when these two are at right angles to each other.

11. From the premises it also appears, that contemporaneous fluents, or quantities that flow or increase together, which are always in a constant ratio to each other, have their fluxions also in the same constant ratio, at every position. For, let AP and BQ be two contemporaneous fluents, described in the same time by the motion of the points P and Q, the contemporaneous positions being P, Q, and p, q; and let AP be to BQ, or Ap to Bq, constantly in the ratio of 1 to n.



Then . . . . is $n \times$ AP = BQ,
and $n \times$ Ap = Bq;
therefore, by subtraction, $n \times$ Pp = Qq;
that is, the fluxion . . Pp : fluxion Qq :: 1 : n,
the same as the fluent AP : fluent BQ :: 1 : n,
or, the fluxions and fluents are in the same constant ratio.

But if the ratio of the fluents be variable, so will that of the fluxions be also, though not in the same variable ratio with the former, at every position.

NOTATION AND DEFINITIONS.

12. To apply the foregoing principles to the determination of the fluxions of algebraic quantities, by means of which those of all other kinds are assigned, it will be necessary first to premise the notation commonly used in this science, with some observations.

First, the *final letters* of the alphabet z, y, x, u, &c. are used, as in the Geometry of co-ordinates, p. 186, to denote *variable or flowing quantities;* and the *initial letters*, a, b, c, d, &c. to denote constant or *invariable ones;* Thus, the variable base AP of the flowing rectangular figure ABQP, in art. 6, may be represented by x; and the invariable altitude PQ, by a: also, the variable base or absciss AP, of the figures in art. 8, may be represented by x, the variable ordinate PQ, by y; and the variable curve or line AQ, by z.

Secondly, that the *fluxion of a quantity* denoted by a single letter, is usually represented by the same letter with a point over it: Thus, the fluxion of x is expressed by $\dot{x}$, the fluxion of y by $\dot{y}$, and the fluxion of z by $\dot{z}$. Sometimes, however, the fluxion of a variable quantity, especially if it be a compound one, is denoted by the Greek character ϕ before it Thus, the fluxion of xy may be denoted by $\phi\ (xy)$; the fluxion of xyz by $\phi\ (xyz)$. As to the fluxions of constant or invariable quantities, as of a, b, c, &c. they are equal to nothing, because they do not flow or change their magnitude.

Thirdly, that the *increments of variable or flowing quantities*, are also denoted by the same letters with a small $'$ over them: Thus, the increments of x, y, z, are x', y', z'.

Fourthly, when an expression is a function of a single variable quantity, and its fluxion is divided by the fluxion of that variable, the quotient is called the *fluxional coefficient.*

Fifthly, when an expression is a function of several variables, and the fluxion is taken on the supposition of only one of them being variable, the result is called *the partial fluxion of that expression taken with respect to that specified variable:* and the fluxional expression divided by the fluxion of that variable is called *the partial fluxional coefficient, taken with respect to that variable quantity*, or *variable*, as it is specifically called *.

Sixthly. The sum of all the practical fluxions, or of all the partial fluxional coefficients, constitutes what is called the *total fluxion* or the *total fluxional coefficient* respectively.

Seventhly. The fluxion of the principal variable (or of that in terms of which all the rest are expressed by means of the given equation) is always considered to be *uniform* or *constant*; and is taken as the *unit* by which the rate of flowing of all the others is to be measured.

13. From these notations, and the foregoing principles, the quantities, and their fluxions, there considered, will be denoted as below. Thus, in all the foregoing figures, put

the variable or flowing line . . AP $= x$,
in art. 6, the constant line . . PQ $= a$,
in art. 8, the variable ordinate PQ $= y$,
also, the variable line or curve . AQ $= z$;

Then shall the several fluxions be thus represented, namely,

$x =$ Pp the fluxion of the line AP,
$a\dot{x} =$ PQqp the fluxion of ABQP in art. 6,
$y\dot{x} =$ PQrp the fluxion of APQ, in art. 8,

* Indeed it has now become almost universal in the modern analysis to employ both the words *constant* and *variable* as substantives.

$\dot{z} = Qq = \sqrt{(\dot{x}^2 + \dot{y}^2)}$ the fluxion of AQ; and

$a\dot{x} = Pr$ the fluxion of the solid in art. 7, if a denote the constant generating plane PQR; also

$nx = BQ$ in the figure to art. 11, and

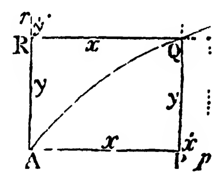
$n\dot{x} = Qq$ the fluxion of the same.

14. The principles and notation being now laid down, we may proceed to the practice and rules of this doctrine; which consist of two principal parts, called the Direct and Inverse Method of Fluxions; namely, the direct Method, which consists in finding the fluxion of any proposed fluent or flowing quantity; and the inverse method, which consists in finding the fluent of any proposed fluxion. As to the former of these two problems, it can always be determined, and that in finite algebraic terms; but the latter, or finding of fluents, can only be effected in some certain cases, except by means of infinite series. First, then, of

THE DIRECT METHOD OF FLUXIONS.

To find the Fluxion of the Product or Rectangle of two Variable Quantities

15. Let ARQP $= xy$, be the flowing or variable rectangle, generated by two lines PQ and RQ, moving always perpendicular to each other, from the positions AR and AP; denoting the one by x, and the other by y; supposing x and y to be so related, that the curve line AQ may always pass through the intersection Q of those lines, or the opposite angle of the rectangle.



Now, the rectangle consists of the two trilinear spaces APQ, ARQ, of which, the fluxion of the former is PQ $\times$ Pp, or $y\dot{x}$,

that of the latter is . RQ $\times$ Rr, or $x\dot{y}$, by art. 8;

therefore the sum of the two $\dot{x}y + x\dot{y}$, is the fluxion of the whole rectangle xy or ARQP.

The same otherwise.

16. Let the sides of the rectangle x and y, by flowing, become $x + x'$ and $y + y'$: then the product of these two, or $xy + xy' + yx' + x'y'$ will be the new or contemporaneous value of the flowing rectangle PR or xy: subtract the one value from the other, and the remainder, $xy' + yx' + x'y'$, will be the increment generated in the same time as x' or y'; of which the last term $x'y'$ is nothing, or indefinitely small, in respect of the other two terms, because x' and y' are indefinitely small in respect of x and y; which term being therefore omitted, there remains $xy' + yx'$ for the value of the increment; and hence, by substituting $\dot{x}$ and $\dot{y}$ for x' and y', to which they are proportional, there arises $\dot{x}y + y\dot{x}$ for the true value of the fluxion of xy; the same as before.

17. Hence may be easily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of xyz, or $uxyz$, or $vuxyz$, &c. And first, for the fluxion of xyz: put $p = xy$, and the whole given fluent $xyz = q$, or $q = xyz = pz$. Then, taking the fluxions of $q = pz$, by the last article, they are $\dot{q} = \dot{p}z + p\dot{z}$; but $p = xy$, and so $\dot{p} = \dot{x}y + x\dot{y}$ by the

same article; substituting therefore these values of p and $\dot{p}$ instead of them, in the value of $\dot{q}$, this becomes $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$ the fluxion of xyz required; which is therefore equal to the sum of the products, arising from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of $uxyz$, the continual product of four variable quantities; put this product, namely, $uxyz$, or $qu = r$, where $q = xyz$ as above. Then, taking the fluxions by the last article, $\dot{r} = \dot{q}u + q\dot{u}$; which, by substituting for q and $\dot{q}$ their values as above, becomes $\dot{r} = \dot{u}xyz + u\dot{x}yz + ux\dot{y}z + uxy\dot{z}$, the fluxion of $uxyz$ as required: consisting of the fluxion of each quantity, drawn into the products of the other three.

In the very same manner it is found, that the fluxion of $vuxyz$ is $\dot{v}uxyz + v\dot{u}xyz + vu\dot{x}yz + vux\dot{y}z + vuxy\dot{z}$; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. Hence is easily derived the fluxion of any power of a variable quantity, as of x^2, or x^3, or x^4, &c. For, in the product or rectangle xy, if $x = y$, then is $xy = xx$ or x^2, and also its fluxion $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$ or $2x\dot{x}$, the fluxion of x^2.

Again, if all the three, x, y, z be equal; then is the product of the three $xyz = x^3$; and consequently, its fluxion $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$ $3x^2\dot{x}$, the fluxion of x^3.

In the same manner, it will appear that

the fluxion of x^4 is $= 4x^3\dot{x}$ and
the fluxion of x^5 is $= 5x^4\dot{x}$, and, in general,
the fluxion of x^n is $= nx^{n-1}\dot{x}$;

where n is any positive whole number whatever.
That is, the fluxion of any positive integral power, is equal to the fluxion of the root ($\dot{x}$), multiplied by the exponent of the power (n), and by the power of the same root whose index is less by 1, (x^{n-1})*

* In the text, the fluxion of the product of two, three, or more, variable quantities is found, and thence, by supposing them to become equal, the fluxions of the square, cube, &c. of a variable quantity, are inferred. Sometimes, the investigation commences with the fluxion of a square, and proceeds thence to that of a rectangle.

Let $x - s$ and x be two states of the same line generated by an equable motion; then, while the line $x - s$ by flowing equably becomes x, its square $(x - s)^2$ will become x^2. That is, while the space s is described equably by the flowing line, the space $x^2 - (x - s)^2 = 2sx - s^2$ will be described by the flowing square of that line, and this latter is the space which *would* have been generated in the same time by a certain magnitude (whether assignable or not) moving uniformly. Hence, the fluxion of the flowing magnitude $(x - s)$, is to the fluxion of the flowing magnitude $(x - s)^2$, as s to $2sx - s^2$, or as 1 to $2x - s$; and as this must obtain in all possible values of $x - s$, it must obtain in the ultimate state, when $(x - s)$ by flowing, becomes x; and then, s vanishing, the ratio becomes 1 to $2x$. That is, the ratio of the fluxions of x and x^2, is that of 1 to $2x$. Consequently, if $\dot{x}$ denote the fluxion of x, then will $2x\dot{x}$ denote the fluxion of x^2.

The fluxion of the square of a quantity being thus found, that of any product is easily assigned. Thus, to determine the fluxion of the product of xy: Put

And thus, the fluxion of $a + cx$ being $c\dot{x}$,
that of $(a + cx)^2$ is $2c\dot{x} \times (a + cx)$ or $2ac\dot{x} + 2c^2x\dot{x}$,
that of $(a + cx^2)^2$ is $4cx\dot{x} \times (a + cx^2)$ or $4acx\dot{x} + 4c^2x^3\dot{x}$,
that of $(x^2 + y^2)^2$ is $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$,
that of $(x + cy^2)^3$ is $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$.

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of $\frac{x}{y}$. For, put the quotient or fraction $\frac{x}{y} = q$; then multiplying by the denominator, $x = qy$; and, taking the fluxions, $\dot{x} = \dot{q}y + q\dot{y}$, $= \dot{x} - q\dot{y}$; and, by division,

$$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} = \text{(by substituting the value of } q\text{, or } \frac{x}{y}\text{)},$$

$$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}, \text{ the fluxion of } \frac{x}{y}, \text{ as required.}$$

That is, the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the square of the denominator. So that the fluxion of $\frac{ax}{y}$ is $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{a\dot{x}y - ax\dot{y}}{y^2}$.

20. Hence too is easily derived the fluxion of any negative integer power of a variable quantity, as of x^{-n}, or $\frac{1}{x^n}$, which is the same thing. For here the numerator of the fraction is 1, whose fluxion is nothing; and therefore, by the last article, the fluxion of such a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the square of the said denominator. That is, the fluxion of x^{-n}, or $\frac{1}{x^n}$ is $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$ or $-\frac{n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; or the fluxion of any negative integer power of a variable quantity, as x^{-n}, is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power less by 1; the same rule as for positive powers.

The same thing is otherwise obtained thus: Put the proposed fraction, or quotient $\frac{1}{x^n} = q$; then is $qx^n = 1$; and, taking the fluxions, we have $\dot{q}x^n + qnx^{n-1}\dot{x} = 0$; hence $\dot{q}x^n = -qnx^{n-1}\dot{x}$; divide by x^n, then $\dot{q} = -\frac{qn\dot{x}}{x} =$ (by substituting $\frac{1}{x^n}$ for q), $\frac{-n\dot{x}}{x^{n+1}}$ or $= -nx^{-n-1}\dot{x}$; the same as before.

Hence the fluxion of x^{-1} or $\frac{1}{x}$ is $-x^{-2}\dot{x}$, or $-\frac{\dot{x}}{x^2}$,

Put $x + y = s$; then $\phi(x + y) = \dot{x} + \dot{y} = \dot{s}$,
also, $x^2 + 2xy + y^2 = s^2$; $\therefore 2xy = s^2 - x^2 - y^2$,
and $xy = \frac{1}{2}s^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$,

$$\begin{aligned}\therefore \text{ by the above } \phi(xy) &= s\dot{s} - x\dot{x} - y\dot{y},\\ &= s(\dot{x} + \dot{y}) - \dot{x}x - y\dot{y},\\ &= (x + y)(\dot{x} + \dot{y}) - x\dot{x} - y\dot{y},\\ &= x\dot{x} + x\dot{y} + y\dot{x} + y\dot{y} - x\dot{x} - y\dot{y},\\ &= \dot{x}y + x\dot{y},\end{aligned}$$

agreeing with the result in art. 15.

that of . . . x^{-2} or $\frac{1}{x^2}$ is $-2x^{-3}\dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

that of . . . x^{-3} or $\frac{1}{x^3}$ is $-3x^{-4}\dot{x}$ or $-\frac{3\dot{x}}{x^4}$,

that of . . . ax^{-4} or $\frac{a}{x^4}$ is $-4ax^{-5}\dot{x}$ or $-\frac{4a\dot{x}}{x^5}$,

that of . . . x^{-n} or $\frac{1}{x^n}$ is $-\frac{n\dot{x}}{x^{n+1}}$,

that of $(a+x)^{-1}$ or $\frac{1}{a+x}$ is $-(a+x)^{-2}\dot{x}$ or $-\frac{\dot{x}}{(a+x)^2}$,

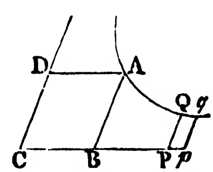
that of $c(a+3x^2)^{-2}$ or $\frac{c}{(a+3x^2)^2}$ is $-12cx\dot{x} \times (a+3x^2)^{-3}$, or $-\frac{12cx\dot{x}}{(a+3x^2)^3}$.

21. Much in the same manner is obtained the fluxion of any fractional power of a fluent quantity, as of $x^{\frac{m}{n}}$, or $\sqrt[n]{x^m}$.

For, put the proposed quantity $x^{\frac{m}{n}} = q$; then, raising each side to the nth power, gives $x^m = q^n$;
taking the fluxions, gives $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$; then dividing by nq^{n-1},
gives $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{m}{n}\, x^{\frac{m}{n}-1}\dot{x}$.
Which is still the same rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be positive or negative, integral or fractional. And hence the fluxion of $ax^{\frac{3}{2}}$ is $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$; that of $ax^{\frac{1}{2}}$ is $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\sqrt{x}}$; and that of

$\sqrt{(a^2-x^2)}$ or $(a^2-x^2)^{\frac{1}{2}}$ is $\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}} \times -2x\dot{x} = -\frac{-x\dot{x}}{\sqrt{(a^2-x^2)}}$.

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities; it remains to determine those of *logarithmic expressions;* and also of *exponential ones,* that is, such powers as have their exponents variable or flowing quantities. And first, for the fluxion of Napier's, or the hyperbolic logarithm.



23. Now, to determine this from the nature of the *hyperbolic spaces.* Let A be the principal vertex of an hyperbola, having its asymptotes CD, CP, with the ordinates DA, BA, PQ, &c. parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known, that any space ABPQ is the log of the ratio of CB to CP, to the modulus ABCD. Now, put $1 =$ CB or BA the side of the square or rhombus DB; $m =$ the modulus, or CB $\times$ BA $\times$ sin. C; or area of DB, or sine of the angle C to the radius 1; also the absciss CP $= x$, and the ordinate PQ $= y$. Then, by the nature of the hyperbola, CP $\times$ PQ is always equal to DB, that is $xy = m$; hence $y = \frac{m}{x}$, and the fluxion of the space, $\dot{x}y$ is $\frac{m\dot{x}}{x} =$ PQqp the fluxion of the log. of x, to the modulus m. And, in the hyperbolic logarithms, the modulus m being unity, therefore $\frac{\dot{x}}{x}$ is the fluxion of

the hyp. log. of x; which is therefore equal to the fluxion of the quantity, divided by the quantity itself*.

Hence the fluxion of the hyp. log.

of $1 + x$ is $\frac{\dot{x}}{1 + x}$,

of $1 - x$ is $\frac{-\dot{x}}{1 - x}$,

of $x + z$ is $\frac{\dot{x} + \dot{z}}{x + z}$.

of $\frac{a + x}{a - x}$ is $\frac{\dot{x}(a - x) + \dot{x}(a + x)}{(a - x)^2} \times \frac{a - x}{a + x} = \frac{2a\dot{x}}{a^2 - x^2}$,

of ax^n is $\frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}$.

24. By means of the fluxions of logarithms, are usually determined those of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. These exponentials are of two kinds, namely, when the root is a constant quantity, as e^x, and when the root is variable as well as the exponent, as y^x.

25. In the first case, put the exponential, whose fluxion is to be found, equal to a single variable quantity z, namely, $z = e^x$; then take the logarithm of each, so shall log. $z = x \times$ log. e; take the fluxions of these, so shall $\frac{\dot{z}}{z} = \dot{x} \times$ log. e, by the last article: hence $\dot{z} = z\dot{x} \times$ log. $e = e^x\dot{x} \times$ log. e, which is the fluxion of the proposed quantity e^x or z; and which therefore is equal to the said given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence also, the fluxion of $(a + c)^{nx}$ is $(a + c)^{nx} \times n\dot{x} \times$ log. $(a + c)$.

26. In like manner, in the second case, put the given quantity $y^x = z$; then the logarithms give log. $z = x \times$ log. y, and the fluxions give $\frac{\dot{z}}{z} = \dot{x}$. log. $y + x . \frac{\dot{y}}{y}$; hence $\dot{z} = z\dot{x}$. log. $y + \frac{zx\dot{y}}{y} =$ (by substituting y^x for z) $y^x\dot{x}$. log. $y + xy^{x-1}\dot{y}$, which is the fluxion of the proposed quantity y^x; and which therefore consists of two terms, of which the one is the fluxion of the given quantity considering the exponent as constant, and the other the fluxion of the same quantity considering the root as constant.

27. The fluxions of the usual trigonometrical quantities, sin. z, cos. z, &c. are easily found by blending these principles with the analytical formulæ at p. 402—6, vol. i., and p. 20 of this volume. We assume the proportionality of the increments, and of their contemporaneous fluxions, and proceed thus:

To find ϕ sin. z, (using ϕ to denote fluxion) we suppose that by a motion of one of the legs including the angle, it becomes $z + x'$ or $z + \dot{z}$. Then ϕ sin. $z =$ sin. $(z + \dot{z}) -$ sin. z. But by equa. 5, p. 20, we have

$$\text{sin. } (z + \dot{z}) = \text{sin. } z \text{ . cos. } \dot{z} + \text{sin. } \dot{z} \text{ cos. } z.$$

But the sine of an arc indefinitely small does not differ sensibly from that arc itself, nor its cosine differ perceptibly from radius; hence we have sin. $\dot{z} = \dot{z}$,

* The fluxion of the exponential might have been found from the developed series, vol. i. p. 243, and thence the fluxion of the logarithm independently of the hyperbola: but as it would have occupied more room without any adequate advantage in point of clearness, the method above given has been employed in preference.

and $\cos.\dot{z} = 1$; and therefore $\sin.(z + \dot{z}) = \sin. z + \dot{z}\cos. z$; whence $\sin.(z + \dot{z}) - \sin. z$, or $\phi(\sin. z) = \dot{z}\cos. z$, viz. the fluxion of the sine of an arc whose radius is unity, is equal to the product of the fluxion of the arc into the cosine of the same arc.

28. In like manner, the fluxion of $\cos. z$, or $\cos.(z + \dot{z}) - \cos. z = \cos. z \cos. \dot{z} - \sin. z \sin. \dot{z} - \cos. z$, or since $\cos.(z + \dot{z}) = \cos. z \cos. \dot{z} - \sin. z \sin. \dot{z}$; therefore, because $\sin. \dot{z} = \dot{z}$, and $\cos. \dot{z} = 1$, we have $\phi \cos. z = \cos. z - \dot{z} \sin. z - \cos. z = -\dot{z}\sin. z$, that is, the fluxion of the cosine of an arc, radius being 1, is found by multiplying the fluxion of the arc (taken with a contrary sign) by the sine of the same arc.

29. By means of these two formulæ, many other fluxional expressions may be found, viz.

$$\phi \cos. mz = - m\dot{z} \sin. mz.$$

$$\phi \sin. mz = + m\dot{z} \cos. mz.$$

$$\phi \tan. z = \frac{\dot{z}}{\cos.^2 z} = \dot{z} \sec.^2 z.$$

$$\phi \operatorname{cotan}. z = -\frac{\dot{z}}{\sin.^2 z} = - \dot{z} \operatorname{cosec}.^2 z.$$

$$\phi \sec. z = \frac{\dot{z} \sin. z}{\cos.^2 z} = \frac{\dot{z} \tan. z}{\cos. z}.$$

$$\phi \operatorname{cosec}. z = - \frac{z \cos. z}{\sin.^2 z} = - \frac{\dot{z} \cot. z}{\sin. \dot{z}}.$$

$$\phi \sin.^m z = m \sin.^{m-1} z\dot{z} \cos. z.$$

$$\phi \cos.^m z = - m \cos.^{m-1} \dot{z} z \sin. \dot{z}.$$

30 Hence, by the way, will flow this useful practical conclusion, that if z be any arc, then

$$\dot{z} = \frac{\phi \sin. z}{\cos. z} = \frac{-\phi \cos. z}{\sin. z} = \cos.^2 z\ \phi \tan. z.$$

$$= \frac{\phi \tan. z}{1 + \tan.^2 z} = - \phi \cot. z \sin.^2 z = \frac{-\phi \cot. z}{1 + \cot.^2 z}.$$

EXERCISE.

The student is required to find the fluxion of the arc in terms of all its trigonometrical functions; and the fluxions of those functions in terms of the arcs.

OF SECOND, THIRD, &c. FLUXIONS.

Having explained the manner of considering and determining the first fluxions of flowing or variable quantities; it remains now to consider those of the higher orders, as second, third, fourth, &c. fluxions.

31. It was remarked, at page 206, that it is usual to consider the fluxion of the principal variable as a *constant rate;* and we have seen that the fluxion of every function whatever has as its multiplier this constant fluxion of the principal variable. The coefficient of this fluxion may, however, and generally does, consist of some new function of the variables, and hence, in all such cases, the fluxion of the function also varies with the value of the principal variable. The fluxion of a function, therefore, admits of a new fluxion; and this is called

the *second fluxion* of the original function. In like manner, should the coefficient of this second fluxion also involve one or more of the variables, it, like the first fluxion, will admit of a new system of changes, or of its appropriate fluxion. The fluxion of this is called the *third fluxion* of the original function. The same reasoning applies to these coefficients continually, so long as they any way involve the variables of the original function.

Now, in the case of the first fluxion, it is composed of two factors, one of which is the fluxion of the principal variable, and the other the fluxional coefficient of the function, of which the former is constant; its fluxion therefore is 0. The fluxion of the fluxion, or second function, is therefore found by multiplying this constant fluxion of the principal variable by the fluxion of its coefficient taken as the other factor of this new function. In the same way, the third fluxion is found from the second, as the second was from the first; and so on to any extent, as long as these successive fluxional coefficients remain variable.

These orders of fluxions are denoted by the same fluent letter with the corresponding number of points over it: namely, two points for the second fluxion, three points for the third fluxion, four points for the fourth fluxion, and so on. So, the different orders of the fluxion of x, are $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c.; where each is the fluxion of the one next before it.

32. This description of the higher orders of fluxions may be illustrated by the figures exhibited in art 8, where, if x denote the absciss AP, and y the ordinate PQ; and if the ordinate PQ or y flow along the absciss AP or x, with a uniform motion; then the fluxion of x, namely, $\dot{x} =$ Pp or Qr, is a constant quantity, or $\ddot{x} = 0$, in all the figures. Also, in fig. 1, in which AQ is a right line, $\dot{y} = rq$, or the fluxion of PQ, is a constant quantity or $\ddot{y} = 0$; for the angle Q, = the angle A, being constant, Qr is to rq, or $\dot{x}$ to $\dot{y}$, in a constant ratio. But in the 2d fig. rq, or the fluxion of PQ, continually increases more and more; and in fig. 3 it continually decreases more and more, and therefore in both these cases y has a second fluxion, being positive in fig. 2, but negative in fig. 3. And so on, for the other orders of fluxions.

Thus if, for instance, the nature of the curve be such, that x^3 is every where equal to a^2y; then, taking the fluxions, it is $a^2\dot{y} = 3x^2\dot{x}$; and, considering $\dot{x}$ always as a constant quantity, and taking always the fluxions, the equations of the several orders of fluxions will be as below, viz.

FLUXIONS.		FLUXIONAL COEFFICIENTS.
First fluxion . .	$a^2\dot{y} = 3x^2\dot{x}$	$\frac{\dot{y}}{\dot{x}} = \frac{3x^2}{a^2}$
Second	$a^2\ddot{y} = 6x\dot{x}^2$	$\frac{\ddot{y}}{\dot{x}^2} = \frac{6x}{a^2}$.
Third	$a^2\dddot{y} = 6\dot{x}^3$	$\frac{\dddot{y}}{\dot{x}^3} = 6$.
Fourth	$a^2\ddddot{y} = 0$	$\frac{\ddddot{y}}{\dot{x}^4} = 0$.

And all the higher fluxions and their coefficients are also severally = 0.

Also the higher orders of fluxions are found in the same manner as the lower ones. Thus,

the first fluxion of y^3 is . . . . $3y^2\dot{y}$;

its 2d flux. or the flux. of $3y^2\dot{y}$, considered as the rectangle of $3y^2$, and $\dot{y}$, is . . . . $3y^2\ddot{y} + 6y\dot{y}^2$:

and the flux. of this again, or the 3d flux. of y^3, is $3y^2\dddot{y} + 18y\dot{y}\ddot{y} + 6\dot{y}^3$.

33. If the function proposed were ax^n, we should find $\phi ax^n = nax^{n-1}\dot{x}$; the factors n, a, and $\dot{x}$ being regarded as constant in the first fluxion $nax^{n-1}\dot{x}$, to obtain the second fluxion it will suffice to make x^{n-1} flow, and to multiply the result by $na\dot{x}$; but $\phi x^{n-1} = (n-1)x^{n-2}\dot{x}$; we have, therefore,

2nd $\phi\, ax^n = n(n-1)ax^{n-2}\dot{x}^2$.

3rd $\phi\, ax^n = n(n-1)(n-2)ax^{n-3}\dot{x}^3$.

4th $\phi\, ax^n = n(n-1)(n-2)(n-3)ax^{n-4}\dot{x}^4$.

&c. = &c.

5th $\phi\, ax^n = n(n-1)(n-2) \ldots\ldots (n-m+1)\,a^{n-m}\dot{x}^m$,

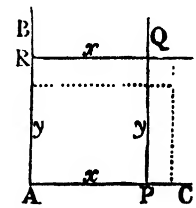
m being supposed not to exceed n: for it is manifest that in the case of n being integral, the function ax^n has only a limited number of fluxions, of which the most elevated is the nth, and which of course is expressed by the formula,

$$n\text{th}\ \phi\, ax^n = n(n-1)(n-2) \ldots\ldots 3 \cdot 2 \cdot 1 \cdot a\dot{x}^n,$$

in which state it admits no longer of being put into fluxions, as it contains no variable quantity, or, in other words, its fluxion is equal to zero.

If, however, n be negative number, fraction, or mixed number, or if it be irrational, then the number of fluxions of it is infinite; for in that case none of the indices, and therefore none of the factors of the coefficient can become *zero*.

34. In the foregoing articles, it has been supposed that the fluents increase, or that their fluxions are positive; but it often happens that some fluents decrease, and that therefore their fluxions are negative: and whenever this is the case, the sign of the fluxion must be changed, or made contrary to that of the fluent. So, of the rectangle xy, when both x and y increase together, the fluxion is $\dot{x}y + x\dot{y}$: but if one of them, as y, decrease, while the other, x, increases; then, the fluxion of y being $-\dot{y}$, the fluxion of xy will in that case be $\dot{x}y - x\dot{y}$. This may be illustrated by the annexed rectangle APQR $= xy$, supposed to be generated by the motion of the line PQ from A towards C, and by the motion of the line RQ from B towards A: For, by the motion of PQ, from A towards C, the rectangle is increased, and its fluxion is $+\dot{x}y$; but, by the motion of RQ, from B towards A, the rectangle is decreased, and the fluxion of the decrease is $x\dot{y}$; therefore, taking the fluxion of the decrease from that of the increase, the fluxion of the rectangle xy, when x increases and y decreases, is $\dot{x}y - x\dot{y}$.



35. We may now collect the principal rules, which have been demonstrated in the foregoing articles, for finding the fluxions of all forms of functions. And hence,

1st. *For the fluxion of any power of a flowing quantity.*—Multiply all together the exponent of the power, the fluxion of the root, and the power next less by 1 of the same root.

2nd. *For the fluxion of the rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper signs.

3d. *For the fluxion of the continual product of any number of flowing quantities.*—Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper signs.

4th. *For the fluxion of a fraction.*—From the fluxion of the numerator drawn

into the denominator, subtract the fluxion of the denominator drawn into the numerator, and divide the result by the square of the denominator.

5th, *Or, the 2d, 3d, and 4th cases may be all included under one, and performed thus.*—Take the fluxion of the given expression as often as there are variable quantities in it, supposing first only one of them variable, and the rest constant; then another variable, and the rest constant; and so on, till they have all in their turns been singly supposed variable; and connect all these fluxions together with their own signs. In other words, add together all the partial fluxions taken with respect to each of the variables separately: the sum is the total fluxion sought.

6th. *For the fluxion of a logarithm*—Divide the fluxion of the quantity by the quantity itself, and multiply the result by the modulus of the system of logarithms.

Note.—The modulus of the hyperbolic logarithms is 1, and the modulus of the common logs. is 0·43429448. (See vol. i. p. 246)

7th. *For the fluxion of an exponential quantity, having the root constant.*—Multiply all together, the given quantity, the fluxion of its exponent, and the hyp. log. of the root.

8th. *For the fluxion of an exponential quantity, having the root variable.*—To the fluxion of the given quantity, found by the 1st rule, as if the root only were variable, add the fluxion of the same quantity found by the 7th rule, as if the exponent only were variable; and the sum will be the fluxion for both of them variable.

Note.—When the given quantity consists of several terms, find the fluxion of each term separately, and connect them all together with their proper signs; also, for the fluxions of trigonometrical formulæ, take the formulæ in articles 27—30.

36. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. Find the fluxion of axy; of $bxyz$; of $cx \times (ax - cy)$; and $x^m y^n$.

2. Find the fluxion of $x^m y^n z^r$; of $(x + y) \times (x - y)$; of $2ax^2$; of $2x^3$; and of $3x^4 y$.

3. Find the fluxion of $4x^{\frac{2}{3}} y^4$; of $ax^2 y - x^{\frac{1}{2}} y^3$; and of $4x^4 - x^2 y + 2byz$.

4. Find the fluxion of $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$; of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$; and of $\frac{1}{\sqrt[n]{x^m}}$ or $\frac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$.

5. Determine the fluxions of the following expressions: $\sqrt{x}$ or $x^{\frac{1}{2}}$; $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$; $\sqrt[3]{x^2}$ or $x^{\frac{2}{3}}$; $\sqrt{x^3}$ or $x^{\frac{3}{2}}$; $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$; and $\sqrt[3]{x^4}$ or $x^{\frac{4}{3}}$.

6. Also those of $\sqrt{(a^2 + x^2)}$ or $(a^2 + x^2)^{\frac{1}{2}}$; $\sqrt{(a^2 - x^2)}$ or $(a^2 - x^2)^{\frac{1}{2}}$; $\sqrt{(2rx - xx)}$ or $(2rx - xx)^{\frac{1}{2}}$; and $\frac{1}{\sqrt{(a^2 - x^2)}}$ or $(a^2 - x^2)^{-\frac{1}{2}}$.

7. Likewise of $(ax - xx)^{\frac{1}{3}}$; $2x\sqrt{a^2 \pm x^2}$; $(a^2 - x^2)^{\frac{3}{2}}$; $\sqrt{xz}$ or $(xz)^{\frac{1}{2}}$; and of $\sqrt{xz - zz}$ or $(xz - zz)^{\frac{1}{2}}$.

8. The fluxions of $-\frac{1}{a\sqrt{x}}$ or $-\frac{1}{a}x^{-\frac{1}{2}}$; $\frac{ax^3}{a + x}$; $\frac{x^m}{y^n}$; $\frac{x + y + z}{x + y}$; and $\frac{c}{xx}$ are required.

9. Find the fourth fluxion of $\frac{3x}{a-x}$; of $\frac{z}{x+z}$; and of $\frac{axy^2}{z}$, x being the principal variable.

10. Also the third fluxion of $\sqrt[3]{(a+bx+cx^2+dx^3)}$ and $\sqrt[n]{(a+bx+cx^2+}$ &c. to $mx^n)$.

11. The second fluxional coefficient of $\frac{3x^{\frac{1}{2}}}{y\sqrt{(x^2-y^2)}}$, and of $\sqrt[n]{\frac{xy}{z}+\frac{yz}{x}+\frac{xz}{y}}$.

12. Determine the fluxion of the hyp. logs. of x^2; of $\sqrt{z}$; of x^m; and of $\frac{2}{x^2}$. And likewise the second fluxions of their common logs., and the fluxion of the hyp. logs. of $\frac{1+x}{1-x}$ and of $\frac{1-x}{1+x}$.

13. Find the fourth fluxions of c^x; of 10^x; of $(a+c)^x$; of 100^{xy}; and of x^x.

14. And the third fluxions of y^{10x}; of x^x; and of $(xy)^{xz}$, x being the principal variable.

15. The fluxion of $x\dot{y}$ is . . . .; and the fluxion of $\dot{x}\dot{y}^2$ is . . . . also
When $\dot{x}$ is constant, find the second fluxions of xy; of xy; and of x^y. And likewise the third fluxions of x^n, when $\dot{x}$ is constant, and the fourth fluxion of x^y when x and y are functions of a third variable.

16. Find the fluxions of $(ax^3+b)^2+2\sqrt{a^2-x^2}\,(x-b)$; $(a+bx^2)^{\frac{5}{4}}$; $a+\frac{4\sqrt{x}}{3+x^2}$; $a+b\sqrt{x}-\frac{c}{x}$; and $\frac{x}{x\pm\sqrt{1-x^2}}$; and the fluxional coefficients of $x(a^2+x^2)\sqrt{a^2-x^2}$; $\log.\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$; $\frac{\sqrt{x^2+2ax}}{\sqrt[3]{x^3-x^2+x}}$, and

17. Find the fluxional coefficients of the following exponential functions:—

$$a^{a^x};\quad c^{x\sqrt{-1}}+c^{-x\sqrt{-1}};\quad a^{b^{(x^2+x)}};\quad x^{x^x}.$$

Also the fluxions of the following logarithmic ones:—
$\log.^2x$ *; $(\log.x^n)^m$; $\log.\{(x+a)^m(x+a')^{m'}(x+a'')^{m''}$ . . . . . to n factors$\}$; $\log.\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$; $\log.^n x$; $\log.^{-1}\frac{\sqrt{1+x^2}}{x}$; $a^{\log x}$; and $e^{\log^x x}$.

18. Prove that in all exponential functions, the nth fluxional coefficient divided by the function itself is constant.

19. Find the fluxion of $a^{\cos. x}$, and the second fluxion of $e^{x \tan x}$, where e = base of hyperbolic logarithms.

20. Find the $2n$th fluxional coefficient of $\sin. x$, and $\cos. x$; and then the $(2n+1)$th; and show the law which prevails amongst them.

21. Find the nth fluxion of uv, of $\frac{u}{v}$, and of u^v, neither variable being the principal one †.

* This notation signifies the logarithm of the logarithm of x; and $\log.^n x$ signifies the same successive taking the logarithm n times: $\log.^{-1}x$ signifies the *reverse operation*.

† That is, the two being functions, not given, of a third variable, and consequently their several fluxions being themselves variable.

22. Find the fluxions of $x^{\log x}$; $(\log. x)^x$; $(\sin. x)^{\sin. x}$; $(\log. \sin. x)^{\log \sin. x}$; and $(\sin. x)^{(\tan x)^{-\sin. x}}$.

23. Find the fluxional coefficient of $\frac{x-a}{\sqrt{y^2+(x-a)^2}} \pm \frac{x+a}{\sqrt{y^2+(x+a)^2}}$.

THE INVERSE METHOD, OR THE FINDING OF FLUENTS.

37. It has been observed, that a Fluent, or Flowing Quantity, is the variable quantity which is considered as increasing or decreasing. Or, the fluent of a given fluxion, is such a quantity, that its fluxion, found according to the foregoing rules, shall be the same as the fluxion given or proposed.

38. It may be further observed, that Contemporary Fluents, or Contemporary Fluxions, are such as flow together, or for the same time.—When contemporary fluents are always equal, or in any constant ratio; then also are their fluxions respectively either equal, or in that same constant ratio. That is, if $x = y$, then is $\dot{x} = \dot{y}$; or if $x : y :: n : 1$, then is $\dot{x} : \dot{y} :: n : 1$; or if $x = ny$, then is $\dot{x} = n\dot{y}$.

39. It is easy to find the fluxions to all the given forms of fluents; but, on the contrary, it is difficult to find the fluents of many given fluxions; and indeed there are numberless cases in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite series. For it is only in certain particular forms and cases that the fluents of given fluxions can be found; there being no method of performing this universally, *à priori*, by a direct investigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for such forms of fluxions as we know, from the direct method, belong to such and such kinds of flowing quantities: and these rules, it is evident, must chiefly consist in performing such operations as are the reverse of those by which the fluxions are found of given fluent quantities. The principal cases of which are as follow.

40. *To find the fluent of a simple fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely substituting the variable or flowing quantity instead of its fluxion; being the result or reverse of the notation only.—Thus,

The fluent of $a\dot{x}$ is ax.
The fluent of $a\dot{y} + 2\dot{y}$ is $ay + 2y$.
The fluent of $\dot{\phi} \sqrt{a^2 + x^2}$ is $\sqrt{a^2 + x^2}$.

41 *When any power of a flowing quantity is multiplied by the fluxion of that quantity.*

Then, having substituted, as before, the flowing quantity, for its fluxion, divide the result by the new index of the power. Or, which is the same thing, take out, or divide by, the fluxion of the root; add 1 to the index of the power; and divide by the index so increased. Which is the reverse of the 1st rule for finding fluxions.

So if the fluxion proposed be . . . . . $3x^5\dot x$.
Leave out or divide by $\dot x$, then it is . . $3x^5$;
add 1 to the index, and it is . . . . . $3x^6$;
divide by the index 6, and it is . . . $\frac{3}{6}x^6$ or $\frac{1}{2}x^6$,
which is the fluent of the proposed fluxion $3x^5\dot x$.

In like manner,

1. The fluent of $2ax\dot x$ is ax^2, and that of $3x^2\dot x$ is x^3.

2. The fluent of $x^{\frac{1}{2}}\dot x + 3y^{\frac{2}{3}}\dot y$ is $\frac{2}{3}x^{\frac{3}{2}} + \frac{9}{5}y^{\frac{5}{3}}$.

3. Find the fluent of $4x^{\frac{1}{2}}\dot x$; of $2y^{\frac{3}{4}}\dot y$; of $az^{\frac{5}{6}}\dot z$; of $x^{n-1}\dot x$; and of $ny^{n-1}\dot y$.

4. Find the fluent of $\frac{\dot z}{z^2}$ or $z^{-2}\dot z$; of $\frac{a\dot y}{y^n}$; of $(a+x)^4\dot x$; of $(a^4+y^4)y^3\dot y$; and of $(a^3+z^3)^4z^2\dot z$.

5. Also the fluents of $(a^n+x^n)^m x^{n-1}\dot x$; $(a^2+y^2)^3y\dot y$; $\frac{z\dot z}{\sqrt{(a^2+z^2}}$; and $\frac{\dot x}{\sqrt{(a-x)}}$.

42. *When the root under a vinculum is a compound quantity; and the index of the part or factor without the vinculum, increased by 1, is some multiple of that under the vinculum :*

Put a single variable letter for the compound root; and substitute its power and fluxion instead of those of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then be applied.

Thus, if the given fluxion be $\dot y = (a^2 + x^{2}{}^{\frac{2}{3}}x^3\dot x$, where 3, the index of the quantity without the vinculum, increased by 1, makes 4, which is just the double of 2, the exponent of x^2 within the vinculum · therefore, putting $\dot z = a^2 + x^2$, thence $x^2 = z - a^2$, the fluxion of which is $2x\dot x = \dot z$; hence then $x^3\dot x = \frac{1}{2}x^2\dot z = \frac{1}{2}\dot z(z - a^2)$, and the given fluxion $\dot y$, or $(a^2+x^2)^{\frac{2}{3}}x^3\dot x$, is $= \frac{1}{2}z^{\frac{2}{3}}\dot z(z-a^2)$ or $= \frac{1}{2}z^{\frac{5}{3}}\dot z - \frac{1}{2}a^2z^{\frac{2}{3}}\dot z$; and hence the fluent y is $= \frac{3}{16}z^{\frac{8}{3}} - \frac{3}{10}a^2z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16}z - \frac{1}{10}a^2)$. Or, by substituting the value of z instead of it, the same fluent is $3(a^2+x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{80}a^2)$, or $\frac{3}{16}(a^2+x^2)^{\frac{5}{3}} \times (x^2 - \frac{3}{5}a^2)$.

In like manner for the following examples.

To find the fluent of $\sqrt{a + cx} \times x^3\dot x$.

To find the fluent of $(a + cx)^{\frac{3}{4}}x^2\dot x$.

To find the fluent of $(a + cx^2)^{\frac{1}{3}} \times dx^3\dot x$.

To find the fluent of $\frac{cz\dot z}{\sqrt{a + z}}$ or $(a + z)^{-\frac{1}{2}}cz\dot z$.

To find the fluent of $\frac{cz^{3n-1}\dot z}{\sqrt{a + z^n}}$ or $(a + z^n)^{-\frac{1}{2}}cz^{3n-1}\dot z$.

To find the fluent of $\frac{\dot z\sqrt{a^2 - z^2}}{z^6}$ or $(a^2 + z^2)^{\frac{1}{2}}z^{-6}\dot z$.

To find the fluent of $\frac{\dot x\sqrt{a - x^n}}{x^{\frac{7}{2}n+1}}$ or $(a - x^n)^{\frac{1}{2}}x^{-\frac{7}{2}n-1}\dot x$.

43. ***When there are several terms, involving two or more variable quantities, having the fluxion of each multiplied by the other quantity or quantities.***

Take the fluent of each term, as if there were only one variable quantity in it,

namely that whose fluxion is contained in it, supposing all the others to be constant in that term; then, if the fluents of all the terms, so found, be the very same quantity in all of them, that quantity will be the fluent of the whole. Which is the reverse of the 5th rule for finding fluxions: Thus, if the given fluxion be $\dot{x}y + x\dot{y}$, then the fluent of $\dot{x}y$ is xy, supposing y constant: and the fluent of $x\dot{y}$ is also xy, supposing x constant: therefore xy is the required fluent of the given fluxion $\dot{x}y + x\dot{y}$.

In like manner,

The fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is xyz; and the fluent of $2xy\dot{x} + x^2\dot{y}$ is x^2y.

Find the fluents of $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2 + 2x^{\frac{1}{2}}y\dot{y}$, and of $\frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$.

The fluent of $\frac{2ax\dot{x}y^{\frac{1}{2}} - \frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$ or $\frac{2ax\dot{x}}{\sqrt{y}} - \frac{ax^2\dot{y}}{2y\sqrt{y}}$ is what?

44. *When the given fluxional expression is in this form* $\frac{\dot{x}y - x\dot{y}}{y^2}$, *namely, a fraction, including two quantities, being the fluxion of the former of them drawn into the latter, minus the fluxion of the latter drawn into the former, and divided by the square of the latter.*

Then, the fluent is the fraction $\frac{x}{y}$, or the former quantity divided by the latter, by the reverse of rule 4, of finding fluxions. That is,

The fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ is $\frac{x}{y}$. And, in like manner,

The fluent of $\frac{2x\dot{x}y^2 - 2x^2y\dot{y}}{y^4}$ is $\frac{x^2}{y^2}$.

Though, indeed, the examples of this case may be performed by the foregoing one. Thus, the given fluxion $\frac{\dot{x}y - x\dot{y}}{y^2}$ reduces to $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$, or $\frac{\dot{x}}{y} - x\dot{y}y^{-2}$; of which,

The fluent of $\frac{\dot{x}}{y}$ is $\frac{x}{y}$ supposing y constant; and

The fluent of $-x\dot{y}y^{-2}$ is also xy^{-1} or $\frac{x}{y}$, when x is constant; therefore, by that case, $\frac{x}{y}$ is the fluent of the whole $\frac{\dot{x}y - x\dot{y}}{y^2}$.

45. *When the fluxion of a quantity is divided by the quantity itself:*

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the same thing, the fluent is equal to 2·30258509 multiplied by the common logarithm of the same quantity, by rule 6, for finding fluxions.

So the fluent of $\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$, is the hyp. log. of x.

The fluent of $\frac{2\dot{x}}{x}$ is $2 \times$ hyp. log. of x, or $=$ hyp. log. x^2.

The fluent of $\frac{a\dot{x}}{x}$, is $a \times$ hyp. log. x, or $=$ hyp. log. of x^a.

The fluent of $\frac{\dot{x}}{a + x}$, is

The fluent of $\frac{3x^2\dot{x}}{a+x^3}$, is

46. *To find fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following table are contained the most usual forms of fluxions that occur in the practical solution of problems, with their corresponding fluents set opposite to them; by means of which, namely, by comparing any proposed fluxion with the corresponding form in the table, the fluent of it will be found.

Many of these fluents may be verified by taking the fluxion of them again, which should, if they be correctly determined, be identical with the given fluxion. A considerable number of these are so taken, by having previously determined the fluxion of the given answer to be that proposed for integration, and thence inferring that, conversely, the fluents are those given in the table. Such fluents are often called "elementary fluents." Such, for instance, are those in the first seven cases in the table; and again in most of those involving circular functions. Several of them, however, are derived from the elementary fluents by different processes, which will be hereafter explained, under the head of "Finding Fluents," at the end of this treatise.

FORMS.	FLUXIONS.	FLUENTS.
1	$x^{n-1}\dot{x}$	$\frac{1}{n}x^n$
2	$(a \pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm \frac{1}{mn}(a \pm x^n)^m$
3	$\frac{x^{mn-1}\dot{x}}{(a \pm x^n)^{m+1}}$	$\frac{1}{mna} \times \frac{x^{mn}}{(a \pm x^n)^m}$
4	$\frac{(a \pm x^n)^{m-1}\dot{x}}{x^{mn+1}}$	$\frac{-1}{mna} \times \frac{(a \pm x^n)^m}{x^{mn}}$
5	$(my\dot{x} + nx\dot{y}) \times x^{m-1}y^{n-1}$, or $\left(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y}\right)x^m y^n$	$x^m y^n$
6	$mx^{m-1}\dot{x}y^n z^r + nx^m y^{n-1}\dot{y}z^r + rx^m y^n z^{r-1}\dot{z}$, or $(m\dot{x}yz + nx\dot{y}z + rxy\dot{z}(x^{m-1}y^{n-1}z^{r-1}$, or $\left(\frac{m\dot{x}}{x} + \frac{ny}{y} + \frac{r\dot{z}}{z}\right)x^m y^n z^r$	$x^m y^n z^r$
7	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. x.
8	$\frac{x^{n-1}\dot{x}}{a \pm x^n}$	$\pm \frac{1}{n}$ log. $(a \pm x^n)$
9	$\frac{x^{-1}\dot{x}}{a \pm x^n}$	$\frac{1}{na}$ log. $\frac{x^n}{a \pm x^n}$

FORMS.	FLUXIONS.	FLUENTS.
10	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a-x^n}$	$\frac{1}{n\sqrt{a}}\log.\frac{\sqrt{a}+\sqrt{x^n}}{\sqrt{a}-\sqrt{x^n}}$
11	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+x^n}$	$\frac{2}{n\sqrt{a}}\tan.^{-1}\sqrt{\frac{x^n*}{a}}$, or $\frac{1}{n\sqrt{a}}\cos.^{-1}\frac{a-x^n}{a+x^n}$
12	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(\pm a+x^n)}}$	$\frac{2}{n}\log.\{\sqrt{x^n}+\sqrt{(\pm a+x^n)}\}$
13	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$	$\frac{2}{n}\sin.^{-1}\sqrt{\frac{x^n}{a}}=\frac{1}{n}\text{vers.}^{-1}\frac{2x^n}{a}$
14	$\frac{x^{-1}\dot{x}}{\sqrt{(a\pm x^n)}}$	$\frac{1}{n\sqrt{a}}\log.\frac{\sqrt{a}-\sqrt{(a\pm x^n)}}{\sqrt{a}+\sqrt{(a\pm x^n)}}$
15	$\frac{x^{-1}\dot{x}}{\sqrt{-a+x^n}}$	$\frac{2}{n\sqrt{a}}\sec.^{-1}\sqrt{\frac{x^n}{a}}=\frac{1}{n\sqrt{a}}\cos.^{-1}\frac{2a-x^n}{x^n}$
16	$\dot{x}\sqrt{dx-x^2}$	$\frac{1}{2}$circ. seg. to diam. d and vers. x
17	$2\dot{x}\sqrt{(a^2-x^2)}$	spher. zone, rad. a, and height from centre x.
18	$c^{nx}\dot{x}$	$\frac{c^{nx}}{n\log. c.}$
19	$\dot{x}y^x\log.y+xy^{x-1}\dot{y}$	y^x
20	$x^{\frac{1}{2}}\dot{x}\sqrt{(bx\pm a)}$	$\left\{+\frac{x^{\frac{1}{2}}(2bx\pm a)\sqrt{(bx\pm a)}}{4b}-\frac{a^2}{4b\sqrt{b}}+\log.\{\sqrt{bx}+\sqrt{(bx\pm a)}\}\right.$
21	$x^{\frac{1}{2}}\dot{x}\sqrt{(a-bx)}$	$+\frac{x^{\frac{1}{2}}(2bx-a)\sqrt{(a-bx)}}{4b}+\frac{a^2}{4b\sqrt{b}}\tan.^{-1}\sqrt{\frac{bx}{a-bx}}$
22	$\frac{\dot{x}\sqrt{(bx\pm a)}}{x^{\frac{1}{2}}}$	$+x^{\frac{1}{2}}\sqrt{(bx\pm a)}\pm\frac{a}{\sqrt{b}}\log.\{\sqrt{bx}+\sqrt{(bx\pm a)}\}$
23	$\frac{\dot{x}\sqrt{(a-bx)}}{x^{\frac{1}{2}}}$	$+x^{\frac{1}{2}}\sqrt{(a-bx}+\frac{a}{\sqrt{b}}\tan.^{-1}\sqrt{\frac{bx}{a-bx}}$

...ined at p. 41 of the present volume. We adopt it here (although we think it somewhat objectionable) because it is getting into universal use.

FORMS.	FLUXIONS.	FLUENTS.
24	$\frac{\dot{x}\sqrt{(a \pm bx)}}{x}$	$2\sqrt{(a \pm bx)} - 2a^{\frac{1}{2}} \log. \frac{\sqrt{a} + \sqrt{(a \pm bx)}}{\sqrt{x}}$
25	$\frac{\dot{x}\sqrt{(bx - a)}}{x}$	$2\sqrt{(bx - a)} - 2a^{\frac{1}{2}} \tan.^{-1} \sqrt{\frac{bx - a}{a}}$
26	$\frac{\dot{x}\sqrt{(a \pm bx^2)}}{x}$	$+\sqrt{(a \pm bx^2)} - a^{\frac{1}{2}} \log. \frac{\sqrt{a} + \sqrt{(a \pm bx^2)}}{x}.$
27	$\frac{\dot{x}\sqrt{(bx^2 - a)}}{x}$	$+\sqrt{(bx^2 - a)} - a^{\frac{1}{2}} \tan.^{-1} \sqrt{\frac{bx^2 - a}{a}}.$
28	$\frac{\dot{x}}{a + bx + cx^2}$	$\frac{2}{\sqrt{(4ac - b^2)}} \tan.^{-1} \frac{b + 2cx}{\sqrt{(4ac - b^2)}}.$
29	$\frac{\dot{x}}{a + bx - cx^2}$	$\frac{2}{\sqrt{(4ac + b^2)}} \log. \frac{\sqrt{(4ac + b^2)} - (b - 2cx)}{\sqrt{(a + bx - cx^2)}}.$
30	$\frac{\dot{x}}{x(a + bx + cx^2}$	$-\frac{1}{a} \log. \frac{\sqrt{(a + bx + cx^2)}}{x}.$ $-\frac{b}{a\sqrt{(4ac - b^2}} \tan.^{-1} \frac{b + 2cx}{\sqrt{(4ac - b^2)}}.$
31	$\frac{x}{x(a + bx - cx^2)}$	$-\frac{1}{a} \log. \frac{\sqrt{a + bx - cx^2)}}{x}.$ $-\frac{b}{a\sqrt{(4ac + b^2)}} \log. \frac{\sqrt{(4ac + b^2)} - (b - 2cx)}{\sqrt{(a + bx - cx^2)}}.$
32	$\frac{x\dot{x}}{a + bx + cx^2}$	$+\frac{1}{2c} \log. (a + bx + cx^2)$ $-\frac{b}{c\sqrt{(4ac - b^2)}} \tan.^{-1} \frac{b + 2cx}{\sqrt{(4ab - b^2)}}.$
33	$\frac{x\dot{x}}{a + bx - cx^2}$	$-\frac{1}{2c} \times \log. (a + bx - cx^2)$ $+\frac{b}{c\sqrt{(4ac + b^2)}} \log. \frac{\sqrt{(4ac + b^2)} - (b - 2cx)}{\sqrt{(a + bx - cx^2)}}.$
34	$\dot{x}\sqrt{(a + bx + cx^2)}$	$+\frac{(2cx + b)\sqrt{(a + bx + cx^2)}}{4c} + \frac{4ac - b^2}{8c\sqrt{c}} \times$ $\log. \left\{2cx + b + 2c^{\frac{1}{2}}\sqrt{(a + bx + cx^2)}\right\}$
35	$\dot{x}\sqrt{(a + bx - cx^2)}$	$+\frac{(2cx - b)\sqrt{(a + bx - cx^2)}}{4} + \frac{4ac + b^2}{8c\sqrt{c}} \times$ $\tan.^{-1} \frac{2cx - b}{2c^{\frac{1}{2}}\sqrt{(a + bx - cx^2)}}$

FORMS.	FLUXIONS.	FLUENTS.
36	$\frac{(A+Bx)\dot{x}}{a+bx+cx^2}$	$\begin{cases} +\frac{B}{2c}\log.\,(a+bx+cx^2) \\ +\frac{2cA-bB}{c\sqrt{(4ac-b^2)}}\tan.^{-1}\frac{b+2cx}{\sqrt{(4ac-b^2)}}. \end{cases}$
37	$\frac{(A+Bx)\dot{x}}{a+bx-cx^2}$	$\begin{cases} -\frac{B}{2c}\log.\,(a+bx-cx^2) \\ +\frac{2cA+bB}{c\sqrt{(4ac+b^2)}}\log.\,\frac{\sqrt{(4ac+b^2)}-(b-2cx)}{\sqrt{(a+bx-cx^2)}}. \end{cases}$
38	$\frac{\dot{x}}{\sqrt{(a+bx+cx^2)}}$	$+\frac{1}{\sqrt{c}}\times\log.\left\{2cx+b+2c^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}.\right\}$
39	$\frac{\dot{x}}{\sqrt{(a+bx-cx^2)}}$	$\frac{1}{\sqrt{c}}\tan.^{-1}\frac{2cx-b}{2c^{\frac{1}{2}}\sqrt{(a+bx-cx)}}.$
40	$\frac{\dot{x}}{x\sqrt{(a+bx+cx^2)}}$	$-\frac{1}{\sqrt{a}}\log.\,\frac{2a+bx+2a^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}}{x}.$
41	$\frac{x}{x\sqrt{(-a+bx+cx^2)}}$	$+\frac{1}{\sqrt{a}}\tan.^{-1}\frac{2a^{\frac{1}{2}}\sqrt{(-a+bx+cx^2)}}{2a-bx}.$

Note.—The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2·302585092994. And the arcs, whose sine, or tangent, &c. are mentioned, have the radius 1, and are those in the common tables of sines, tangents, and secants. Also the numbers m, n, &c., are to be some real quantities, as the forms fail when $m=0$, or $n=0$, &c.

The use of the foregoing Table of Forms of Fluxions and Fluents.

47. In using the foregoing table, it is to be observed, that the first column serves only to show the number of the form; in the second column are the several forms of fluxions, which are of different kinds or classes; and in the third or last column, are the corresponding fluents.

The method of using the table is this. Having any fluxion given, to find its fluent: First, compare the given fluxion with the several forms of fluxions in the second column of the table, till one of the forms be found that agrees with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree or correspond, will give the particular values of the general quantities in the tabular form; then substitute these particular values in the general or tabular form of the fluent, and the result will be the particular fluent of the given fluxion, after it is multiplied by any co-efficient the proposed fluxion may have.

EXAMPLES.

1. To find the fluent of the fluxion $3x^{\frac{5}{3}}\dot{x}$.

This is found to agree with the first form. And, by comparing the fluxions, it appears that $x = x$, and $n - 1 = \frac{5}{3}$, or $n = \frac{8}{3}$; which being substituted in the tabular fluent, or $\frac{1}{n}x^n$, gives, after multiplying by 3 the co-efficient, $3 \times \frac{3}{8}x^{\frac{8}{3}}$, or $\frac{9}{8}x^{\frac{8}{3}}$, for the fluent sought.

2. To find the fluent of $5x^2\dot{x}\sqrt{c^3 - x^3}$, or $5x^2\dot{x}(c^3 - x^3)^{\frac{1}{2}}$.

This fluxion, it appears, belongs to the second tabular form: for $a = c^3$, and $-x^n = -x^3$, and $n = 3$ under the vinculum, also $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the exponent ${}^{n-1}$ of x^{n-1} without the vinculum, by using 3 for n, is $n - 1 = 2$, which agrees with x^2 in the given fluxion: so that all the parts of the form are found to correspond. Then substituting these values into the general fluent, $-\frac{1}{mn}(a - x^n)^m$, it becomes $-\frac{5}{3} \times \frac{2}{3}(c^3 - x^3)^{\frac{3}{2}} = -\frac{10}{9}(c^3 - x^3)^{\frac{3}{2}}$.

3. To find the fluent of $\frac{x^2\dot{x}}{1+x^3}$.

This is found to agree with the 8th form; where .... $\pm x^n = + x^3$ in the denominator, or $n = 3$; and the numerator x^{n-1} then becomes x^2, which agrees with the numerator in the given fluxion; also $a = 1$. Hence then, by substituting in the general or tabular fluent, $\frac{1}{n}$ log. of $a+x^n$, it becomes $\frac{1}{3}$ log. $1+x^3$.

4. To find the fluent of $ax^4\dot{x}$.

5. To find the fluent of $2(10+x^2)^{\frac{2}{3}}x\dot{x}$.

6. To find the fluent of $\frac{a\dot{x}}{(c^2 + x^2)^{\frac{3}{2}}}$.

7. To find the fluent of $\frac{3x^2\dot{x}}{(a - x)^4}$.

8. To find the fluent of $\frac{c^2 - x^2}{x^5}\dot{x}$.

9. To find the fluent of $\frac{1 + 3x}{2x^4}\dot{x}$.

10. To find the fluent of $\left(\frac{3\dot{x}}{x} + \frac{2\dot{y}}{y}\right)x^3y^2$.

11. To find the fluent of $\left(\frac{\dot{x}}{x} + \frac{\dot{y}}{3y}\right)xy^{\frac{1}{3}}$.

12. To find the fluent of $\frac{3\dot{x}}{ax}$ or $\frac{3}{a}x^{-1}\dot{x}$.

13. To find the fluent of $\frac{a\dot{x}}{3 - 2x}$.

14. To find the fluent of $\frac{3\dot{x}}{2x - x^2}$ or $\frac{3x^{-1}\dot{x}}{2 - x}$.

15. To find the fluent of $\frac{2\dot{x}}{x - 3x^3}$ or $\frac{2x^{-1}\dot{x}}{1 - 3x^2}$

16. To find the fluent of $\frac{3x\dot{x}}{1 - x^4}$.

17. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2 - x^{\frac{5}{2}}}$.

18. To find the fluent of $\frac{2x\dot{x}}{1+x^4}$.

19. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2+x^5}$.

20. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1+x^4}}$.

21. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^2-4}}$.

22. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1-x^4}}$.

23. To find the fluent of $\frac{a\dot{x}}{\sqrt{4-x^2}}$.

24. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{1-x^2}}$.

25. To find the fluent of $\frac{a\dot{x}}{\sqrt{ax^2+x^{\frac{1}{2}}}}$.

26. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{x^2-1}}$.

27. To find the fluent of $\frac{a\dot{x}}{\sqrt{(x^{\frac{1}{2}}-ax)^2}}$.

28. To find the fluent of $2\dot{x}\sqrt{2x-x^2}$.

29. To find the fluent of $a^x\dot{x}$.

30. To find the fluent of $3a^{3x}\dot{x}$.

31. To find the fluent of $3z^x\dot{x}\log. z + 3xz^{x-1}\dot{z}$.

32. To find the fluent of $(1+x^3)x\dot{x}$.

33. To find the fluent of $(2+x^4)\,x^{\frac{3}{2}}\dot{x}$.

34. To find the fluent of $x^2\dot{x}\sqrt{(a^2+x^2)}$.

48. *To find Fluents by the Method of Continuation.*

There are certain methods of finding fluents one from another, or of deducing the fluent of a proposed fluxion from another fluent previously known or found. There are hardly any general rules, however, that will suit all cases; but they mostly consist in assuming some quantity y in the form of a rectangle or product of two factors, which are such, that the one of them drawn into the fluxion of the other may be of the form of the proposed fluxion; then taking the fluxion of the assumed rectangle, there will thence be deduced a value of the proposed fluxion in terms that will often admit of finite fluents. The manner in such cases will better appear from the following examples.

Ex. 1. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Here it is obvious that if y be assumed $= x\sqrt{(x^2+a^2)}$, then one part of the fluxion of this product, viz. $x \times$ flux. of $\sqrt{(x^2+a^2)}$, will be of the same form as the fluxion proposed. Putting therefore the assumed rectangle $y = x\sqrt{(x^2+a^2)}$ into fluxions, it is $\dot{y} = \dot{x}\sqrt{(x^2+a^2)} + \frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. But as the former part, viz. $\dot{x}\sqrt{(x^2+a^2)}$, does not agree with any of our preceding forms, which have been

integrated, multiply it by $\sqrt{(x^2+a^2)}$, and subscribe the same as a denominator to the product, by which that part becomes $\frac{x^2+a^2}{\sqrt{(x^2+a^2)}}\dot{x} = \frac{x^2\dot{x}+a^2\dot{x}}{\sqrt{(x^2+a^2}}$; this united with the former part, makes the whole $\dot{y} = \frac{2x^2\dot{x}}{\sqrt{(x^2+a^2)}} + \frac{a^2\dot{x}}{\sqrt{(x^2+a^2)}}$; hence the given fluxion $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{2}\dot{y} - \frac{1}{2}a^2 \times \frac{\dot{x}}{\sqrt{(x^2+a^2)}}$, and its fluent is therefore $\frac{1}{2}y - \frac{1}{2}a^2 \times f\frac{\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{2}x\sqrt{(x^2+a^2)} - \frac{1}{2}a^2 \times$ hyp. log. of $x + \sqrt{(x^2+a^2)}$, by the 12th form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2-a^2)}}$ will be found from that of $\frac{\dot{x}}{\sqrt{(x^2-a^2)}}$ by the same 12th form, and is $= \frac{1}{2}x\sqrt{(x^2-a^2)} + \frac{1}{2}a^2 \times$ hyp. log. $x + \sqrt{(x^2-a^2)}$.

Ex. 3. Also in a similar manner, by the 13th form, the fluent of $\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$ will be found from that of $\frac{\dot{x}}{\sqrt{(a^2-x^2)}}$, and comes out $-\frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a \times$ cir. arc to radius a and sine x.

Ex. 4. In like manner the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found from that of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. Here it is manifest that y must be assumed $= x^3\sqrt{(x^2+a^2)}$, in order that one part of its fluxion, viz. $\dot{x} \times$ flux. of $\sqrt{(x^2+a^2)}$ may agree with the proposed fluxion. Thus, by taking the fluxion, and reducing as before, the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found $= \frac{1}{4}x^3\sqrt{(x^2+a^2)} - \frac{3}{4}a^2 \times f\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Ex. 5. Thus also the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2-a^2)}}$ is $\frac{1}{4}x^3\sqrt{(x^2-a^2)} + \frac{3}{4}a^2 \times f\frac{x^2\dot{x}}{\sqrt{(x^2-a^2)}}$.

Ex. 6. And the $f\frac{x^4\dot{x}}{\sqrt{(a^2-x)}}$, is $-\frac{1}{4}x^3\sqrt{(a^2-x^2)} + \frac{3}{4}a^2 \times f\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$.

In like manner the student may find the fluents of $\frac{x^6\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, $\frac{x^8\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, &c. to $\frac{x^n\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, where n is any even number, each from the fluent of that which immediately precedes it in the series, by substituting for y as before. Thus the fluent of $\frac{x^n\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{n}x^{n-1}\sqrt{(x^2+a^2)} - \frac{n-1}{n}a^2 \times f\frac{x^{n-2}\dot{x}}{\sqrt{(x^2+a^2)}}$.

49. In like manner we may proceed for the series of similar expressions where the index of the power of x in the numerator is some odd number.

Ex. 1. To find the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$. Here assuming $y = x^2\sqrt{(x^2+a^2)}$, and taking the fluxion, one part of it will be similar to the fluxion proposed. Thus $\dot{y} = 2x\dot{x}\sqrt{(x^2+a^2)} + \frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$; hence at once the given fluxion $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}} = \dot{y} - 2x\dot{x}\sqrt{(x^2+a^2)}$; therefore the required fluent is $y - f.\,2x\dot{x}\sqrt{(x^2+a^2)} = x^2\sqrt{(x^2+a^2)} - \frac{2}{3}(x^2+a^2)^{\frac{3}{2}}$, by the 2d form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, is $x^2\sqrt{(x^2-a^2)} - \frac{2}{3}(x^2-a^2)^{\frac{3}{2}}$.

Ex. 3. And the fluent of $\frac{x^3\dot{x}}{\sqrt{(a^2-x^2)}}$, $= -x^2\sqrt{(a^2-x^2)} - \frac{2}{3}(a^2-x^2)^{\frac{3}{2}}$.

Ex. 4. To find the fluent of $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}}$, from that of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$.

Here it is manifest we must assume $y = x^4\sqrt{(x^2+a^2)}$. This in fluxions and reduced gives

$\dot{y} = \frac{5x^5\dot{x}}{\sqrt{(x^2+a^2)}} + \frac{4a^2x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, and hence $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}\dot{y} - \frac{4a^2}{5} \frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$;

and the fluent is $\frac{1}{5}y - \frac{4}{5}a^2 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}x^4\sqrt{(x^2+a^2)} - \frac{4}{5}a^2 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, the fluent of the latter part being the same as in ex. 1, above.

In like manner the student may find the fluents of $\frac{x^5\dot{x}}{\sqrt{(x^2-a^2)}}$ and $\frac{x^5\dot{x}}{\sqrt{(a^2-x)^2}}$. He may then proceed in a similar way for the fluents of $\frac{x^7\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, $\frac{x^9\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, &c. $\frac{x^n\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, where n is any odd number, viz., always by means of the fluent of each preceding term in the series.

50. A similar process may be employed for the fluents of the series of fluxions,

$$\frac{\dot{x}}{\sqrt{(a \pm x)}},\ \frac{x\dot{x}}{\sqrt{(a \pm x)}},\ \frac{x^2\dot{x}}{\sqrt{(a \pm x)}},\ \&c. \ldots \frac{x^n\dot{x}}{\sqrt{(a \pm x)}},$$

using the fluent of each preceding term in the series, as a part of the next term, and knowing that the fluent of the first term $\frac{\dot{x}}{\sqrt{a \pm x}}$ is given, by the 2d form of fluents, $= 2\sqrt{(a \pm x)}$, of the same sign as x.

Ex. 1. To find the fluent of $\frac{x\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{\dot{x}}{\sqrt{(x+a)}} = 2\sqrt{(x+a)} = \text{A}$ suppose. Here it is evident we must assume $y = x\sqrt{(x+a)}$, for then its fluxion $\dot{y} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \dot{x}\sqrt{(x+a)} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \frac{x\dot{x}}{\sqrt{(x+a)}} + \frac{a\dot{x}}{\sqrt{(x+a)}} = \frac{\frac{3}{2}x\dot{x}}{\sqrt{(x+a)}} + a\dot{\text{A}}$; hence $\frac{x\dot{x}}{\sqrt{(x+a)}} = \frac{2}{3}\dot{y} - \frac{2}{3}a\dot{\text{A}}$; and the required fluent is $\frac{2}{3}y - \frac{2}{3}a\text{A} = \frac{2}{3}x\sqrt{(x+a)} - \frac{4}{3}a\sqrt{(x+a)} = (x-2a) \times \frac{2}{3}\sqrt{(x+a)}$.

In like manner the student will find the fluents of $\frac{x\dot{x}}{\sqrt{(x-a)}}$ and $\frac{x\dot{x}}{\sqrt{(a-x)}}$.

Ex. 2. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{x\dot{x}}{\sqrt{(x+a)}} = \dot{\text{B}}$. Here y must be assumed $= x^2\sqrt{(x+a)}$; for then taking the fluent and reducing, there is found $\frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}\dot{y} - \frac{4}{5}a\dot{\text{B}}$; therefore $f\frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}y - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a(x-2a) \times \frac{2}{3}\sqrt{(x+a)} = (9x^2 - 4ax + 8a^2) \times \frac{2}{15}\sqrt{(x+a)}$.

In the same manner the student will find the fluents of $\frac{x^2\dot{x}}{\sqrt{(x-a)}}$ and of $\frac{x^2\dot{x}}{\sqrt{(a-x)}}$. And in general, the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x+a)}}$ being given $=$ C, he will find the fluent of $\frac{x^n\dot{x}}{\sqrt{(x+a)}} = \frac{2}{2n+1}x^n\sqrt{(x+a)} - \frac{2n}{2u+1}a\text{C}$.

51. In a similar way we might proceed to [illegible] a denominator fluxions by means of other fluents in the table of form[illegible] such as $x\dot{x}\sqrt{(dx-x^2)}$, $x^2\dot{x}\sqrt{(dx-x^2)}$, $x^3\dot{x}\sqrt{(dx-x^2)}$, [illegible] this the fluent of $\dot{x}\sqrt{(dx-x^2)}$, the fluent of which, by the 16th tabu[illegible] the circular semisegment, to diameter d and versed sine x, or the half or trilineal segment contained by an arc with its right sine and versed sine, the diameter being d.

Ex. 1. Putting then said semiseg. or fluent of $\dot{x}\sqrt{(dx-x^2)} = A$, to find the fluent of $x\dot{x}\sqrt{(dx-x^2)}$. Here assuming $y = (dx-x^2)^{\frac{3}{2}}$, and taking the fluxions, they are, $\dot{y} = \frac{3}{2}(d\dot{x} - 2x\dot{x})\sqrt{(dx-x^2)}$; hence $x\dot{x}\sqrt{(dx-x^2)} = \frac{1}{2}d\dot{x}\sqrt{(dx-x^2} - \frac{1}{3}\dot{y} = \frac{1}{2}d\dot{A} - \frac{1}{3}\dot{y}$; therefore the required fluent, $fx\dot{x}\sqrt{(dx-x^2)}$, is $\frac{1}{2}dA - \frac{1}{3}y = \frac{1}{2}dA - \frac{1}{3}(dx-x^2)^{\frac{3}{2}} = B$ suppose.

Ex. 2. To find the fluent of $x^2\dot{x}\sqrt{(dx-x^2)}$, having that of $x\dot{x}\sqrt{(dx-x^2)}$ given $= B$. Here assuming $y = x(dx-x^2)$, then taking the fluxions, and reducing, there results $\dot{y} = (\frac{5}{2}dx\dot{x} - 4x^2\dot{x})\sqrt{(dx-x^2)}$; hence $x^2\dot{x}\sqrt{(dx-x^2)} = \frac{5}{8}dx\dot{x}\sqrt{(dx-x^2)} - \frac{1}{4}\dot{y} = \frac{5}{8}d\dot{B} - \frac{1}{4}\dot{y}$, the flu. therefore of $x^2\dot{x}\sqrt{(dx-x^2)}$ is $\frac{5}{8}dB - \frac{1}{4}y = \frac{5}{8}dB - \frac{1}{4}x\,dx - x^2)^{\frac{3}{2}}$.

Ex. 3. In the same manner the series may be continued to any extent; so that in general, the fluent of $x^{n-1}\sqrt{(dx-x^2)}$ being given $= C$, then the next, or the fluent of $x^n\dot{x}\sqrt{(dx-x^2)}$ will be $\frac{2n+1}{n+2} - \frac{1}{2}dC - \frac{1}{n+2}x^{n-1}(dx-x2)^{\frac{3}{2}}$.

52. To find the fluent of such expressions as $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax),}}$ a case not included in the table of forms already given.

Put the proposed radical $\sqrt{(x^2 \pm 2ax)} = z$, or $x^2 \pm 2ax = z^2$; then, completing the square, $x^2 \pm 2ax + a^2 = z^2 + a^2$, and the root is $x \pm a = \sqrt{(z^2+a^2)}$. The fluxion of this is $\dot{x} = \frac{z\dot{z}}{\sqrt{(z^2+a^2)}}$; therefore $\frac{\dot{x}}{\sqrt{(x^2 \pm 2ax)}} = \frac{\dot{z}}{\sqrt{(z^2+a^2)}}$; the fluent of which, by the 12th form, is the hyp. log. of $z + \sqrt{(z^2+a^2)}$ = hyp. log. of $x \pm a + \sqrt{(x^2 \pm 2ax)}$, the fluent required.

Ex. 2. To find now the fluent of $\frac{x\dot{x}}{\sqrt{(x^2+2ax)}}$, having given, by the above example, the fluent of $\frac{\dot{x}}{\sqrt{(x^2+2ax)}} = A$ suppose. Assume $\sqrt{(x^2+2ax)} = y$; then its fluxion is $\frac{x\dot{x}+a\dot{x}}{\sqrt{x^2+2ax}} = \dot{y}$; therefore $\frac{x\dot{x}}{\sqrt{(x^2+2ax)}} = \dot{y} - \frac{\dot{x}}{\sqrt{(x^2+2ax)}} = \dot{y} - a\dot{A}$; the fluent of which is $y - aA = \sqrt{(x^2+2ax)} - aA$, the fluent sought.

Ex. 3. Thus also, this fluent of $\frac{x\dot{x}}{\sqrt{(x^2+2ax)}}$ being given, the fluent of the next in the series, or $\frac{x^2\dot{x}}{\sqrt{(x^2+2ax)}}$ will be found, by assuming $x\sqrt{(x^2+2ax)} = y$; and so on for any other of the same form. As, if the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x^2+2ax)}}$ be given $= C$; then, by assuming $x^{n-1}\sqrt{(x^2+2ax)} = y$, the fluent of $\frac{x^n\dot{x}}{\sqrt{(x^2+2ax)}} = \frac{1}{n}x^{n-1}\sqrt{(x^2+2ax)} - \frac{2n-1}{n}aC$.

Ex. 4. In like manner, the fluent of $\frac{\dot{x}}{\sqrt{(x^2 - 2ax)}}$ being given, as in the first example, that of $\frac{x\dot{x}}{\sqrt{(x^2 - 2ax)}}$ may be found; and thus the series may be continued exactly as in the 3d example, only taking $-2ax$ for $+2ax$.

53. Again, having given the fluent of $\frac{\dot{x}}{\sqrt{(2ax - x^2}}$, which, by the table is $\frac{1}{a} \times$ circular arc to radius a and versed sine x, the fluents of $\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}$, $\frac{x^2\dot{x}}{\sqrt{(2ax - x^2}}$, &c. $\frac{x^n\dot{x}}{\sqrt{(2ax - x^2)}}$, may be assigned by the same method of continuation. Thus,

Ex. 1. For the fluent of $\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}$, assume $\sqrt{(2ax - x^2)} = y$; the required fluent will be found $= -\sqrt{(2ax - x^2)} + \text{A}$, or arc to radius a and vers. x.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(2ax - a^2)}}$ is $f\,\frac{\frac{3}{2}ax\dot{x}}{\sqrt{(2ax - x^2}} - \frac{1}{2}x \times \sqrt{(2ax - x^2)} = \frac{3}{2}a\text{A} - \frac{3a + x}{2}\sqrt{(2ax - x^2)}$, where A denotes the arc mentioned in the last example.

Ex. 3. And in general the fluent of $\frac{x^n\dot{x}}{\sqrt{(2ax - x^2)}}$ is $\frac{2n - 1}{n}\,a\text{C} - \frac{1}{n}\,x^{n-1} \times \sqrt{(2ax - x^2)}$, where C is the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(2ax - x^2)}}$, the next preceding term in the series.

54. Thus, also, the fluent of $\dot{x}\sqrt{(x - a)}$ being given, $= \frac{2}{3}(x - a)^{\frac{3}{2}}$, by the 2d form, the fluents of $x\dot{x}\sqrt{(x - a)}$, $x^2\dot{x}\sqrt{(x - a)}$, &c. . . $x^n\dot{x}\sqrt{(x - a)}$, may be found. And in general, if the fluent of $x^{n-1}\dot{x}\sqrt{(x - a)} = \text{C}$ be given; then by assuming $x^n(x - a)^{\frac{3}{2}} = \dot{y}$, the fluent of $x^n\dot{x}\sqrt{(x - a)}$ is found $= \frac{2}{2n + 3}\,x^n(x - a)^{\frac{3}{2}} + \frac{2na}{2n + 3}\,\text{C}$.

55. Also, given the fluent of $(x - a)^m\dot{x}$, which is $\frac{1}{m + 1}(x - a)^{m+1}$ by the 2d form, the fluents of the series $(x - a)^m x\dot{x}$, $(x - a)^m x^2\dot{x}$, &c. . . . $(x - a)^m x^n\dot{x}$ can be found. And in general, the fluent of $(x - a)^m x^{n-1}\dot{x}$ being given $= \text{C}$; then by assuming $(x - a)^{m+1}x^n = y$, the fluent of $(x - a)^m x^n\dot{x}$ is found $= \ldots\ldots$ $\frac{x^n(x - a^{m+1} + na\text{C}}{m + n + 1}$.

Also, by the same way of continuation, the fluents of $x^n\dot{x}\sqrt{(a \pm x)}$ and of $x^n\dot{x}\,(a \pm x)^m$ may be found.

56. When the fluxional expression contains a trinomial quantity, as $\sqrt{(b + cx + x^2)}$, this may be reduced to a binomial, by substituting another letter for the unknown one x, connected with half the coefficient of the middle term with its sign. Thus, put $z = x + \frac{1}{2}c$: then $z^2 = x^2 + cx + \frac{1}{4}c^2$; therefore $z^2 - \frac{1}{4}c^2 = x^2 + cx$, and $z^2 + b - \frac{1}{4}c^2 = x^2 + cx + b$ the given trinomial; which is $= z^2 + a^2$, by putting $a^2 = b - \frac{1}{4}c^2$.

Ex. 1. To find the fluent of $\frac{3\dot{x}}{\sqrt{(5 + 4x + x^2)}}$.

Here $z = x + 2$; then $z^2 = x^2 + 4x + 4$, and $z^2 + 1 = 5 + 4x + x^2$, also $\dot{x} = \dot{z}$; theref. the proposed fluxion reduces to $\frac{3\dot{z}}{\sqrt{(1 + z^2)}}$; the fluent of which, by the 12th form in the table is 3 hyp. log. of $z + \sqrt{(1 + z)} = 3$ hyp. log. $x + 2 + \sqrt{(5 + 4x + x^2)}$.

Ex. 2. To find the fluent of $\dot{x}\sqrt{(b + cx + dx^2)} = \dot{x}\sqrt{d} \times \sqrt{\left(\frac{b}{d} + \frac{c}{d}x + x^2\right)}$.

Here assuming $x + \frac{c}{2d} = z$; then $\dot{x} = \dot{z}$, and the proposed fluxion reduces to $\dot{z}\sqrt{d} \times \sqrt{\left(z^2 + \frac{b}{d} - \frac{c^2}{4d^2}\right)} = \dot{z}\sqrt{d} \times \sqrt{(z^2 + a^2)}$, putting a^2 for $\frac{b}{d} - \frac{c^2}{4d^2}$; and the fluent will be found by a similar process to that employed in example 1, art. 40.

Ex. 3. In like manner, for the fluxion of $x^{n-1}\dot{x}\sqrt{(b + cx^n + dx^{2n})}$, assuming $x^n + \frac{c}{2d} = z$, $nx^{n-1}\dot{x} = \dot{z}$, and $x^{n-1}\dot{x} = \frac{1}{n}\dot{z}$; hence $x^{2n} + \frac{c}{d}x^n + \frac{c^2}{4d^2} = z^2$, and $\sqrt{(dx^{2n} + cx^n + b)} = \sqrt{d} \times \sqrt{\left(x^{2n} + \frac{c}{d}x + \frac{b}{d}\right)} = \sqrt{d} \times \sqrt{\left(z^2 + \frac{b}{d} - \frac{c^2}{4d^2}\right)} = \sqrt{d} \times \sqrt{(z^2 \pm a^2)}$, putting $\pm a^2 = \frac{b}{d} - \frac{c^2}{4d^2}$; hence the given fluxion becomes $\frac{1}{n}\dot{z}\sqrt{d} \times \sqrt{(z^2 \pm a^2)}$, and its fluent as in the last example.

Ex. 4. Also, for the fluent of $\frac{x^{n-1}\dot{x}}{b + cx + dx^2}$; assume $x^n + \frac{c}{2d} = z$, then the fluxion may be reduced to the form $\frac{1}{dn} \times \frac{\dot{z}}{z^2 \pm a^2}$, and the fluent found as before.

The student who may wish to see more on this branch, may profitably consult Dr. Dealtry's very methodical and ingenious treatise on Fluxions, from which several of the foregoing cases and examples have been taken or imitated.

To find Fluents by Infinite Series.

57. When a given fluxion, whose fluent is required, is so complex, that it cannot be made to agree with any of the forms in the foregoing table of cases, nor made out from the general rules before given; recourse may then be had to the method of infinite series; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite series of simple terms, by the methods given for that purpose in the first volume, viz. either by division or extraction of roots, or by the binomial theorem, &c.; and multiply every term by the fluxional letter, and by such simple variable factor as the given fluxional expression may contain. Then take the fluent of each term separately, by the foregoing rules, connecting them all together by their proper signs; and the series, after it is multiplied by any constant factor or co-efficient which may be contained in the given fluxional expression, will be the fluent sought.

58. It is to be noted, however, that the quantities must be so arranged, as that the series produced may be a converging one, rather than diverging: and this is effected by placing the greater terms foremost in the given fluxion. When these are known or constant quantities, the infinite series will be an ascending one; that is, the powers of the variable quantity will ascend or increase; but if the variable quantity be set foremost, the infinite series produced will be a

descending one, or the powers of that quantity will decrease always more and more in the succeeding terms, or increase in the denominators of them, which is the same thing.

For example, to find the fluent of $\dfrac{1 - x}{1 + x - x^2}\dot{x}$.

Here, by dividing the numerator by the denominator, the proposed fluxion becomes $\dot{x} - 2x\dot{x} + 3x^2\dot{x} - 5x^3\dot{x} + 8x^4\dot{x} -$ &c.; then the fluents of all the terms being taken, give $x - x^2 + x^3 - \frac{5}{4}x^4 + \frac{8}{5}x^5 -$ &c. for the fluent sought.

Again, to find the fluent of $\dot{x}\sqrt{1 - x^2}$.

Here, by extracting the root, or expanding the radical quantity $\sqrt{(1 - x^2}$, the given fluxion becomes $\dot{x} - \frac{1}{2}x^2\dot{x} - \frac{1}{8}x^4\dot{x} - \frac{1}{16}x^6\dot{x} -$ &c. Then the fluents of all the terms, being taken, give $x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 -$ &c. for the fluent sought.

When the value of x is greater than unity, the greatest power of the variable must come first in the series; but when it is less than unity the least must commence the series.

OTHER EXAMPLES.

1. To find the fluent of $\dfrac{bx\dot{x}}{a - x}$ both in an ascending and descending series.
2. To find the fluent of $\dfrac{b\dot{x}}{a + x}$ in both series.
3. To find the fluent of $\dfrac{3\dot{x}}{(a + x)^2}$.
4. To find the fluent of $\dfrac{1 - x^2 + 2x^4}{1 + x - x^4}x$.
5. Given $\dot{z} = \dfrac{b\dot{x}}{a^2 + x^2}$, to find z.
6. Given $\dot{z} = \dfrac{a^2 + x^2}{a + x}\dot{x}$, to find z.
7. Given $\dot{z} = 3\dot{x}\sqrt{a + x}$, to find z.
8. Given $\dot{z} = 2\dot{x}\sqrt{a^2 + x^2}$, to find z.
9. Given $\dot{z} = 4\dot{x}\sqrt{a^2 - x^2}$, to find z.
10. Given $\dot{z} = \dfrac{5a\dot{x}}{\sqrt{x^2 - a^2}}$, to find z.
11. Given $\dot{z} = 2\dot{x}\sqrt[3]{a^3 - x^3}$, to find z.
12. Given $\dot{z} = \dfrac{3a\dot{x}}{\sqrt{ax - xx}}$ to find z.
13. Given $\dot{z} = 2\dot{x}\sqrt[3]{x^3 + x^4 + x^5}$, to find z.
14. Given $\dot{z} = 5\dot{x}\sqrt{ax - xx}$, to find z.

To correct the Fluent of any given Fluxion.

59. The fluxion found from a given fluent is always perfect and complete; but the fluent found from a given fluxion is not always so; as it very often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c.; for, the fluent of any given fluxion, as $\dot{x}$, may be either x, which is found by the rule, or it may be $x + c$, or $x - c$, that is, x plus or minus some constant quantity c; because both x and $x \pm c$ have the same fluxion $\dot{x}$, and the finding of the constant quantity c, to be added or

subtracted with the fluent as found by the foregoing rules, is called *correcting the fluent*.

Now this correction can only be determined *from the nature of the problem in hand*, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, supposed to be found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but unknown, quantity c, with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantities, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of c, the constant quantity of the correction.

EXAMPLES.

60. *Ex.* 1. To find the correct fluent of $\dot{z} = ax^3\dot{x}$.

The general fluent is $z = ax^4$, or $z = ax^4 + c$, taking in the correction c: which correction can only be ascertained as the nature of the inquiry where the fluent occurs fixes it.

Thus, if it be known that z and x begin together, or that z is $= 0$, when $x = 0$; then writing 0 for both x and z, the general equation becomes $0 = 0 + c$, or $= c$; so that, the value of c being 0, the correct fluents are $z = ax^4$.

But if z be $= 0$, when x is $= b$, any known quantity; then substituting 0 for z, and b for x, in the general equation, it becomes $0 = ab^4 + c$, and hence we find $c = - ab^4$; which being written for c in the general fluential equation, it becomes $z = ax^4 - ab^4$, for the correct fluents.

Or, if it be known that z is $=$ some quantity d, when x is $=$ some other quantity as b; then substituting d for z, and b for x, in the general fluential equation $z = ax^4 + c$, it becomes $d = ab^4 + c$; and hence is deduced the value of the correction, namely, $c = d - ab^4$; consequently, writing this value for c in the general equation, it becomes $z = ax^4 - ab^4 + d$, for the correct equation of the fluents in this case.

61. And hence arises another easy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or position; then subtract the sides of the resulting particular equation from the corresponding sides of the general one, and the remainders will give the correct equation of the fluents sought.

So, the general equation being $z = ax^4$;
write d for z, and b for x, then $d = ab^4$;
hence, by subtraction, $z - d = ax^4 - ab^4$,
or $z = ax^4 - ab^4 + d$, the correct fluents as before.

Ex. 2. To find the correct fluents of $\dot{z} = 5x\dot{x}$; z being $= 0$ when x is $= a$.

Ex. 3. To find the correct fluents of $\dot{z} = 3\,\dot{x}\sqrt{a + x}$; z and x being $= 0$ at the same time.

Ex. 4. To find the correct fluent of $\dot{z} = \dfrac{2a\dot{x}}{a + x}$; supposing z and x to begin to flow together, or to be each $= 0$ at the same time.

Ex. 5. To find the correct fluents of $\dot{z} = \frac{2\dot{x}}{a^2 + x^2}$; supposing z and x to begin together.

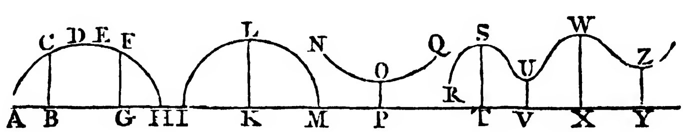
OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

62. MAXIMUM, denotes the greatest state or quantity attainable in any given case, or the greatest value of a variable quantity: by which it stands opposed to Minimum, which is the least possible quantity in any case.

Thus the expression or sum $a^2 + bx$, evidently increases as x, or the term bx, increases; therefore the given expression will be the greatest, or a maximum, when x is the greatest, or infinite: and the same expression will be a minimum, or the least, when x is the least, or nothing.

Again, in the algebraic expression $a^2 - bx$, where a and b denote constant or invariable quantities, and x a flowing or variable one, it is evident that the value of this remainder or difference, $a^2 - bx$, will increase, as the term bx, or as x, decreases; therefore the former will be the greatest, when the latter is the smallest; that is, $a^2 - bx$ is a maximum, when x is the least, or nothing at all; and the difference is the least, when x is the greatest.

63. Some variable quantities increase continually; and so have no maximum, but what is infinite. Others again decrease continually; and so have no minimum, but what is of no magnitude, or nothing. But, on the other hand, some variable quantities increase only to a certain finite magnitude, called their maximum, or greatest state, and after that they decrease again. While others decrease to a certain finite magnitude, called their minimum, or least state, and afterwards increase again. And, lastly, some quantities have several maxima and minima.



Thus, for example, the ordinate BC of the parabola, or such-like curve, flowing along the axis AB from the vertex A, continually increases, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreases to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increases to a certain magnitude, namely, the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreases again to nothing, at the point M. And in the curve NOQ, the ordinate decreases only to the position OP, where it is least, or a minimum; and after that it continually increases towards Q. But in the curve RSU, &c. the ordinates have several maxima, as ST, WX, and several minima, as VU, YZ, &c.

64. Now, because the fluxion of a variable quantity is the rate of its increase or decrease; and because the maximum or minimum of a quantity neither increases nor decreases, at those points or states; therefore such maximum or

minimum has no fluxion, or the fluxion is then equal to nothing *. From which we have the following rule.

To find the Maximum or Minimum.

65. From the nature of the question or problem, find an algebraical expression for the value, or general state, of the quantity whose maximum or minimum is required; then take the fluxion of that expression, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter †, and other common *constant* quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

[This test of the maximum and minimum admits, however, of an exception; viz.—when the given equation and its fluxional co-efficient, resolved for any specified value of one of them, have a common measure in terms of the other and constant quantities. Indeed, referring to vol. i. p. 229, we see at once that the derived equation, by means of which equal roots of the equation are determined, is only the *fluxional coefficient of the given equation*, though there obtained by a different process of reasoning. In this case the result is, *in general*, neither a maximum nor a minimum; and hence this test ought to be applied before we pronounce on the character of the point in question.]

So, if it be required to find the maximum state of the compound expression $100x - 5x^2 \pm c$, or the value of x when $100x - 5x^2 \pm c$ is a maximum. The fluxion of this expression is $100\dot{x} - 10x\dot{x}$; which being made $=0$, and divided by $10\dot{x}$, the equation is $10 - x = 0$; and hence $x = 10$. That is, the value of x is 10, when the expression $100x - 5x^2 \pm c$ is the greatest. As is easily tried: for if 10 be substituted for x in that expression, it becomes $\pm c + 500$: but if, for x, there be substituted any other number, whether greater or less than 10, that expression will always be found to be less than $\pm c + 500$, which is therefore its greatest possible value, or its maximum.

66. It is evident that if a maximum or minimum be any way compounded with, or operated on, by a given constant quantity, the result will still be a maximum or minimum. That is, *if a maximum or minimum be increased, or decreased, or multiplied, or divided, by a given quantity, or any given power or root of it be taken*; the result will still be a maximum or minimum. Thus, if x be a maximum or minimum, then also is $x + a$, or $x - a$, or ax, or $\frac{x}{a}$, or x^n, or $\sqrt[n]{x}$, still a maximum or minimum. Also, *the logarithm of the same will be a maximum*

* In the equations of curve lines, (which *any* equation between two variables may be supposed to be,) the abscissa x is *usually* taken as the independent variable, and y the dependent one. When y attains its maximum or minimum state, its fluxion is zero. Hence the equation expressing this circumstance, combined with its simultaneous equation of the curve, enables us to determine the position of the points (that is, the values of x and y) at which this takes place.

In precisely the same way, if we take y as the independent variable, and put the fluxion of x equal to zero, the maximum and minimum values of x will be found.

† For since the fluxion of the independent variable is constant, it cannot become 0; and hence the condition can only be fulfilled by taking its coefficient equal to zero.

When the common factor involves *either of the variables* in any mode of combination, the expression contains rational factors, and is separable into parts, *each* of which contains as many solutions as that factor has (by the theory of equations, vol. i.) dimensions of the variables. Whenever this is the case, the solution of the problem becomes *comparatively* easy.

or a minimum. And therefore, *if any proposed maximum or minimum can be made simpler by performing any of these operations, it is better to do so, before the expression is put into fluxions.*

67. When the expression for a maximum or minimum contains several variable letters or quantities; take the fluxion of it as often as there are variable letters; supposing first one of them only to flow, and the rest to be constant; then another only to flow, and the rest constant; and so on for all of them: then putting each of these fluxions $= 0$, there will be as many equations as unknown letters, from which these may be all determined. For the fluxion of the expression must be equal to nothing in each of these cases; otherwise the expression might become greater or less, without altering the values of the other letters, which are considered as constant.

So, if it be required to find the values of x and y, when $4x^2 - xy + 2y$ is a minimum. Then we have,

First, . . . $8x\dot{x} - \dot{x}y = 0$, and $8x - y = 0$, or $y = 8x$.
Secondly, $2\dot{y} - x\dot{y} = 0$, and $2 - x = 0$, or $x = 2$.
And hence y or $8x = 16$.

68. *To find whether a proposed quantity admits of a Maximum or a Minimum.*

Every algebraic expression does not admit of a maximum or minimum, properly so called; for it may either increase continually to infinity, or decrease continually to nothing; and in both these cases there is neither a proper maximum nor minimum; for the true maximum is that finite value to which an expression increases, and after which it decreases again: and the minimum is that finite value to which the expression decreases, and after that it increases again. Therefore, when the expression admits of a maximum, its fluxion is positive before the point, and negative after it; but when it admits of a minimum, its fluxion is negative before, and positive after it. Hence then, taking the fluxion of the expression a little before the fluxion is equal to nothing, and again a little after the same; if the former fluxion be positive, and the latter negative, the middle state is a maximum; but if the former fluxion be negative, and the latter positive, the middle state is a minimum.

So, if we would find the quantity $ax - x^2$ a maximum or minimum; make its fluxion equal to nothing, that is, ... $a\dot{x} - 2x\dot{x} = 0$, or $(a - 2x)\dot{x} = 0$; dividing by $\dot{x}$, gives $a - 2x = 0$, or $x = \frac{1}{2}a$ at that state. Now, if in the fluxion $(a - 2x)\dot{x}$, the value of x be taken rather less than its true value, $\frac{1}{2}a$, that fluxion will evidently be positive; but if x be taken somewhat greater than $\frac{1}{2}a$, the value of $a - 2x$, and consequently of the fluxion, is as evidently negative. Therefore, the fluxion of $ax - x^2$ being positive before, and negative after the state when its fluxion is $= 0$, it follows that at this state the expression is not a minimum, but a maximum.

Again, taking the expression $x^3 - ax^2$, its fluxion $3x^2\dot{x} - 2ax\dot{x} = (3x - 2a)x\dot{x} = 0$; this divided by $x\dot{x}$ gives $3x - 2a = 0$, and $x = \frac{2}{3}a$, its true value when the fluxion of $x^3 - ax^2$ is equal to nothing. But now to know whether the given expression be a maximum or a minimum at that time, take x a little less than $\frac{2}{3}a$ in the value of the fluxion $(3x - 2a)\, x\dot{x}$, and this will evidently be negative; and again, taking x a little more than $\frac{2}{3}a$, the value of $3x - 2a$, or of the fluxion, is as evidently positive. Therefore the fluxion of $x^3 - ax^2$ being negative before that fluxion is $= 0$, and positive after it, it follows that in this state the quantity $x^3 - ax^2$ admits of a minimum, but not of a maximum.

The expression of this rule may perhaps be rather more convenient thus:—

Find the second fluxional co-efficient, and insert the values of the variables

already determined, in the result. Then if the value of this expression be positive, those values indicate the minimum, and if negative, the maximum.

SOME EXAMPLES FOR PRACTICE.

1. Of all triangles, ACB, constructed on the same base AB, and having the same given perimeter, to determine that whose area or surface is the greatest.

Let s denote the semiperimeter, b the base AB, x the side AC, then BC will $= 2s - b - x$. Therefore putting Δ for the surface, we have (vol. i. p. 417)

$$\Delta^2 = s\,(s - b)\,(s - x)\,(b + x - s).$$

Expressing this equation logarithmically, we have, $2 \log. \Delta = \log. s + \log. (s - b) + \log. (s - x) + \log. (b + x - s)$ which (art. 58) is to be a max. or when put into fluxions equal to *zero*.

$$\text{Hence } \frac{2\dot{\Delta}}{\Delta} = \frac{-\dot{x}}{s - x} + \frac{\dot{x}}{b + x - s};$$

Or, dividing by $2\dot{x}$, and multiplying by Δ,

$$\frac{\dot{\Delta}}{\dot{x}} = \frac{\Delta}{2}\left(\frac{1}{b + x - s} - \frac{1}{s - x}\right) = 0.$$

Now, here it is evident, since Δ must be a max. that $\frac{\Delta}{2}$ cannot $= 0$; consequently the second factor must: that is,

$$\frac{1}{b + x - s} - \frac{1}{s - x} = 0, \text{ or } b + x - s = s - x.$$

Therefore $2s - b - x\ x$, or AC = BC; that is, the triangle must be isosceles, and its equal sides are each equal to $s - \frac{b}{2}$.

Cor. Hence it follows that of all *isoperimetrical* triangles, the one which has the greatest surface is equilateral. A truth, indeed, which may be readily shown by a direct investigation.

2. Amongst all parallellopipedons of given magnitude, whose planes are respectively perpendicular to one another, to determine that which has the least surface.

Let x, y, and z, be the measure of the three edges of the required parallelopipedon. Then, since the magnitude is given,

we have $xyz = a$, a given magnitude;
and $2xy + 2xz + 2yz =$ a minimum.

Here, substituting for z, and dividing by 2, there results

$$xy + x \,.\, \frac{a}{xy} + y \,.\, \frac{a}{xy} = \text{a min.}$$

$$\text{or, } u = xy + \frac{a}{y} + \frac{a}{x} = \text{a min.}$$

Therefore adopting the principle of art. 59,

$$\left.\begin{aligned} \frac{\dot{u}}{\dot{x}} &= y - \frac{a}{x^2} = 0 \\ \text{and } \frac{\dot{u}}{\dot{y}} &= x - \frac{a}{y^2} = 0 \end{aligned}\right\}\text{must both obtain.}$$

$$\text{Hence } y = \frac{a}{x^2} = a \div \left(\frac{a}{y^2}\right)^2 = a \,.\, \frac{y^4}{a^4} = \frac{y^4}{a}.$$

Consequently $\left.\begin{array}{r} y = a^{\frac{1}{3}} \\ x = a^{\frac{1}{3}} \\ \therefore z = a^{\frac{1}{3}} \end{array}\right\}$ and thus it appears that the required parallelopipedon is a *cube*.

3. Divide a given arc A into two such parts, that the mth power of the sine of one part, multiplied into the nth power of the sine of the other part shall be a maximum.

Let x and y be the parts: then $x + y = \text{A}$, and $\sin.^m x \times \sin.^n y = $ a max.

$$\text{In logs.}\quad m \log. \sin. x + n \log. \sin. y = \text{a max.}$$

$$\text{Hence, (art. 66)}\ \frac{m\dot{x} \cos. x}{\sin. x} + \frac{n\dot{y} \cos. y}{\sin. y} = 0.$$

$$\text{But } \dot{y} = -\dot{x} \therefore \frac{m\dot{x} \cos. x}{\sin. x} - \frac{n\dot{x} \cos. y}{\sin. y} = 0.$$

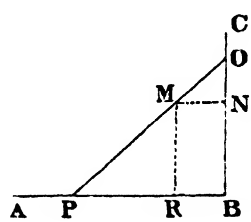
$$\text{Hence } m \cot. x = n \cot. y, \text{ or } m \tan. y = n \tan. x.$$

$$\therefore \frac{m}{n} = \frac{\tan. x}{\tan. y} \ldots \text{and } \frac{m+n}{m-n} = \frac{\tan. x + \tan. y}{\tan. x - \tan. y} = \frac{\sin. (x+y)}{\sin. (x-y)}.$$

Hence x and y become known: and the same principle is evidently applicable to three or more arcs, making together a given arc.

4. To find the longest straight pole that can be put up a chimney, whose height RM $= a$, from the floor to the mantel, and depth MN $= b$, from front to back, are given.

Here the longest pole that can be put up the chimney is, in fact, the *shortest* line PMO, which can be drawn through M, and terminated by BA and BC.



Here if ON be denoted by x, and PO by y, we have ON : NM :: MR : RP $\therefore$ RP $= \frac{ab}{x}$: and hence $y = (1 + \frac{a}{x})\ (b^2 + x^2)^{\frac{1}{2}}$. The fluxional co-efficient is $\frac{x^3 - ab^2}{x^2(b^2 + x^2)^{\frac{1}{2}}}$; which being put $= 0$, gives $x^3 = ab^2$, and $x = \sqrt[3]{ab^2}$.

Hence PO $= (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$. Whether this be a maximum or a minimum will be seen below. When $a = b$, PO $= a \cdot 2^{\frac{3}{2}} = a\sqrt{8}$.

Otherwise,

Denote the inclination of the pole to the horizon, viz. the angle OPB by θ. Then PM $= a$ cosec θ. and MO $= b$ sec. θ. Hence PO $= a$ cosec. $\theta + b$ sec. θ is to be a *max*. Its fluxional co-efficient is

$$-\frac{a \cos. \theta}{\sin.^2\theta} + \frac{b \sin. \theta}{\cos.^2\theta} = 0, \text{ or } \tan. \theta = \sqrt[3]{\frac{a}{b}}\text{; and hence}$$

$$\text{PO} = a\sqrt{1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}} + b\sqrt{1 + \frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}} = \pm\{a^{\frac{2}{3}} + b^{\frac{2}{3}}\}^{\frac{3}{2}}.$$

The double sign signifies that the line PO may be measured from either extremity, P or O.

Next, to find whether it is a *max.* or a *min.* find the second fluxional co-efficient.

Now the first fluxional coefficient is either $\tan.\theta = \sqrt[3]{\frac{a}{b}}$ or $\cot.\theta = \sqrt[3]{\frac{b}{a}}$. The fluxional coefficient of the first of these is $\sec.^2\theta$ and that of the second is $-\operatorname{cosec.}^2\theta$. The first form having the value of θ above found inserted, is positive, and the second, negative. Whence we learn that the length of PO above found is both a *max.* and a *min.* We must, therefore, recur to the *geometrical* conditions of the problem to ascertain which it is in the present case.

Now, the condition which we have used is that the line PO passes through M and is terminated by the lines AB and BC. But that line may be increased without limit, since when it becomes parallel to either of the lines AB or BC, it is infinite. The line found above is, then, the *least* that can be drawn through M.

If a pole longer than that now found were placed in the chimney, it must in its progress have coincided in length and position with PO; and as these two conditions are contradictory, PO is the longest pole that could fulfil the conditions.

5. Divide a line, or any other given quantity a, into two parts, so that their rectangle or product may be the greatest possible.

6. Divide the given quantity a into two parts such, that the product of the m power of the one by the n power of the other may be a maximum.

7. Divide the given quantity a into three parts such, that the continual product of them all may be a maximum.

8. Divide the given quantity a into three parts such, that the continual product of the nth power of the 1st, the mth power of the 2d, and the pth power of the 3d, may be a maximum.

9. Determine a number such, that the difference between its m power and n power shall be the greatest possible.

10. Divide the number 80 into two such parts, x and y, that $2x^2 + xy + 3y^2$ may be a minimum.

11. Find the greatest rectangle that can be inscribed in a given triangle.

12. Find the greatest rectangle that can be inscribed in the quadrant of a given circle: and, also, in any given segment of it.

13. Find the least right-angled triangle that can circumscribe the quadrant of a given circle or ellipse.

14. Find the greatest rectangle inscribed in, and the least isosceles triangle circumscribed about the greatest semi-ellipse, which can be cut from a cone whose base is 12, and altitude 9 inches: and likewise about the greatest parabola or hyperbola that can be cut from the same cone.

15. Find the equation of the straight line which bisects a given triangle, and has its portion intercepted by the sides a minimum.

16. Find the dimensions of the cone whose volume is given and whose whole surface is a maximum.

17. Inscribe the greatest cylinder in a given cone; or to cut the greatest cylinder out of a given cone.

18. Determine the dimensions of a rectangular cistern, capable of containing a given quantity a of water, so as to be lined with lead at the least possible expense.

19. Required the dimensions of a cylindrical tankard, to hold one quart of ale measure, that can be made of the least possible quantity of silver, of a given thickness.

20. Cut the greatest parabola and the greatest ellipse from a given cone.

21. Cut the greatest ellipse from a given cone.

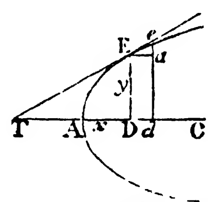
22. Find the value of x when x^x is a minimum: also find the least value a can have in the equation $x^x = a$, and the corresponding value of x.

23. Find the number which bears to its logarithm the least ratio possible.

24. Required an arc such that the rectangle under its tangent and the cosine of the double arc may be a maximum or a minimum.

THE METHOD OF TANGENTS; OR, OF DRAWING TANGENTS TO CURVES.

69. The Method of Tangents, is a method of determining the length of the tangent and subtangent at any given point of any curve of which the equation is given. Or, *vice versa*, the nature of the curve, from the equation of the tangent being given.



If AE be any curve, and E be any point in it, to which it is required to draw a tangent TE. Draw the ordinate ED: then if we can determine the subtangent TD, limited between the ordinate and tangent, in the axis produced, by joining the points T, E, the line TE will be the tangent sought.

70. Let dae be another ordinate, indefinitely near to DE, meeting the curve, or tangent produced in e; and let Ea be parallel to the axis AD. Then is the elementary triangle Eea similar to the triangle TDE; and

$\frac{ea}{a\mathrm{E}} = \frac{\mathrm{ED}}{\mathrm{DT}}$ = tan. DTE, or tan. DTE $= \frac{\dot{y}}{\dot{x}}$. Hence DT $= y$ cot. DTE $= \frac{y\dot{x}}{\dot{y}}$; which is therefore the general value of the subtangent sought; where x is the absciss AD, and y the ordinate DE, of the given point of the curve.

Hence we have this general rule.

GENERAL RULE.

71. By means of the given equation of the curve, when put into fluxions, find the value of either $\frac{\dot{y}}{\dot{x}}$, or of $\frac{\dot{x}}{\dot{y}}$, which value substitute for it in the expression

DT $= \frac{y\dot{x}}{\dot{y}}$, and, when reduced to its simplest terms, it will be the value of the subtangent sought. Then for the *given values* of the co-ordinates * of E calculate DT, and join TE. It will be the tangent sought.

EXAMPLES.

1. Let the proposed curve be that which is defined, or expressed, by the equation $ax^2 + xy^2 - y^3 = 0$.

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* That is, the value of *one* of them being given, and the other found from the equation of the curve, this given corresponding value of the first being previously substituted for it.

Here the fluxional equation of the curve is

$$2ax\dot{x} + y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0;\ \text{then by transposition,}$$

$$2ax\dot{x} + y^2\dot{x} = 3y^2\dot{y} - 2xy\dot{y};\ \text{and hence, by division,}$$

$$\frac{\dot{x}}{\dot{y}} = \frac{3y^2 - 2xy}{2ax + y^2};\ \text{consequently}\ \frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - 2xy^2}{2ax + y^2},$$

which is the value of the subtangent TD sought, and the tangent may be drawn.

2. To draw a tangent to a circle; the equation of which is $ax - x^2 = y^2$; where x is the absciss, y the ordinate, and a the diameter.

3. To draw a tangent to a parabola; its equation being $px = y^2$; where p denotes the parameter of the axis.

4. To draw a tangent to an ellipse; its equation being $c^2(ax - x^2) = a^2y^2$; where a and c are the two axes.

5. To draw a tangent to an hyperbola; its equation being $c^2(ax + x^2) = a^2y^2$; where a and c are the two axes.

6. To draw a tangent to the hyperbola referred to the asymptote as an axis; its equation being $xy = a^2$; where a^2 denotes the rectangle of the absciss and ordinate answering to the vertex of the curve.

72. There is another method of effecting the solution of this problem, slightly different in its details though the same in principle. It has, however, several advantages in an analytical point of view over that just given; though in general it is inferior to it in point of facility in actual construction.

When the value of x or y is given, which designates the given point E, the other value can be found from the equation of the curve, as already explained. Hence for any given point the co-ordinates are given; and as the fluxional co-efficient is composed wholly of these and given constant quantities, the value of this also is known for any given point. But from the preceding investigation it appears that this co-efficient expresses the natural trigonometrical tangent of the angle which the tangent of the curve makes with the axis of the principal variable: and it has been shown how to find the equation of the straight line which passes through a given point, and makes a given angle with the axis (vide p. 190.), and hence we see how to form the equation of the tangent at any given point of a given curve. If this equation be formed, the intersection of the tangent with the axes of co-ordinates can be found, and the line itself constructed.

If now we sub-accentuate the xy of the curve (to designate their being given) thus $x_, y_,$, and denote by x and y the co-ordinates of the points of the tangent, we have (ibid.) the equation of the linear tangent represented by

$$(x - x_,)\,\dot{y}_, = (y - y_,)\,\dot{x}_,\ \text{or}$$

$$y - y_, = \frac{\dot{y}_,}{\dot{x}_,}(x - x_,).$$

For example, to find the equation of the tangent to any given point $x_,y_,$ of a conic section, whose general equation is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

In the first place, taking the fluxion: then resolving we have for $\dot{x}$ and $\dot{y}$,

$$(2ay + bx + d)\,\dot{y} + (2cx + by + ef)\,\dot{x} = 0,$$

or, accenting x and y, we have, finally,

$$y - y_, = -\frac{2cx_, + by + e}{2ay_, + bx_, + d}(x - x_,)$$

agreeing in all respects with that found at page 196, by an investigation, which is in appearance different from this. The attentive student will, however, readily

discover that the investigations in both cases are based on the same principle—the doctrine of limiting ratios.

7. Find the equations of the tangents at any given point, $x_,y_,$ of the curves defined at p. 201—203.

OF THE NORMALS TO CURVE LINES.

73. The *normal* being the line drawn from any point of a curve perpendicular to the curve, or, which is the same thing, to its coincident tangent at that point, its equation is obviously found by the process explained at p. 240; and the tangent of its inclination to either axes is $-\frac{\dot{x}_,}{\dot{y}_,}$, where $x_,y_,$ designates the given point. The equation of the normal therefore is

$$y - y_, = -\frac{\dot{x}_,}{\dot{y}_,}(x - x_,)$$

Thus the normal to a line of the second order is

$$y - y_, = \frac{2ay_, + bx_, + d}{2cx_, + by + e}(x - x_,).$$

74. The *subnormal* being the portion of the abscissa intercepted between the ordinate and normal its value is readily found to be

from which, as in case of the tangent, the normal may be actually constructed by the analogous process to that described at page 240.

For example, in the parabola referred to the diameter and vertex, we have $y^2 = 4ax$. Hence $\frac{y_,\dot{y}_,}{\dot{x}_,} = 2a$; or the subnormal is constant and equal to half the parameter, agreeing with the geometrical determination before given.

Examples.—The student must find the equations of the normals, and the values of the sub-normals to the curves defined at pages 201—203.

OF ASYMPTOTES.

75. It has been seen in treating of the hyperbola (see props. A, B, C, . . pages 129—134) that it is possible for the curve to approach continually nearer to a certain straight line, and yet never meet it, though it approach nearer to it than any distance which can be assigned, however small. But the hyperbola is only one of a very numerous class of curves which possesses this property; and the straight line is only one of a very numerous class of lines towards which this indefinitely close approach may be made. The conjugate hyperbolas, for instance, so approach each other: and there is, indeed, no line whatever to which innumerable asymptotic curves cannot be drawn. Of rectilinear asymptotes, alone we propose to give here a succinct account; and must refer the inquiring

reader for further details to works expressly devoted to the Geometry of Curve Lines.

The idea that the asymptote may be considered as the tangent to an infinitely distant point in the curve is the foundation of this inquiry, and at once suggests the method of conducting it.

(1.) If any finite value of one variable renders the corresponding value of the other infinite, the curve has an asymptote parallel to the axis of that other.

(2.) If for infinite values of $x_{,}$ or $y_{,}$ the equation of the tangent be such as to give an intersection with the axes of co-ordinates (one or both), the curve has a corresponding asymptote.

The application of this principle is facilitated by considering that the parts of the axes intercepted by the tangent are respectively

$$x_{,} - \frac{y_{,}\dot{x}_{,}}{\dot{y}_{,}} \text{ and } y_{,} - \frac{x_{,}\dot{y}_{,}}{\dot{x}_{,}}.*$$

76. In illustration of the former of these methods take the curve whose equation is $xy = a^2$. Then $y = \frac{a^2}{x}$, and manifestly as x decreases towards 0, y increases towards infinity, and when $x = 0$, y is infinite. The axis of y is therefore itself an asymptote. In the same manner the axis of x may be shown to be an asymptote. This is in fact the common hyperbola referred to its asymptotes.

Again, taking the *witch* whose equation is $y = d\sqrt{\frac{d-x}{x}}$, we find the tangent to the circle at B is an asymptote to the curve, the value of x being then 0, and the value of y infinite.

Some of the other curves described at page 201—203, have asymptotes which be found in the same manner.

77. In illustration of the second method, let us take the hyperbola referred to its principal axes. Its equation being $y = \pm\frac{b}{a}(x^2 - a^2)^{\frac{1}{2}}$ we have $\frac{\dot{y}}{\dot{x}} = \frac{\pm bx}{a\sqrt{x^2 - x^2}}$. Hence the expressions above given for the intercepts of the axes become $x_{,} - \frac{x^2 - a^2}{x}$ or $\frac{a^2}{x}$, and $\mp\frac{b}{a}\cdot\frac{a^2}{\sqrt{x^2 - a^2}}$, both of which are 0 when $x = \frac{1}{0}$ or infinite, and hence there are two asymptotes equally inclined to the axis of x.

To find the equation of the asymptote, we have $\frac{\dot{y}_{,}}{\dot{x}_{,}} = \pm\frac{b}{a}\cdot\frac{1}{\sqrt{1 - \frac{a^2}{x_{,}^2}}}$ which when $x_{,}$ is infinite becomes $\pm\frac{b}{a}$; and the equation is $y = \pm\frac{b}{a}\cdot x$.

EXAMPLES.

1. Find the rectilinear asymptotes to the curves denoted by the following equations:—

(1.) $y^3 - 3axy + y^3 = 0$; (2) $y^3 - 2xy^2 + x^2y - a^3 = 0$;

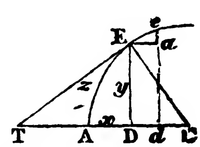
* There is also a third method sometimes employed, which consists in developing the expression for one variable in *descending* powers of the other. It is not generally so easy of application as that explained here, although, in some few cases, it gives great facility.

(3.) $y^6 - 2a^2x^2y + x^6 = 0$; (4.) $xy^2 - cy = ax^3 + bx^2 + cx + d$;
(5.) $y^4 - 2x^2y^2 - x^4 + 2axy^2 - 5ax^3 = 0$; and
(6.) $x^4 - 2ay^3 - 3a^2y - 2a^2x^2 + a^4 = 0$.

2. Find by the general method which of the curves defined at page 201—203 have asymptotes, and determine their equations.

OF RECTIFICATIONS; OR, TO FIND THE LENGTHS OF CURVE LINES.

78. Rectification, is the finding the length of a curve line, or finding a right line equal to a proposed curve.



By art. 10 it appears, that the elementary triangle Eae, formed by the increments of the absciss, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypothenuse; and therefore the square of the latter is equal to the sum of the squares of the two former, that is $\text{E}e^2 = \text{E}a^2 + ae^2$. Or, substituting, for the increments, their proportional fluxions, it is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; where z denotes any curve line AE, x its absciss AD, and y its ordinate DE. Hence this rule.

RULE.

79. From the given equation of the curve put into fluxions, find the value of $\dot{x}^2$ or $\dot{y}^2$, which value substitute instead of it in the equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; then the fluents, being taken, will give the value of z, or the length of the curve, in terms of the absciss or ordinate.

EXAMPLES.

1. To find the length of the arc of a circle, in terms both of the sine, versed sine, tangent, and secant.

The equation of the circle may be expressed in terms of the radius, and either the sine, or the versed sine, or tangent, or secant, &c. of an arc. Let therefore the radius of the circle be CA or CE $= r$, the versed sine AD (of the arc AE) $= x$, the right sine DE $= y$, the tangent TE $= t$, and the secant CT $= s$; then, by the nature of the circle, there arises these equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2t^2}{r^2 + t^2} = \frac{s^2 - r^2}{s^2} r^2.$$

Then, by means of the fluxions of these equations, with the general fluxional equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itself; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - xx}} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r^2\dot{t}}{r^2 + t^2} = \frac{r^2\dot{s}}{\sqrt{s^2 - r^2}}\,\text{*}.$$

* These formulæ are, obviously, analogous to those given in art. 46, p. 221, and are so many forms of fluxions whose fluents become known. Thus the fluent of an expression, such as

Hence the value of the curve, from the fluent of each of these, expressed in series, gives the four following forms, in series, viz. putting $d = 2r$ the diameter, the curve is

$$z = (1 + \frac{x}{2.3d} + \frac{3x^2}{2.4.5d^2} + \frac{3.5x^3}{2.4.6.7d^3} + \&c.)\sqrt{dx},$$

$$= (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \&c)\,y.$$

$$= (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \&c.)\,t,$$

$$= (\frac{s-r}{s} + \frac{s^3 - r^3}{2.3s^3} + \frac{3(s^5 - r^5)}{2.4.5s^5} + \&c.)\,r.$$

Now, it is evident, that the simplest of these series, is the third in order, or that which is expressed in terms of the tangent. That form will therefore be the fittest by which to calculate an example in numbers. And for this purpose it will be convenient to assume some arc whose tangent, or at least the square of it, is known to be some small simple number. Now, the arc of 45 degrees, it is known, has its tangent equal to the radius; and therefore, taking the radius $r = 1$, and consequently the tangent of 45°, or t, $= 1$ also, in this case the arc of 45° to the radius 1, or the arc of the quadrant to the diameter 1, will be equal to the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ &c.

But as this series converges very slowly, it will be proper to take some smaller arc, that the series may converge faster; such as the arc of 30 degrees, the tangent of which is $= \sqrt{\frac{1}{3}}$, or its square $t^2 = \frac{1}{3}$: which being substituted in the series, the length of the arc of 30° comes out

$(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c)\sqrt{\frac{1}{3}}$. Hence, to compute these terms in numbers, after the first, the succeeding terms will be found by dividing, always by 3, and these quotients again by the odd numbers, 3, 5, 7, 9, &c.; and again adding every other term together, into two sums, the one the sum of the positive terms, and the other the sum of the negative ones; then lastly, the one sum taken from the other, leaves the length of the arc of 30 degrees; which being the 12th part of the whole circumference when the radius is 1, or the 6th part when the diameter is 1, consequently 6 times that arc will be the length of the whole circumference to the diameter 1. Therefore, multiplying the first term $\sqrt{\frac{1}{3}}$ by 6, the product is $\sqrt{12} = 3{\cdot}4641016$; and hence the operation, true to 7 places of decimals, will be conveniently made as follows:

$\frac{r\dot{x}}{\sqrt{(2rx - x^2)}}$, is a circular arc whose radius is $= r$ and versed sine $= x$. The fluent of an expression such as $\frac{r^2\dot{t}}{r^2 + t^2}$ is a circular arc whose radius is $= r$ and tangent $= t$ · and so of the rest.

Conversely, the same formulæ, or those just referred to, serve to assign the relative magnitudes of the *differences* in any parts of a table of natural sines, of natural tangents, &c. Thus, $\dot{t} = \frac{r^2 + t^2}{r^2}\dot{z} = \dot{z}\,\{\text{sec. tan.}^{-1}\,t\}^2$ consequently, the tabular differences of the tangents vary as the squares of the secants. Hence, those differences, at 0°, at 45°, and at 60°, are as 1^2, $(\sqrt{2})^2$, and 2^2, or as 1, 2, and 4. This suggests an application of these formulæ which will often be found useful.

		+ Terms.	— Terms.
1)	3·4641016 (	3·4641016	
3)	1·1547005 (		0·3849002
5)	3849002 (	769800	
7)	1283001 (		183286
9)	427667 (	47519	
11)	142556 (		12960
13)	47519 (	3655	
15)	15840 (		1056
17)	5280 (	311	
19)	1760 (		93
21)	587 (	28	
23)	196 (		8
25)	65 (	3	
27)	22 (		1
		+3·5462332	—0·4046406
		—0·4046406	

So that at last 3·1415926 is the whole circumference to the diameter 1 *.

Scholium.—Dr. Hutton paid great attention to this problem, and investigated several series for the more ready computation of the circumference of the circle, which were published originally in the *Philosophical Transactions,* and subsequently in his *Mathematical Tracts* in 8vo. One of the best is here subjoined.

The method consists in finding out such small arcs as have for tangents some small and simple vulgar fractions, the radius being denoted by 1, and such also that some multiple of those arcs shall differ from an arc of 45°, the tangent of which is equal to the radius, by other small arcs, which also shall have tangents denoted by other such small and simple vulgar fractions. For it is evident, that if such a small arc can be found, some multiple of which has such a proposed difference from an arc of 45°, then the length of these two small arcs will be easily computed from the general series, because of the smallness and simplicity of their tangents; after which, if the proper multiple of the first arc be increased or diminished by the other arc, the result will be the length of an arc of 45°, or one-eighth of the circumference. And the manner in which he discovered such arcs is this:

Let T, t, denote any two arcs, of which T is the greater, and t the less: then it is known that the tangent of the difference of the corresponding arcs is equal to $\frac{T-t}{1+Tt}$. Hence, if t, the tangent of the smaller arc, be successively denoted by each of the simple fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$. &c. the general expression for the tangent of the difference between the arcs will become respectively $\frac{2T-1}{2+T}$ $\frac{3T-1}{3+T}$, $\frac{4T-1}{4+T}$, $\frac{5T-1}{5+T}$, &c.; so that if T be expounded by any given number, then these expressions will give the tangent of the difference of the arcs in known numbers, according to the values of t, severally assumed respectively. And if, in the first place, T be equal to 1, the tangent of 45°, the fore-

* For this value, true to 100 places of decimals; and indeed for many curious and important investigations in reference to rectifications, quadratures, &c. see *Hutton's Mensuration.*

going expressions will give the tangent of an arc, which is equal to the difference between that of 45° and the first arc; or that of which the tangent is one of the numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. Then, if the tangent of this difference, just now found, be taken for T, the same expressions will give the tangent of an arc, equal to the difference between that of 45° and the triple of the first arc. And again, taking this last found tangent for T, the same theorem will produce the tangent of an arc equal to the difference between that of 45° and the quadruple of the first arc; and so on, always taking for T the tangent last found, the same expressions will give the tangent of the difference between the arc of 45° and the next greater multiple of the first arc; or that of which the tangent was at first assumed equal to one of the small numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. This operation, being continued till some of the expressions give such a fit, small, and simple fraction as is required, is then at an end; for we have then found two such small tangents as were required, viz. the tangent last found, and the tangent first assumed.

The Doctor exemplifies this method by a variety of substitutions, and thus obtains a collection of very valuable series; of which, however, it may suffice to present one or two in this place. Thus, in the case of $t = \frac{1}{4}$, the expression $\frac{4T-1}{4+T}$ gives, for the successive tangents $\frac{3}{5}$, $\frac{7}{23}$, $\frac{5}{99}$, $-\frac{79}{401}$, &c. of which the third is a convenient number, and gives for A, the arc of 45°,

$$A = \left\{ \begin{array}{l} \frac{3}{4} \times \left(1 - \frac{1}{3 \cdot 16} + \frac{1}{5 \cdot 16^2} - \frac{1}{7 \cdot 16^3} + \&c.\right) \\ + \frac{5}{99} \times \left(1 - \frac{5^2}{3 \cdot 99^2} + \frac{5^4}{5 \cdot 99^4} - \frac{5^6}{7 \cdot 99^6} + \&c.\right) \end{array} \right.$$

This is, obviously, a very compendious series for operation, since 99 is resolvable into the two simple factors 9 and 11.

Another excellent series is the following:

$$A = \left\{ \begin{array}{l} \frac{4}{5} \times \left(1 + \frac{4}{3 \cdot 10} + \frac{8\alpha}{5 \cdot 10} + \frac{12\beta}{7 \cdot 10} + \&c.\right) \\ - \frac{7}{50} \times \left(1 + \frac{4}{3 \cdot 100} + \frac{8\alpha}{5 \cdot 100} + \frac{12\beta}{7 \cdot 100} + \&c.\right) \end{array} \right.$$

Where α, β, γ, δ, &c. denote always the preceding terms in each series.

For other valuable series the reader may consult the paper itself.

2. To find the length of a parabola.

3. To find the length of a semicubical parabola, whose equation is $ax^2 = y^3$.

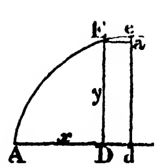
4. To find the length of an elliptical curve.

5. To find the length of an hyperbolic curve.

Other examples may be found at page 201—203, upon which the student may exercise himself in the rectifications of the several curves in general expressions, and also for the *whole length* of those whose values are finite.

OF QUADRATURES; OR, FINDING THE AREAS OF CURVES.

80. The quadrature of curves is the measuring their areas, or finding a square, or other right-lined space, equal to a proposed curvilinear one.



By art. 9, it appears, that any flowing quantity being drawn into the fluxion of the line along which it flows, or in the direction of its motion, there is produced the fluxion of the quantity generated by the flowing. That is, DE × Dd or $y\dot{x}$ is the fluxion of the area ADE. Hence this rule.

RULE.

81. From the given equation of the curve, find the value either of $\dot{x}$ or of y; which value substitute instead of it in the expression $y\dot{x}$; then the fluent of that expression, being taken, will be the area of the curve sought.

EXAMPLES.

1. To find the area of the common parabola.

The equation of the parabola being $ax = y^2$; where a is the parameter, x the absciss AD, or part of the axis, and y the ordinate DE.

From the equation of the curve is found $y = \sqrt{ax}$. This substituted in the general fluxion of the area $y\dot{x}$ gives $\dot{x}\sqrt{ax}$ or $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ the fluxion of the parabolic area, and the fluent of this, or $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$, is the area of the parabola ADE, which is therefore equal to $\frac{2}{3}$ of its circumscribing rectangle.

2. To square the circle, or find its area.

The equation of the circle being $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$, where a is the diameter; by substitution, the general fluxion of the area $y\dot{x}$, becomes $\dot{x}\sqrt{ax - x^2}$, for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity $\sqrt{ax - x^2}$ is thrown into a series, by extracting the root, when the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times (1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c.);$$

and then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times (\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \frac{1.3.5x^4}{4.6.8.11a^4} - \&c.);$$

for the general expression of the semisegment ADE.

And when the point D arrives at the extremity of the diameter, then the space becomes a semicircle, and $x = a$; and then the series above becomes barely

$$a^2(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5}{4.6.8.11} - \&c.)$$

for the area of the semicircle whose diameter is a.

If, instead of taking the equation of the circle having the origin of the co-ordinates at the circumference, the equation $x^2 + y^2 = r^2$ be taken, regarding the origin of the co-ordinates at the centre; and if, still farther, r be taken $= 1$, then $y = \sqrt{1 - x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \&c.$ Taking this value of y for it in the expression $y\dot{x}$, the correct fluent will be

$$x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152} - \frac{7x^{11}}{2816},$$

for the area of the portion CFBD (fig. p. 383, vol. i.) Now, if arc BD $= 30°$, then CF $= x = \frac{1}{2}$, and the sum of the series $= \cdot4783057$. From which deducting the area of the triangle CFB $= \frac{1}{4} \cdot \frac{1}{2}\sqrt{3} = \cdot2165063$, there remains $\cdot2617994$ for the area of the sector CBD. Twelve times this, or 3·1415928; &c. expresses the area of the circle whose diameter is 2.

3. To find the area of any parabola, whose equation is $a^m x^n = y^{m+n}$.

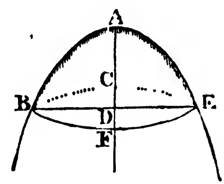
4. To find the area of an ellipse.

5. To find the area of an hyperbola.

6. To find the area between the curve and asymptote of an hyperbola.

7. To find the like area in any other hyperbola whose general equation is $x^m y^n = a^{m+n}$.

TO FIND THE SURFACES OF SOLIDS OF REVOLUTION.



82. In the solid formed by the rotation of any curve about its axis, the surface may be considered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the solid. Therefore, as the fluxion of any generated quantity, is produced by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the surface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the surface BAE, is equal to the fluxion of AE drawn into the circumference BCEF, whose radius is the ordinate DE.

But if π be $= 3{\cdot}141593$, the circumference of a circle whose diameter is 1, $x =$ AD the absciss, $y =$ DE the ordinate, and $z =$ AE the curve; then $2y =$ the diameter BE, and $2\pi y =$ the circumference BCEF; also, AE $= \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: therefore $2\pi y\dot{z}$ or $2\pi y\sqrt{\dot{x}^2 + \dot{y}^2}$ is the fluxion of the surface. And consequently if, from the given equation of the curve, the value of $\dot{x}$ or $\dot{y}$ be found, and substituted in this expression $2\pi y\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluent of the expression being then taken, will be the surface of the solid required.

EXAMPLES.

1. To find the surface of a sphere, or of any segment.

In this case, AE is a circular arc, whose equation is $y^2 = ax - x^2$, or $y = \sqrt{(ax - x^2)}$.

The fluxion of this gives $\dot{y} = \dfrac{a - 2x}{2\sqrt{ax - x^2}}\dot{x} = \dfrac{a - 2x}{2y}\dot{x}$;

hence $\dot{y}^2 = \dfrac{a^2 - 4ax + 4x^2}{4y^2}\dot{x}^2 = \dfrac{a^2 - 4y^2}{4y^2}\dot{x}^2$; consequently $\dot{x}^2 + \dot{y}^2 = \dfrac{a^2\dot{x}^2}{4y^2}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \dfrac{a\dot{x}}{2y}$.

This value of $\dot{z}$, the fluxion of a circular arc, may be found more easily thus: In the fig. to art. 64, the two triangles EDC, E*ae*, are equiangular, being each of them equiangular to the triangle ETC: conseq. ED : EC :: E*a* : E*e*, that is, $y : \frac{1}{2}a :: \dot{x} : \dot{z} = \dfrac{a\dot{x}}{2y}$, the same as before.

The value of $\dot{z}$ being found, by substitution is obtained $2\pi y\dot{z} = a\pi\dot{x}$ for the fluxion of the spherical surface, generated by the circular arc in revolving about

the diameter AD. And the fluent of this gives $a\pi x$ for the said surface of the spherical segment BAE.

But πa is equal to the whole circumference of the generating circle; and therefore it follows, that the surface of any spherical segment, is equal to the same circumference of the generating circle, drawn into x or AD, the height of the segment.

Also when x or AD becomes equal to the whole diameter a, the expression πax becomes πaa or πa^2, or 4 times the area of the generating circle, for the surface of the whole sphere.

And these agree with the rules before found in Mensuration of Solids.

2. To find the surface of a spheroid.

3. To find the surface of a paraboloid.

4. To find the surface of an hyperboloid.

TO FIND THE CONTENTS OF SOLIDS OF REVOLUTION.

83. Any solid which is formed by the revolution of a curve about its axis (see last fig.), may also be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And therefore the area of that circle being drawn into the fluxion of the axis will produce the fluxion of the solid. That is, flux. of AD × area of the circle BCF, whose radius is DE, or diameter BE, is the fluxion of the solid, by art 9.

85. Hence if AD $= x$, DE $= y$, $\pi = 3{\cdot}141593$; because πy^2 is equal to the area of the circle BCF; therefore $\pi y^2\dot{x}$ is the fluxion of the solid. Consequently, if, from the given equation of the curve, the value of either y^2 or x be found, and that value substituted for it in the expression $\pi y^2\dot{x}$, the fluent of the resulting quantity, being taken, will be the solidity of the figure proposed.

EXAMPLES.

1. To find the solidity of a sphere, or any segment.

The equation to the generating circle being $y^2 = ax - x^2$, where a denotes the diameter, by substitution, the general fluxion of the solid $\pi y^2\dot{x}$, becomes $\pi ax\dot{x} - \pi x^2\dot{x}$, the fluent of which gives $\frac{1}{2}\pi ax^2 - \frac{1}{3}\pi x^3$, or $\frac{1}{6}\pi x^2(3a - 2x)$, for the solid content of the spherical segment BAE, whose height AD is x.

When the segment becomes equal to the whole sphere, then $x = a$, and the above expression for the solidity, becomes $\frac{1}{6}\pi a^3$ for the solid content of the whole sphere.

And these deductions agree with the rules before given and demonstrated in the Mensuration of Solids.

2. To find the solidity of a spheroid.

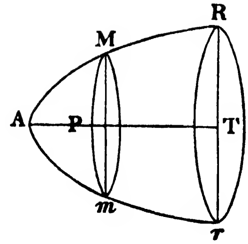
3. To find the solidity of a paraboloid.

4. To find the solidity of an hyperboloid.

5. To find the solidity of a body, or segment, or frustum, produced by the revolution upon its axis of any curve denoted by the general equation

$$y^2 = A + Bx + Cx^2.$$

Where AP $= x$, PM $= y$; and taking the several cases when A, B, or C, become equal to nothing, and those in which they have finite values.



Here the fluxion of the solid is

$$\pi y^2\dot{x} = (A + Bx + Cx^2)\pi\dot{x},$$

of which the fluent

$$(A + \tfrac{1}{2}Bx + \tfrac{1}{3}Cx^2)\pi x$$

is the general expression for the solidity.

In this, substituting for Bx, its value $y^2 - A - Cx^2$, in the primitive equation, we have

$$\text{solidity} = (y^2 + A - \tfrac{1}{3}Cx^2)\tfrac{1}{2}\pi x. \quad \text{(I.)}$$

Or, substituting for Cx^2, its value $y^2 - A - Bx$, in the primitive equation, we have

$$\text{solidity} = (y^2 + 2A + \tfrac{1}{2}Bx)\tfrac{1}{3}\pi x. \quad \text{(II.)}$$

Obtain, from the particular equation of the curve whose rotation produces the specified solid, the values of A, B, and C, and substitute them in either equa. I. or II. (whichever most simplifies the operation); so shall a theorem for the particular case be obtained.

1. *For the Cone.*

Here the equation of the right line is $y = ax$, whence $y^2 = a^2x^2$; therefore $A = 0$, $B = 0$, $C = a^2$. Hence equa. II. becomes $y^2 \times \frac{1}{3}\pi x = \frac{1}{3}\pi xy^2$, solidity.

2. *For the Frustum of a Cone.*

Let y be the radius of the larger bounding circle, r that of the smaller; then $y - r = ax$, and $y^2 - 2ry + r^2 = a^2x^2$,

$$\text{or } y^2 - 2r(r + ax) + r^2 = a^2x^2,$$
$$\text{or } y^2 - 2r^2 - 2arx + r^2 = a^2x^2,$$
$$\text{and } y^2 = r^2 + 2arx + a^2x^2.$$

Here $A = r^2$, $B = 2Ar$, $C = a^2$: and, by substituting in equa. II., we have $(y^2 + 2r^2 + arx)\frac{1}{3}\pi x =$ solidity; or, putting for ax, its value $y - r$, $(y^2 + 2r^2 + (y - r)r)\frac{1}{3}\pi x = (y^2 + 2r^2 + ry - r^2)\frac{1}{3}\pi x = (y^2 + ry + r^2)\frac{1}{3}\pi x$, solidity.

3. *For the Cylinder.*

If $y = r$, the conic frustum becomes a cylinder, and the preceding expression becomes $3y^2 \times \frac{1}{3}\pi x = \pi y^2 x$, solidity.

4. *For a Spheric Segment.*

Here, if r be the radius of the sphere, $y^2 = 2rx - x^2$. Hence $A = 0$, $B = 2r$, and $C = -1$. These values of the coefficients being substituted for them in equations I. and II., we have

$$\left.\begin{array}{r} (y^2 + \frac{1}{3}x^2)\frac{1}{2}\pi x \\ \text{or } (y^2 + rx)\frac{1}{3}\pi x \end{array}\right\} \text{solidity.}$$

5. *For the whole Sphere.*

Here $x = 2r = d$, and y^2 vanishes: so that the last of the preceding values becomes $rx \times \frac{1}{3}\pi x = \dfrac{2r^2 \cdot 2\pi r}{3} = \pi r^3 = \frac{4}{3} \cdot \frac{1}{8}\pi d^3 = \frac{1}{6}\pi d^3$, solidity.

6. *For the Segment of a Spheroid.*

Let T and K be the transverse and conjugate arcs. Then $y^2 = \dfrac{K^2}{T^2}(tx - x^2) = \dfrac{K^2x}{T} - \dfrac{K^2x^2}{T^2}$: where $A = 0$, $B = \dfrac{K^2}{T}$, and $C = -\dfrac{K^2}{T^2}$. Substituting these in equation II., we have $(y^2 + \dfrac{K^2}{2T}x)\frac{1}{3}\pi x$, solidity.

7. *For the whole Spheroid.*

Here $x = T$, y vanishes, and the preceding expression becomes

$$\frac{K^2T}{2T} \cdot \tfrac{1}{3}\pi T = \tfrac{1}{6}\pi K^2T = \tfrac{2}{3} \text{ circumscrib. cylinder.}$$

8. *For the middle Frustum of a Spheroid.*

Let $2t$ = transverse, $2c$ = conjug.; then $y^2 = c^2 - \frac{c^2}{t^2}x^2$.

Hence $A = c^2$, $B = 0$, $C = -\frac{c^2}{t^2}$. These values substituted in equa. II. give $(y^2 + 2c^2)\,\pi x$, solidity.

9. *For a Paraboloidal Frustum.*

Let r = radius of the lesser base, then $y^2 = r^2 + px$. Here $A = r^2$, $B = p$, $C = 0$: and these substituted in equa. I. give $(r^2 + y^2)\frac{1}{2}\pi x$, solidity.

10. *For a complete Paraboloid.*

Here $y^2 = px$, or r vanishes; and the above becomes $\frac{1}{2}\pi y^2 x$ = solidity.

11. *For the Hyperboloidal Segment.*

Let $2t$ = transverse, $2c$ conjugate, then $y^2 = \frac{2c^2x}{t} + \frac{c^2}{t^2}x^2$.

Here $A = 0$, $B = \frac{2c^2}{t}$, $C = \frac{c^2}{t^2}$. And from equation II. $(y^2 + \frac{c^2}{t}x)\frac{1}{3}\pi x =$ solidity of the segment.

MECHANICS.

Definitions and Preliminary Notions.

1. *Mechanics* is the science of equilibrium and of motion.

2. Every cause which moves, or tends to move a body, is called a *force.*

3. When the forces that are applied simultaneously to a body, destroy or annihilate each other's effects, then there is *equilibrium.*

4. *Statics* has for its object the equilibrium of forces applied to *solid* bodies.

5. By *Dynamics* we investigate the circumstances of the motion of solid bodies.

6. *Hydrostatics* is the science in which the equilibrium of fluids is considered.

7. *Hydrodynamics* is that in which the circumstances of their motion are investigated.

According to this division, *Pneumatics*, which relates to the properties of *elastic fluids*, is a branch of *Hydrostatics.*

For farther elucidation the following definitions, also, may advantageously find a place here, viz.

8. The *magnitude of a body* is the volume of space contained within its boundaries; and its determination is effected by the rules for mensuration.

Body is either Hard, Soft, or Elastic. A Hard Body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A Soft Body is that whose parts yield to any stroke or impression, without restoring themselves again; the figure of the body remaining altered. And an Elastic

Body is that whose parts yield to any stroke, but which presently restore themselves again, and the body remains the same figure as before the stroke.

We know of no bodies that are absolutely, or perfectly, either hard, soft, or elastic; but all partaking these properties, more or less, in some intermediate degree.

Bodies are also either Solid or Fluid. A Solid Body is that whose parts are not easily moved among one another, and which retains any figure given to it. But a Fluid Body is that whose parts yield to the slightest impression, being easily moved among one another; and its surface, when left to itself, is always observed to settle in a smooth plane at the top.

9. The *density of a body* is the quantity of matter contained within a volume of given dimensions; as for instance a cubic foot. It is proportional to the *gravity* or *weight* of the body, and is usually measured by means of it.

[Some one kind of body is taken as a standard,—as a cubic foot of water, which at the temperature of 40° of Fahrenheit's thermometer weighs 1000 avoirdupois ounces. This weight is taken as the *unit* of weights; and all other bodies are expressed in terms of this unit.]

Thus, in two spheres, or cubes, &c. of equal size or magnitude; if the one weigh only one pound, but the other two pounds; then the density of the latter is double the density of the former; if it weigh three pounds, its density is triple; and so on.

10. The *mass of a body* is the quantity of matter in it: and it is estimated by multiplying the number of units of magnitude into the number of units of density.

11. Motion is a continual and successive change of place.—If the body move equally, or pass over equal spaces, in equal times, it is called Equable or Uniform Motion. But if it increase or decrease, it is Variable Motion; and it is called Accelerated Motion in the former case, and Retarded Motion in the latter.—Also, when the moving body is considered with respect to some other body at rest, it is said to be Absolute Motion. But when compared with others in motion, it is called Relative Motion.

12. Velocity, or Celerity, is an affection of motion, by which a body passes over a certain space in a certain time. Thus, if a body in motion pass uniformly over 40 feet in 4 seconds of time, it is said to move with the velocity of 10 feet per second; and so on.

13. All forces are measured by the effects which they, unobstructed by foreign forces, *produce*; or, when obstructed, the effects which they *tend to produce*.

14. Momentum, or Quantity of Motion, is the power or force in moving bodies, by which they continually tend from their present places, or with which they strike any obstacle that opposes their motion.

15. Forces are distinguished into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percussion, or for a certain time as a permanent force.

16. Accelerative, or Retardive Force, is commonly understood to be that which affects the velocity only: or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly, and to the mass or body moved inversely. So, if a body of 2 pounds weight be acted on by a motive force of 40; then the accelerating force is 20. But if the same force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter case is only 10; and so is but half the former, and will produce only half the velocity.

17. Gravity, or Weight, is that force by which a body endeavours to fall downwards It is called Absolute Gravity, when the body is in empty space; and Relative Gravity, when immersed in a fluid.

18. Specific Gravity is the relation of the weights of different bodies of equal magnitude; and so is proportional to the density of the body.

19. *Resistance* is the opposition which one body offers to the motion of another already impressed with a force which, but for that opposition, would move in the direction occupied by the resisting body.

20. *Inertia* is the opposition offered by any body to a change of state by means of a force impressed upon it.

21. *Friction* is the resistance offered by the surface of one body to the motion of another drawn along it, and pressing upon it by means of its weight or force.

NEWTONIAN AXIOMS.

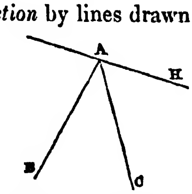
22. Every body has a natural tendency to continue in its present state, whether it be at rest, or moving uniformly in a right line.

23. The change or Alteration of Motion, by an external force, is always proportional to that force, and in the direction of the right line in which it acts.

24. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal changes of motion are produced in bodies acting on each other; and these changes are directed towards opposite or contrary parts.

STATICS.

25. All forces may be represented in *magnitude* and *direction* by lines drawn in the directions in which they act, and taken from any scale, proportional to those forces. Thus, if two forces act upon a body, the point of application of which is A, in the directions AB, AC; and the ratio of the forces be the same with the ratio of AB to AC, the lines AB, AC are said to represent those forces in *magnitude* and *direction*. It is this susceptibility of being represented by lines that brings forces and their operation within the province of Geometry.



26. Forces may also be represented by algebraical symbols, and their direction by trigonometrical ones. Thus the forces above represented may be denoted by b and c, and the angles which their lines of direction make with any given line AH, may be denoted by B and C, or by ϕ and θ respectively. This is analogous to *geometry of co-ordinates*.

27. The name *resultant* is given to a force which is equivalent to two or more forces acting at once upon a point, or upon a body; these separate forces being named *constituents*, *components*, or *composants*.

28. The operation by which the *resultant* of two or more forces applied to the

same point, or line, or body, is determined, is called the *composition of forces;* the inverse problem is called the *decomposition*, or the *resolution of forces.*

In the composition of forces three cases may occur.

1. The forces may all act in the same straight line in either the same or opposite directions.
2. The forces may be applied to different points of the same body, and act in parallel lines.
3. The forces may act in different directions, but be applied to the same point, or in lines tending to the same points.

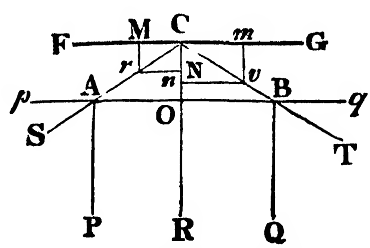
These are all the cases in which equilibrium can take place amongst forces applied to a body.

29. First, *when the Forces act all in the same Straight Line.*

The resultant of two or more forces which act upon the same line, in the same direction, is equal to their sum; and if some forces act in one direction, and others in a direction immediately opposite, the resultant will be equal to the excess of the sum of the forces which act in one direction above the sum of those which act in the opposite direction.

30. Second—*The Composition and Resolution of Parallel Forces.*

Prop. If to the extremities of an inflexible right line AB, are applied two forces, P, and Q, whose directions are parallel and whose actions concur:—1st, The direction of the resultant, R, of those two forces is parallel to the right lines AP, BQ, and is equal to their sum. 2dly, That resultant divides the line AB into two parts reciprocally proportional to the two forces.



1. It is manifest that if two new forces, p and q, equal and in opposite directions are applied to the line AB, they will make no change in the state of the system; so that the resultant of the four forces p, q, P, Q, will be the same as that of the resultant of the two original forces, P, Q. Suppose, now, that S is the resultant of the two forces p, P, while T is that of the two forces q, Q. Moreover, the conjoint action of the forces P and p must produce a resultant lying in the plane PAp, and within the angle PAp: and in like manner the resultant of Q and q must lie in the plane and within the angle QBq. But the lines AP, BQ are parallel, by hypothesis, and hence are in the same plane; and the line AB is in the same plane, since two of its points A and B are in it. Hence the two lines AS, BQ are in the same plane.

Again, the lines AP, BQ being parallel, the angles PAB, QBA are together equal to two right angles, and hence the angles SAB, TBA are greater than two right angles. The lines SA, TB, being produced, will, therefore, meet in some point C.

It is obviously of no consequence, so far as effect is concerned, whether we suppose the force S to be applied at A or at C; and similarly the force T may be applied either at B or C. Let us then conceive them both to be applied at C in the directions CA, CB. Through this point let FG be drawn parallel to AB, and suppose each of the forces S and T resolved into two forces directed respectively in FG and CR, which can always be done, since we only restore the

components P, p and Q, q for the resultants S and T. The forces, according to FG, being equal to p and q respectively, and applied in opposite directions, destroy each other's effects: the remaining forces, therefore, lying the same way on CR must be added together for the resultant, by the first case, which thus is equal to P + Q. The first part of the proposition is therefore proved.

2. In order to establish the second part of the proposition, let MC, CN, be lines in proportion to each other as the forces p, P; and mC, Cn, respectively proportional as q, Q: and draw Nr, nv, parallel to AB.

Then, by the sim. triangles CNr, COA; Cnv, COB $\left\{ \begin{array}{l} P : p :: CN : Nr :: CO : OA \\ q : Q :: nv \;\; nC : OB : OC \end{array} \right.$

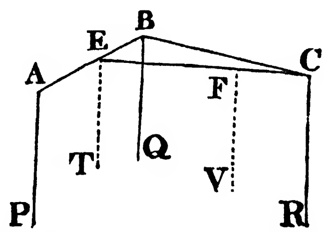
Consequently, $P . q : p . Q :: CO . OB \;\; CO . OA$.

or, since $p = q$, it is reduced to $P : Q :: OB \;\; OA$. Q. E. D.

Corol. 1. If P = Q, BO = OA.

Corol. 2. When a single force R is applied to a point O, of an inflexible straight line AB, we may always resolve it, or conceive it resolved, into two others, which being applied to the two points A and B, in directions parallel to R, shall produce the same effect.

31. Prop. Any number of parallel forces, P, Q, R, S, &c. acting in the same sense, and their points of application being connected in an invariable manner; to determine their resultant.



Determining first, by the preceding prop. the resultant T of two of the forces P and Q, we shall have T = P + Q;

$$P + Q : Q :: AB : AE.$$

Thus, we may substitute for the forces P and Q. the single force T whose value and point of application are known. Draw EC from that point of application to the point C, at which another force, R, is applied. Compounding the forces T and R, their resultant V will be = T + R = P + Q + R; and its point of application, F, such that

$$P + Q + R : R :: EC : EF.$$

A similar method may, obviously, be pursued for any number of parallel forces: the equations assuming this character—

$$Rx = Fd + F'd' + F''d'' \text{ \&c.} \quad Ry = F\delta + F'\delta' + F''\delta'' \text{ \&c.}$$

$$R = F + F' + F'' + \text{\&c.}$$

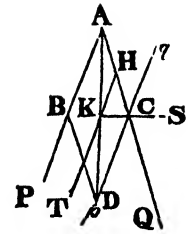
32. If parallel forces act in opposite directions; some, for example, upwards, others downwards; find the resultants of the first and of the second class separately, the general resultant will be expressed by the difference of the two former.

33. The point through which the resultant of parallel forces passes, is called *the centre of parallel forces*. If the forces, without ceasing to be respectively parallel, and without changing either their magnitudes or their points of application, assume another general direction, the centre of those forces will still be the same, because the magnitudes and relations, on which its *position* depends, remain the same.

34. Third: *Of Concurring Forces, or forces applied at the same point.*

Prop. The resultant of two forces P and Q acting in one plane, will be represented in direction and in magnitude, by the diagonal of the parallelogram constructed on the directions of those forces.

1. *In direction.* Take, on the directions AP, AQ, of the forces P, Q, distances AB, AC, proportional to those forces respectively. Suppose that the force Q is applied at the point C, and that at the same point two other forces p, q, equal to each other, act *in opposite directions,* parallel to the line AP, each of those forces being, also equal to Q.



The effect of the four forces P, Q, p, q, will evidently be the same as that of the primitive forces P, Q; since the other two annihilate each other's effects.

The forces Q, q, will have a resultant S, whose direction CS, will bisect the angle, QCq, made by the direction of the other two; since no reason can be assigned why it should be more inclined toward one than the other.

The forces P, p, acting in parallel directions, would have a resultant, T, whose direction TH (art. 30.) would be parallel to them, and pass through a point, H, such that P : p :: HC : HA.

Now, the point K, where the directions CS, TH of these two resultants intersect, will evidently be a point in the direction of the resultant of the *four* forces P, p, Q, q; and, consequently, of the original forces P, Q.

But the triangle CHK is isosceles: for, since HT, Cp, are parallel, the alternate angles DCK, HKC, are equal, and DCK, HCK, are equal, because SC bisects the angle QCq: hence, HCK = HKC, and HK = HC.

But, from what has preceded, P : Q :: HC : HA; and therefore

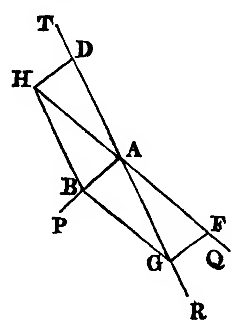
P : p or Q :: HK : HA.

From B drawing BD parallel to AC, we shall have

P : Q :: AB : AC :: CD : AC,
whence CD : AC :: HK : HA;

a proportion which indicates that the three points, A, K, D, are in one line,—viz. the diagonal of the parallelogram ABCD, by the construction.

2. *In magnitude.* For, with regard to the forces P, Q, represented in magnitude and direction by AB and AF, let T be opposed to those two forces so as to keep the whole system in equilibrio: then it will of necessity be equal and opposite to their resultant, R, whose direction is AG. Now, if we suppose that the force Q is in equilibrio with the two forces P and T (which is consistent with our first hypothesis) the resultant of these latter will fall in the prolongation of QA, and will be represented by AH = AF. Also, if HD be drawn parallel to AB, and HB be joined, it will be equal and parallel to AG; and we shall have

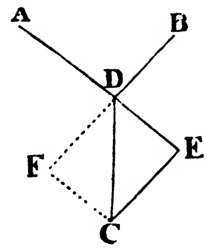


P : T :: AB : AD.

Consequently, since AB represents, or measures, the force P, AD will represent or measure the force T; and as that force is in equilibrio with the two forces P and Q, or with their resultant, R, this latter will be represented by AG = AD; that is, by the diagonal of the parallelogram ABGF. Q. E. D.

35. *Corol.* 1. If three forces, as A, B, C, acting simultaneously in the same plane, keep one another in equilibrio, they will be respectively proportional to the three sides, DE, EC, CD, of a triangle which are drawn parallel to the directions of the forces AD, DB, CD.

For, producing AD, BD, and drawing CF, CE, parallel to them, then the force in CD is equivalent to the two AD, BD, by the supposition; but the force CD is also equivalent to the two ED and CE or FD; therefore, if CD represent the force C, then ED will represent its opposite force A, and CE, or FD, its opposite force B. Consequently the three forces A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

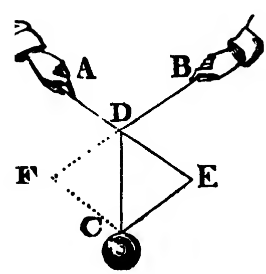


36. *Corol.* 2. Because the three sides CD, CE, DE, are proportional to the sines of their opposite angles E, D, C; therefore the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the direction of the other two.

37. *Corol.* 3. The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of any other triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

38. *Corol.* 4. If any number of forces be kept in equilibrio by their actions against one another; they may be all reduced to two equal and opposite ones.—For, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane: and so on, till at last they are all reduced to the action of only two opposite forces; which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.

39. *Corol.* 5. If one of the forces, as C, be a weight, which is sustained by two strings drawing in the directions DA, DB; then the force or tension of the string AD is to the weight C, or tension of the string DC, as DE to DC; and the force or tension of the other string BD, is to the weight C, or tension of CD, as CE to CD.



40. *Corol.* 6. Since in any triangle CDE we have, by the principles of trigonometry,

$$DC^2 = DE^2 + EC^2 - 2DE \cdot EC \cos. DEC,$$

it follows, that if F, f, be two forces that act simultaneously in directions which make an angle A, then we may find the magnitude of the resultant, R, by the equation

$$R = \sqrt{(F^2 + f^2 + 2Ff \cos. A)}.$$

EXAMPLES.

1. Two forces whose intensities are 10 and 12 act upon a point, and make angles with a given line of 25° and 30°. Find the magnitude and position of the resultant, as well when the directions are both on the same side of the given line, as when they are on different sides.

2. Three equal forces act in the same plane, and under equal angles of 20°; find the magnitude of their resultant.

3. The resultant of two forces is 25, and the angles which it forms with them are 30° and 60° respectively. Find those components.

4. The ratio of three forces is 3 : 4 : 5, the last of which is the resultant of the other two, and their sum is 96 : find the angles which the components make with the resultant, and the magnitudes of all three.

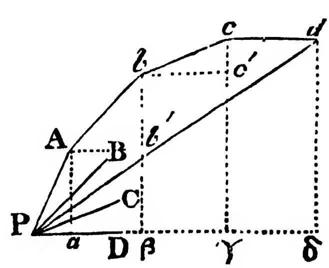
5. Show that if three forces, whose directions concur in one point, are represented by the three contiguous edges of a parallelopiped, their resultant will be represented, both in magnitude and direction, by the diagonal drawn from the point of concourse, to the opposite angle of the parallelopiped.

6. Prove that if any number of forces be applied to a point, so as to produce equilibrium, they can be reduced to two equal and opposite forces, estimated along *any* given line passing through that point.

41. [*Remark.*—The properties, in this proposition and its corollaries, hold true of all similar forces whatever, whether they be instantaneous or continual, or whether they act by percussion, drawing, pushing, pressing, or weighing; and are of the utmost importance in mechanics and the doctrine of forces.]

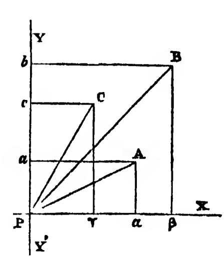
42. Prop. To find the resultant of several forces concurring in one point, and acting in one plane.

1st. *Graphically.*—Let, for example, four forces, A, B, C, D, act upon the point P, in magnitudes and directions represented by the lines PA, PB, PC, PD.



From the point A draw Ab parallel and equal to PB; from b draw bc parallel and equal to PC; from c draw cd parallel and equal to PD: and so on, till all the forces have thus been brought into the construction. Then join Pd, which will represent both the magnitude and the direction of the required resultant.

This is, in effect, the same thing as finding the resultant of two of the forces A and B; then blending that resultant with a third force C; their resultant with a fourth force D, and so on; as will be evident on completing the several parallelograms.



2d. *By computation.*—Take any two lines at right angles to one another, as XX′ and YY′, passing through P, the point of the application of the forces. Let PA, PB, PC, &c. be the directions of the forces, and proportional to them in magnitudes. Complete the parallelograms $a\alpha$, $b\beta$, $c\gamma$, &c having their sides parallel to the two lines XX′ and YY′.

Then the positions of the two lines XX′ and XX′ being given with respect to one of the forces, the angles which those directions form with the axes of co-ordinates XX′ and YY′ are also given. But PA is the resultant of Pα and Pa, PB of Pβ, Pb, PC of Pγ, Pc, &c.: and the forces PA, PB, PC, &c. are resolved into pairs of components, which are in the

direction of the two lines XX′ and YY′. Hence Pa + Pb + Pc, is the sum of the forces which act in the direction YY′, and Pa + Pβ + Pγ is the sum of those which act in the direction of XX′. We have, therefore, to calculate this sum for the given case of the direction and magnitude of the forces.

Now Pa = Aa = PA sin. APa, Pa = PA cos. APa, and so of the others. Hence the sum of the forces resolved in XX′ is equal to

$$\text{A. cos. AP}\alpha + \text{B. cos. BP}\beta + \text{C. cos. CP}\gamma + \ldots\ldots\,;$$

and the sum of those resolved in YY′ is similarly equal to

$$\text{A. sin. AP}\alpha + \text{B. sin. BP}\beta + \text{C. sin. CP}\gamma + \ldots\ldots\ldots$$

The resultant of these two forces, thus calculated, in reference to the rectangular co-ordinates (XX′, YY′) is easily obtained.

Had we employed the construction in the last article, we should have had those sums respectively equal to Pδ and δd: and then

$$\frac{d\delta}{\text{P}\delta} = \text{tan. } d\text{P}\delta, \text{ and P}d = \text{P}\delta \text{ sec. } d\text{P}\delta.$$

Had the lines of reference, or co-ordinate axes, not passed through P, still the same process is applicable by means of the formulæ given at pp. 192, 3, for the transformation of co-ordinates.

In a manner precisely similar, if the forces be not all in the same plane, we may resolve each of them into three forces parallel to three axes of co-ordinates at right angles to one another; and then find the magnitude and position of the diagonal of the rectangular parallelopiped on whose sides are the three sums of the components parallel to each axis *.

Had the axes of co-ordinates been oblique instead of rectangular, the same mode of composition would have been applicable; but the actual calculation more complex and laborious.

The numerical computation of the first part generally is best effected by means of a table of natural sines and cosines. The latter by logarithms.

Numerical example. Let the four forces A, B, C, D, be respectively 25, 28, 30, and 20 ; and the angles APB, BPC, CPD, each 20°. Required the magnitude and direction of the resulting force, and the angle which it makes with each force.

43. *Remark.* Connected with this subject is the doctrine of ***moments***; for an elucidation of which, however, the student should consult some of the books written expressly on mechanics, as those by *Marrat*, *Gregory*, or *Poisson*.

* Two of the angles α, β, γ, which a line forms with the three rectangular axes are sufficient for fixing the direction of the line; since they are connected by the equation

$$\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1.$$

The same may be established also by geometrical considerations. Both modes of proof will be a useful exercise for the student.

THE MECHANICAL POWERS, &c.

44. Weight and Power, when opposed to each other, signify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

45. A *Machine*, or *Engine*, is any mechanical instrument contrived to move bodies: and it is composed of the mechanical powers.

46. The *Mechanical Powers*, are certain simple instruments, commonly employed for raising greater weights, or overcoming greater resistances, than could be effected by the natural strength without them. These are usually accounted six in number, viz. the *Lever*, the *Wheel-and-axle*, the *Pulley*, the *Inclined Plane*, the *Wedge* and the *Screw*.

Of these, the *lever*, *wheel-and-axle*, and the *pulley*, form one class, dependent on the same principles; and the other class comprises the inclined *plane*, the *wedge*, and the *screw*, which depend upon another set of principles slightly different from the former.

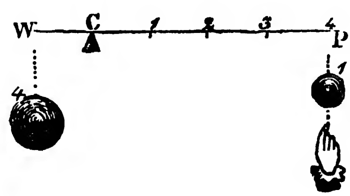
47. The *Centre of Motion*, is the fixed point about which a body moves; and the *Axis of Motion*, is the fixed line about which it moves.

48. The *Centre of Gravity*, is a certain point, on which a body being freely suspended, it will rest in any position.

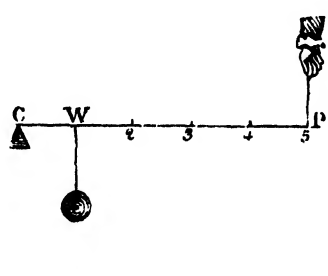
OF THE LEVER.

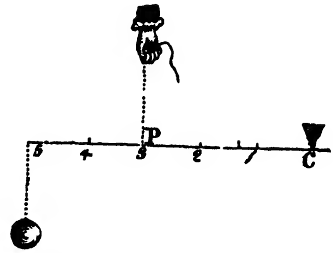
49. A Lever is any inflexible rod, bar, or beam, which serves to raise weights, while it is supported at a point by a fulcrum or prop, which is the centre of motion. The lever is supposed to be void of gravity or weight, to render the demonstrations easier and simpler. There are three kinds of levers.

50. A Lever of the First kind has the prop C between the weight W and the power P. And of this kind are balances, scales, crows, hand-spikes, scissors, pincers, &c.

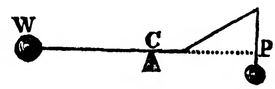


51. A Lever of the Second kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.





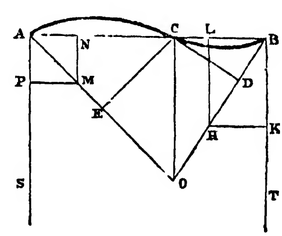
52. A Lever of the Third kind has the power between the weight and the prop. Such as tongs, the bones and muscles of animals, a man rearing a ladder, &c.



53. Any of these kinds of levers may be entirely straight, or any way curved or bent between their fulcra and the points of application of the forces. Thus, a lever of the first kind is exhibited in the annexed figure; such as a hammer drawing a nail, &c. This is sometimes, though improperly, called a fourth kind of lever.

54. In all these instruments the power may be represented by a weight, which is its most natural measure, acting downward; but having its direction changed, when necessary, by means of a fixed pulley.

55. Prop. When the weight and power keep the lever in equilibrio, they are to each other reciprocally as the distances of their lines of direction from the prop. That is, P : W :: CD : CE; where CD and CE are perpendicular to BO and AO, the directions of the two weights, or the weight and power at B and A.



Let A and B be the points of application, C the fulcrum, and AO, BO the directions of the forces.

In AO, BO take AM and BH proportional to the two forces acting at A and B in the directions of those lines, and draw AS, HL, BT, MN perpendicular to AB, and HK, MP parallel to it. Draw also the perpendiculars CE, CD, from C to AO and BO.

Then the forces AM, BH, being each resolved into two, parallel and perpendicular to the line ACB, will be proportional to AN, AP and BL, BK respectively. The parts AN, BL, acting in the direction of the line ACB, do not tend to turn the lever about C; and the effective forces are therefore BK and AP.

Now AP = AM cos. PAM = AM sin. CAO $= f$,

And BK = HB cos. HBK = HB sin. CBO $= f_,$ suppose, which are the forces acting perpendicularly on the ends of the lever.

But when equilibrium takes place, we have

CA $. f =$ CB $. f_,$ or in this case

CA . AM sin. CAO = CB . BH sin. CBO, or

AM . CA sin. CAO = BH . CB sin. CBO; or finally,

AM . CE = BH . CD, or P : W :: CD : CE.

56. *Cor.* 1. When the angle A is = the angle B, then is CD : CE :: CB : CA :: P : W. Or when the two forces act perpendicularly on the lever, as two weights, &c.; then, in case of an equilibrium, D coincides with B, and E with P; consequently then the above proportion becomes also P : W :: CB :

CA, or the distances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to those forces.

Cor. 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the sine of the angle of direction CAE. For the perp. CE is as CA . sin. $\angle$ A.

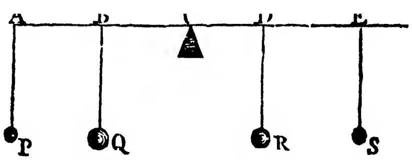
Cor. 3. Because the product of the extremes is equal to the product of the means, therefore the product of the power into the distance of its direction, is equal to the product of the weight into the distance of its direction.

That is, P . CE = W . CD.

Cor. 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will describe circles whose radii are the distances CD, CE; and since the circumferences or spaces described are as the radii and also as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the masses and velocities, are as the masses and radii; that is, as P . CE and W . CD, which are equal by cor. 3.

Cor. 5. In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop, any one is as the distance of the other two.

Cor. 6. If several weights P, Q, R, S, act on a straight lever, and keep it in equilibrio; then the sum of the products on one side of the prop, will be equal to the sum on the other side, made by multiplying each weight by its distance; namely, P . AC + Q . BC = R . DC + S . EC.



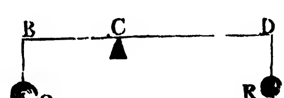
For, the effect of each weight to turn the lever, is as the weight multiplied into its distance; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal. The same would also follow from art. 31.

Cor. 7. If the intensities of any number of forces acting on a lever be given, and the distances of their points of application from a given point, the position of the fulcrum may be computed.

For, let F be the point, FE, FD, FB, FA the distances of the points of application of S, R, Q, P. Then by cor. 6. P . AC + Q . BC = R . DC + S . EC. But

$$AC = AF - FC,\ BC = BF - FC,\ CD = FC - FD,\ CE = FC - FE. \text{ Hence}$$
$$P\,(AF - FC) + Q\,(BF - FC) = R\,(FC - FD) + S\,(FC - FE), \text{ or}$$
$$P\,.\,AF + Q\,.\,BF + R\,.\,DF + S\,.\,EF = FC\,(P + Q + R + S)$$
$$\text{or } FC = \frac{P\,.\,AF + Q\,.\,BF + R\,.\,DF + S\,.\,EF}{P + Q + R + S}.$$

Cor. 8. Because, when two weights Q and R are in equilibrio, Q : R :: CD : CB;



therefore, by composition, Q + R : Q :: BD : CD,
and, Q + R : R :: BD : CB.

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.

57. *Ex.* 1. Suppose CA (fig. to Cor. 6) $= 9$, CB $= 5$, CD $= 4$, CE $= 10$, P $=$ 3lbs. Q $=$ 10lbs. R $=$ 7lbs. Required S when the whole is in equilibrio.

Ex. 2. The arms of a bent lever are equal in length, but make an angle of 135°. Suppose the lever to rest upon a fulcrum at its angular point, what must be the ratio of the weights suspended at the ends of the two arms, so that one arm shall be horizontal? And what will be the position of the lever (in a state of balanced rest) when 5lbs. are suspended at the end of one arm, and 3lbs. at that of the other?

Ex. 3. A bent lever whose arms are inclined at an angle of 60° are to one another as $10 : 7$. It was loaded with a weight of 100lbs. at the end of the longer arm, and by means of a weight at the other extremity, was brought into equilibrium with the legs equally inclined to the horizon, what was that weight?

Ex. 4. A straight lever resting on a fulcrum one-third of its own length from one extremity B, is pressed by forces of 40 and 70 lbs. at A and B respectively. The first force makes with the prolongation of the lever an angle of 120°: what must be the direction of the other force to keep the lever horizontal?

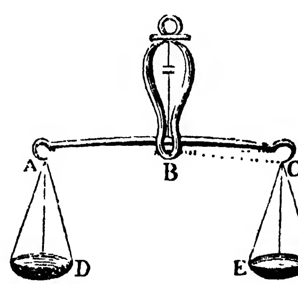
Ex. 5. The distances of the points A, D, E from the fulcrum of the lever on one side are $\frac{19}{4}$, $\frac{5}{2}$, and $\frac{5}{4}$, and the distances of B and F on the other side are 10 and $\frac{13}{2}$. At A, D, E, F are placed the weights 10, 2, 5, and 18, and at B a force is applied in a direction which would meet the vertical line through the fulcrum at the distance 20 from that fulcrum. What was that force? And did its line of direction meet the vertical above or below the fulcrum?

6. A bent lever whose arms are as $10 : 7$ and inclined to one another in an angle of 127° 15′ is loaded with a weight of 12·5 lbs. on the longer arm, and made by means of a weight at the other arm to take such a position that the longer arm shall make with the vertical line through the fulcrum an angle double of that which the shorter arm makes. Find that weight.

7. A given segment of a circle a is at liberty to *roll* on a horizontal line so as to take a position of equilibrium, when to the extremities are attached two weights which are in the ratio of b to c. Find the point on which it rests.

SCHOLIUM.

58. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of goods. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:



1*st*, That the points of suspension of the scales and the centre of motion of the beam, A, B, C, should be in a straight line: 2*d*, That the arms AB, BC, be of an equal length: 3*d*, That the centre of gravity be in the centre of motion B, or a little below it: 4*th*, That they be in equilibrio when empty: 5*th*, That there be as little friction as possible at the centre B. A defect in any of these properties, makes the scales either imperfect or false. But it often happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibrio when empty; but when they are charged with any weights, so as to

be still in equilibrio, those weights are not equal; but the deceit will be detected by changing the weights to the contrary sides, for then the equilibrium will be immediately destroyed.

59. To find the true weight of any body by such a false balance:—First weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these two weights, will be the true weight required. For, if any body b weigh W pounds or ounces in the scale D, and only w pounds or ounces in the scale E: then we have these two equations, namely,

$$AB \cdot b = BC \cdot W,$$
$$\text{and } BC \cdot b = AB \cdot w;$$

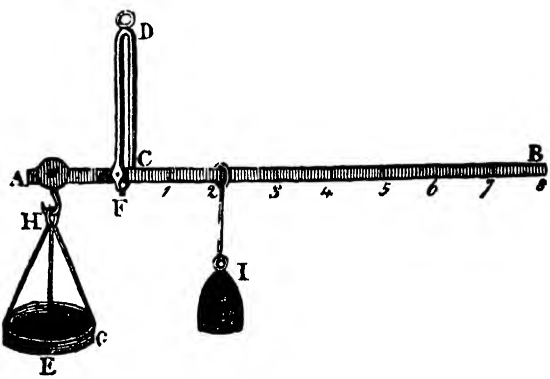
the product of the two is $AB \cdot BC \cdot b^2 = AB \cdot BC \cdot Ww$;

hence then . . . . $b^2 = Ww$,

and . . . . $b = \sqrt{Ww}$,

the mean proportional, which is the true weight of the body b.

60. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived, that one weight only may serve to weigh a great many, by sliding it backward and forward, to different distances, on the longer arm of the lever; and it is thus constructed:



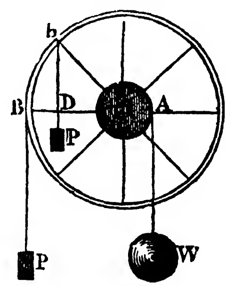
Let AB be the steelyard, and C its centre of motion, whence the divisions must commence if the two arms just balance each other: if not, slide the constant moveable weight I along from B towards C, till it just balance the other end without a weight, and there make a notch in the beam, marking it with a cypher 0. Then hang on at A a weight W equal to I, and slide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and sliding I back to balance it, there mark it with 2. Do the same at 3, 4, 5, &c. by making W equal to 3, 4, 5, &c times 1; and the beam is finished. Then, to find the weight of any body b by the steelyard: take off the weight W, and hang on the body b at A; then slide the weight I backward and forward till it just balance the body b, which suppose to be at the number 5; then is b equal to 5 times the weight of I. So, if I be one pound, then b is 5 pounds; but if I be 2 pounds, then b is 10 pounds; and so on.

61. In all calculations respecting the steelyard, the weights of the unequal arms must be taken into account, and treated as weights placed at the middles of the arms, when, as is generally the case, they are of the same thickness through their whole length. Where they are not equally thick through their whole length, their centres of gravity (see a future page) must be found, and their masses conceived to act at those points.

OF THE WHEEL AND AXLE.

62. PROP. In the wheel-and-axle, the weight and power will be in equilibrio when the power P is to the weight W reciprocally as the radii of the circles where they act; that is, as the radius of the axle CA, where the weight hangs, to the radius of the wheel CB, where the power acts That is, P : W :: CA : CB.

Here the chord, by which the power P acts, goes about the circumference of the wheel, while that of the weight W goes round its axle, or another smaller wheel, attached to the larger, and having the same axis or centre C. So that BA is a lever moveable about the point C, the power P acting always at the distance BC, and the weight W at the distance CA; therefore P : W :: CA : CB.



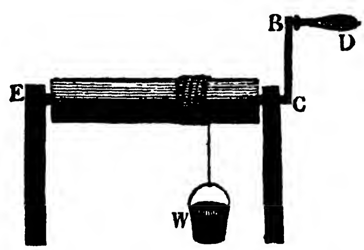
This is evidently only a variety of the lever so contrived as to always keep the arms of the same lengths respectively.

63. *Corol* 1. If the wheel be put in motion; then, the spaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much slower, as it is heavier than the power; so that what is gained in power, is lost in time. And this is the universal property of all machines and engines.

64. *Corol.* 2 If the power do not act at right angles to the radius C*b*, but obliquely; draw CD perpendicular to the direction of the power; then, by the nature of the lever, P : W :: CA : CD.

SCHOLIUM.

65. To this mechanical power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or spindle CE, by the handle CBD; the power applied at B is to the weight W on the roller as the radius of the roller is to the radius CB of the handle.



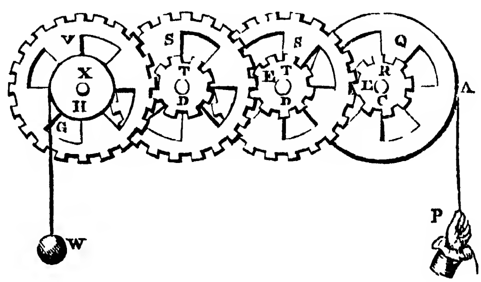
66. And the same for all cranes, capstans, windlasses, and such like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; so that they are always in the reciprocal ratio of their velocities And to the same principle may be referred the gimblet and auger for boring holes.

67. All this, however, is on supposition that the ropes or cords, sustaining the weights, are of no sensible thickness. For, if the thickness be considerable, or if there be several folds of them, over one another, on the roller or barrel; then we must measure to the middle of the outermost rope, for the radius of the roller; or, to the radius of the roller, we must add half the thickness of the cord, when there is but one fold.

68. The wheel and axle has a great advantage over the simple lever, in point of convenience. For a weight can be raised but a little way by the lever;

whereas, by the continual turning of the wheel and roller, the weight may be raised to any height, or from any depth.

69. By increasing the number of wheels, too, the power may be multiplied to any extent, making always the less wheels to turn greater ones, as far as we please: and this is commonly called tooth-and-pinion work, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in case of an equilibrium, the power is to the weight, as the continual product of the radii of all the axles to that of all the wheels. So, if the power P turn the wheel Q, and this turn the small wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V, and this turn the axle X, which raises the weight W.



Then if W_1 denote the force exercised by the pinion C upon the wheel D, W_2 the force exercised by the pinion D upon the wheel F, and so on through any number of corresponding wheels and pinions, till we come to the axle F and weight W, we shall have

$$\left.\begin{array}{l} P : W_1 :: CE : CA \\ W_1 : W_2 :: DF : DE \\ W_2 : W :: GH : HF, \&c. \end{array}\right\} \text{Hence, by composition of ratios,}$$

$$P : W :: CE . DF . GH, \&c. : CA . DE . HF, \&c.;$$

$$\text{or } \frac{P}{W} = \frac{\text{continued product of the radii of all the pinions}}{\text{continued product of the radii of all the wheels}}.$$

And in the same proportion is the velocity of W slower than that of P. Thus, if each of three wheels be to its axle, as 10 to 1; then $P : W :: 1^3 : 10^3$ or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then, when put in motion, the power will move 1000 times faster than the weight.

EXAMPLES.

1. Let the length of the handle CB (art. 65.) be 15 inches, the radius of the axle 3 inches. Required P the power to balance a weight W of 180lbs.

2. Let the radii of the four axles (in the preceding figure) be 2, 4, 5, and 3 inches respectively; the radii of the wheels 10, 12, 8, and 15, respectively. Then, what weight, W, will be balanced by a power, P, of 30lbs.?

3. A power of 5lbs. keeps in equilibrio a weight of 150lbs., by means of a wheel whose diameter is 10 feet. Find the diameter of the axle.

4. In a combination of wheels and pinions, in which the teeth of each pinion are applied to the circumference of the next wheel, the radii of the several pinions are 1, 1·5, 2, 2·5, and 3, and the radii of the wheels are 8, 10, 12, 15, and 18. What is the ratio between the power and the weight?

5. It is required to add to the last system two equal wheels, and two equal pinions, so that the ratio of the power to the weight may be as $(\cdot 1)^4$ to 23328, what is the ratio of the radii of the added wheels and pinions?

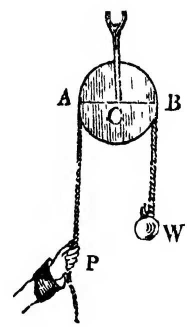
6. A power, P, acting by means of a rope going over a wheel whose diameter

is 6 feet, supports a weight of 320lbs.: the diameter of the axle is 8 inches, and the rope is 2 inches thick. What is the value of P, supposing the thickness of the rope taken into account?

7. There are x wheels and axles, the diameter of each wheel 8 times that of its axle, and the weight sustained is 32768 times the power. What is x?

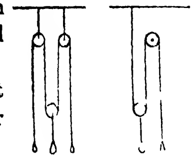
8. Four wheels, A, B, C, D, whose diameters are 10, 8, 6, 4 feet respectively, are put in motion by a power of 100lbs. applied at the circumference of the wheel A: these wheels act upon each other by means of three smaller wheels, whose diameter is 10 inches, and the last wheel turns an axle D whose diameter is 12 inches. What weight may be sustained by a rope going over this axle?

OF THE PULLEY.



70. A Pulley is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined with others to increase the power. It is also either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power. In other words, if while a pulley turns on its axis, that axis remains in its place, the pulley is called *a fixed pulley*; if the axis ascends or descends, it is a *moveable pulley* *. A fixed pulley confers no mechanical *power*, but simply a mechanical *advantage*.

71. Prop. By means of a single moveable pulley, each portion of the thread being vertical, a weight may be supported by two forces, each equivalent to half the weight; or by two threads, each passing over a fixed pulley, and connected with another weight equal to half the first; or one of them connected with such a weight, and the other to a fixed point.



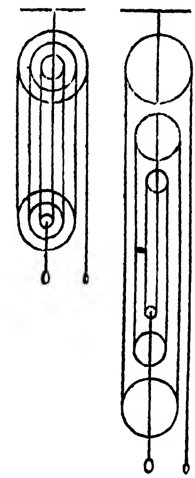
For it is obvious that each thread supports an equal part of the weight, and the substitution of equivalent weights, or of a fixed point, will not impair the equilibrium.

72. Prop. If several moveable pulleys be connected with a weight, and parallel portions of the same thread act upon them all, there will be an equilibrium when the weight attached to the thread is to the weight attached to the pullies, as one to the number of threads at the lower block.

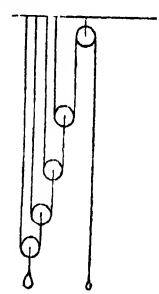
For the force being equably communicated throughout the length of the thread, each portion will co-operate equally in supporting the weight, and will support that portion of it which is to the whole as 1 to the number of threads;

* The comparison of a pulley to a lever, is both unnecessary and imperfect.

consequently a weight equal to that portion will retain any part of the thread in equilbrium, and with it the whole thread, and the whole weight. And if the radii of the pulleys be taken in arithmetical progression, their angular velocity may be made equal, and they may be fixed to the same axis.



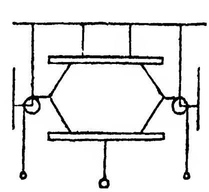
Note.—In pulleys arranged as in the right hand diagram of this prop. the portions of thread or cord passing over the several upper and lower pulleys, would have variations of velocity, not adequately provided for by the sizes of those pulleys; and thus the system would be exposed to great friction. To remedy this, Mr. *White*, of Chevening, proposed the arrangement indicated by the left-hand diagram. The wheels in the upper block, as well as those in the lower, each are attached to one axis, and those wheels or grooves differ in size in proportion to the quantity of rope that must pass over them successively. The successive radii of the coils on the lower block are as the numbers 1, 3, 5, 7, &c., while those on the upper are as 2, 4, 6, 8, &c. The cord, being passed successively over the grooves of such wheels, would be thrown off exactly in the same manner as if every groove were upon a separate wheel, and each wheel revolved independently of the others. This arrangement is good in theory; but there is such considerable difficulty in making the successive grooves precisely of the requisite dimensions, especially as the radius of the cord must in each case be added to that of the pulley, that it is very rarely adopted in practice.



73. Prop. If one end of a thread, supporting a moveable pulley, be fixed, and the other attached to another moveable pulley, and the threads of this pulley be similarly arranged, the weight will be counterpoised by a power which is found by halving it as many times as there are moveable pullies.

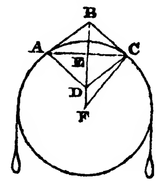
This proposition is evident from a consideration of the figure, and the law of the single moveable pulley.

74. Prop. If two threads be attached to a weight and passed over fixed pulleys, there will be an equilibrium when the distance of the weight from the horizontal line is to its distance from either pulley, as the weight to the sum of the equal forces acting on the threads.



By producing the oblique lines, and crossing them with a vertical one, a triangle will be formed of which the sides will represent the forces; whence the truth of the proposition will appear. And if the weights are unequal, their situation may be determined by the same general law. In the same manner the force may be found, which is requisite for sustaining a weight, by inflecting a thread connected in any manner with it, as by means of a lever.

75. Prop. When a thread is coiled round a cylinder, the pressure on any part of the circumference is to the tension as its length to the radius; when the direction of the line is oblique, the pressure on the whole circumference is to the tension as the circumference to the radius; and the tension of the oblique line is to a force straining it in the direction of the cylinder, as the length of a coil to the length of the axis.



Let AB and BC be tangents of the small arc AC; then if BC, and BA represent the force of tension, at A and C, the diagonal of the parallelogram BD will be the joint result; but BD = 2BE, and by the properties of similar triangles

BE : BC :: EC : CF; BD : BC :: AC : CF.

If the position of the thread be oblique, we shall find by the composition of forces, supposing it uncoiled, and its extremities retained in a line parallel to the axis, that its tension is to a force acting in the direction of the axis, as the oblique length of any portion to its height. Now this tension produces on any small oblique portion of the circumference, a pressure equal to that which would be produced on the corresponding transverse portion by an equal force acting transversely; for the versed sine of the arc is the same in both cases; consequently the pressure on the whole circumference is equal to that which would be produced by the same tension acting transversely.

THE INCLINED PLANE.

76. The Inclined Plane, is a plane inclined to the horizon, or making an angle with it. It is often reckoned one of the simple mechanic powers; and the double inclined plane makes a wedge. It is employed to advantage in raising heavy bodies in certain situations, diminishing their weights by laying them on the inclined planes.

[It may here be premised that a pressure may be regarded as a force so counteracted by another force that no motion ensues. Thus, we continually exert a pressure by means of our weight, upon the ground on which we stand, or the chair on which we sit; but at the instant when we are falling or leaping, we neither exert nor experience any pressure.

It may also be premised as a general principle, that the pressure exercised by any force upon any surface is in the direction of a perpendicular drawn to the plane from the point of application of the force.]

77. Prop. If a weight be sustained on an inclined plane, by a force that acts parallel to the plane, the weight, power, and pressure on the plane are respectively, as the length, height, and base of the plane; or as BC, BA, AC, in the annexed figure.

For draw AE perpendicular to the plane BC or to the line DW. Then we are to consider that the body W is sustained in equilibrio by three forces, viz.—

1. Its own weight acting perpendicular to the horizon AC, or parallel to AB.

2. The power P acting in WD parallel to the plane;

3. The reaction of the plane perpendicular to the surface, or parallel to EA.

But when a body is kept in equilibrio, the forces are proportional to the sides of any triangle which is formed by lines perpendicular to the directions of the forces (art. 37.): and in the present case the sides of the triangle ABC are perpendicular to the directions of the forces, and consequently the three forces are proportional to its sides. That is, denoting the weight, power, and pressure by W, P and p respectively, we have

$$W : P : p :: BC : BA : AC.$$

These may be written as equations thus:

$$\frac{W}{P} = \frac{BC}{BA}, \quad \frac{W}{p} = \frac{BC}{CA}, \text{ and } \frac{P}{p} = \frac{BA}{AC}.$$

Again, since the triangle is right angled at A, we have

$$\begin{aligned} W &= P \text{ cosec. } C = p \text{ sec. } C \\ P &= W \sin. \ C = p \tan. \ C \\ p &= P \cot. \ C = W \cos. C \end{aligned}$$

EXAMPLES.

Ex. 1. A power of 1lb. acting parallel to a plane, supports a weight of 2lbs. Required the inclination of the plane to the horizon, and the pressure upon the plane.

2. The weight, power, and pressure on an inclined plane are respectively as 13, 5, and 12. Required the angle of the plane's inclination to the horizon.

3. Two inclined planes whose lengths are 10 feet and 8 feet are placed together with their summits united, and such that the horizontal distance of their other extremities was 12 feet. A body whose weight was 100lbs. was supported on the shorter plane by a weight resting on the other plane, by means of a pulley at their common summit. What was that weight, the two portions of the connecting cord being respectively parallel to the two planes?

4. A power of p pounds acting parallel to an inclined plane supports a weight of q pounds. Find an expression for the inclination of the plane.

5. A weight of p pounds acting parallel to an inclined plane sustains another of q pounds acting vertically, and the base of the plane was a feet. Determine the height and length of that plane.

6. Two inclined planes have angles of elevation of $\beta_{,}$ and $\beta_{,,}$ degrees, and have their common line of section parallel to the horizon. A weight of p pounds is supported by a weight of q pounds on the other. Find the ratio of p to q.

7. The altitudes of two planes are $a_{,}$ and $a_{,,}$, and their lengths $b_{,}$ and $b_{,,}$, and they are joined at their intersection with the horizon. These planes support a block of stone of c cwt.: what part of its weight is sustained by each plane?

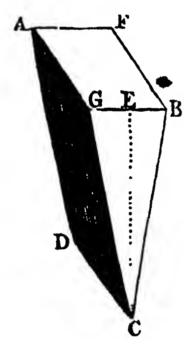
8. When the power, instead of acting parallel to the plane, makes a given angle with it, show that P : W :: sin. inclination of the plane : sin. of the angle which the direction of the power makes with the plane.

78. *Scholium.* The Inclined Plane comes into use in some situations in which the other mechanical powers cannot be conveniently applied, or in combination

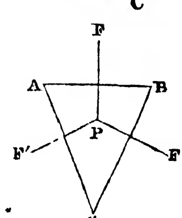
with them. As, in sliding heavy weights either up or down a plank or other plane laid sloping: or letting large casks down into a cellar, or drawing them out of it. Also, in removing earth from a lower situation to a higher by means of wheelbarrows, or otherwise, as in making fortifications, &c., inclined planes, made of boards, are employed. Rail-roads, on inclined planes, serve often to convey coals from the mouth of a mine.

Of all the various directions of drawing bodies up an inclined plane, or sustaining them on it, the most favourable is where it is parallel to the plane BC, and passing through the centre of the weight; a direction which is easily given to it, by fixing a pulley at D, so that a cord passing over it, and fixed to the weight, may act or draw parallel to the plane. In every other position, it would require a greater power to support the body on the plane, or to draw it up. For if one end of the line be fixed at W, and the other end inclined down towards B, below the direction WD, the body would be drawn down against the plane, and the power must be increased in proportion to the greater difficulty of the traction. And, on the other hand, if the line were carried above the direction of the plane, the power must be also increased; but here only in proportion as it endeavours to lift the body off the plane.

THE WEDGE.



79. A wedge is a piece of wood or metal, having a rectangular head as AGBF, two triangular parallel faces, one of which is GBC, and two rectangular faces which meet, one of them being ADCG. Or, it may be regarded as a triangular prism, one of whose bases is GBC. It takes specific names, as isosceles, scalene, &c. from the corresponding character of the triangle.



80. Prop. When three forces acting perpendicularly upon the faces and head of a wedge, meet in a point, they are respectively as the faces upon which their action is exerted.

Let the forces F, F′, F″, acting perpendicularly upon AB, AC, BC, meet in P. Then, by the doctrine of the parallelogram of forces (art. 36.)

$$
\begin{array}{llll}
 & F & : F' & : F'' \\
:: & \sin. F'PF'' & : \sin. FPF'' & : \sin. FPF' \\
:: & \sin. C & : \sin. A & : \sin. B \\
:: & AB & : BC & : AC.
\end{array}
$$

81. If the forces act obliquely upon either or all the faces, let them be reduced to the perpendicular direction; and the proposition will then apply.

82. Scholium.—Various other theories of the wedge are advanced by different

authors, but on account of the irregularities produced by friction, they are but of little value except as geometrical propositions.

In the wedge, the friction against the sides is very great, at least equal to the force to be overcome, because the wedge retains any position to which it is driven; and therefore the resistance is doubled by the friction. But then the wedge has a great advantage over all the other powers, arising from the force of percussion or blow with which the back is struck, which is a force incomparably greater than any dead weight or pressure, such as is employed in other machines. And accordingly we find it produces effects vastly superior to those of any other power; such as the splitting and raising the largest and hardest rocks, the raising and lifting the largest ship, by driving a wedge below it, which a man can do by the blow of a mallet: and thus it appears that the small blow of a hammer, on the back of a wedge, is incomparably greater than any mere pressure, and will overcome it.

OF THE SCREW.

83. The Screw is one of the six mechanical powers, chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

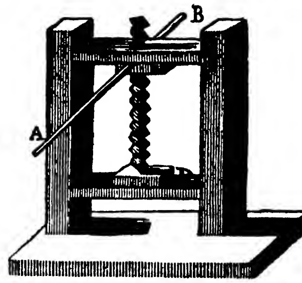
The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

84. Prop. The energy of a power applied to turn a screw round, is to the force with which it presses upward or downward, setting aside the friction, as the distance between two threads, is to the circumference where the power is applied.

The screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the circumference of the screw; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference of the screw; therefore the power is to the pressure, as the distance of two threads is to that circumference. But, by means of a handle or lever, the gain in power is increased in the proportion of the radius of the screw to the radius of the power, or length of the handle, or as their circumferences. Therefore, finally, the power is to the pressure, as the distance of the threads, is to the circumference described by the power.

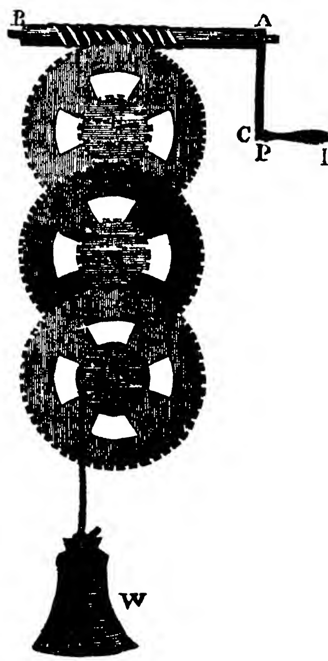
85. *Corol.* When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

SCHOLIUM.



86. Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder; and let the screw be turned by a handle of 4 feet long, from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the screw will press on the board at D, when the man turns the handle at A and B, with his whole force. Then the diameter AB of the power being 4 feet, or 48 inches, its circumference is 48 × 3·1416 or $150\frac{4}{5}$ nearly; and the distance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the pressure, as 1 to $603\frac{1}{5}$; but the power is equal to 150lb.; theref. as 1 : $603\frac{1}{5}$: : 150 : 90480; and consequently the pressure at D is equal to a weight of 90480 pounds, independent of friction.

87. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turns a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle C; supposing the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.



Here 20 × 3·1416 × 2 = 125·664, is the circumference of the power.

And 125·664 to $\frac{1}{2}$, or 251·328 to 1, is the force of the screw alone.

Also, 18 to 2, or 9 to 1, being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1^3, or 729 to 1, is the power gained by the wheels.

Consequently 251·328 × 729 to 1, or $183218\frac{1}{9}$ to 1 nearly, is the ratio of the power to the weight, arising from the advantage both of the screw and the wheels.

But the power is 150lb.; therefore 150 × $183218\frac{1}{9}$, or 27482716 pounds, is the weight the man can sustain, which is equal to 12269 tons weight.

But the power has to overcome, not only the weight, but also the friction of the screw, which is very great, in some cases equal to the weight itself, since it is sometimes sufficient to sustain the weight, when the power is taken off.

88. Upon the same principle the advantage of any other combination of the mechanical powers may be computed: allowance, however, being always to be made for stiffness of cords, friction, and other causes of resistance.

EXAMPLES.

1. A screw, the distance between whose spirals is one inch, is turned on its axis by a lever whose length is two feet, reckoning from the axis of the screw; what force could be sustained by it when a power of 30lbs. acts at the extremity of the lever?

2. A person who could just lift 60lbs. was able by means of a lever 3 feet long acting as the handle of a screw, of two inches diameter, to sustain a ton weight: what is the distance between the threads of that screw, estimated parallel to the axis? and what angle did they make with a line traced on the surface of the cylinder parallel to the axis?

3. The distance between two contiguous threads of a screw is two inches, and the arm to which the power is applied is 20 inches. Determine the power and weight, when a weight of 20lbs. more will just produce equilibrium.

THE CENTRE OF GRAVITY.

89. Def. The Centre of Gravity of any body or system of bodies is that point about which the body or system, acted upon only by the force of gravity, will balance itself in all positions; or it is a point which when supported, the body or system will be supported, however it may be situated in other respects.

The centre of gravity of a body is not always *within* the body itself: thus the centre of gravity of a ring is not in the substance of the ring, but in the axis of its circumscribing cylinder; and the centre of gravity of a hollow staff, or of a bone, is not in the matter of which it is constituted, but somewhere in its imaginary axis. Every body, however, has a centre of gravity, and so has every system of bodies.

90. It is a fact established by general observation in all ages and all countries, that whenever bodies are unsupported or left to themselves, they begin to move downwards in vertical lines, and continue thus to move until they meet with something which interrupts their motion or prevents their further descent. This is observed to take place not only with respect to large and very ponderous bodies, but to smaller ones, and even to the most minute particles into which they can be separated, provided they are not so small as to elude the observation of our senses. And if certain substances, such as smoke, and vapours, &c. seem to contradict this universal fact; it is because they are only in *appearance* left to themselves, while in reality they are supported, and put into an ascending motion, by the action of the fluids, &c. that compose the atmosphere which surrounds the earth. All bodies, and their most intimate particles, tend towards a point which is either accurately or very nearly the centre of the terraqueous globe; yet this tendency is certainly not essential to matter, it is an effort which matter of itself is not able to make, being indifferent to either motion or rest: we are authorized, then, to conclude that this tendency to motion is caused by a power not existing in the matter on which our observations are made, but in something exterior; and this force, without attempting to explain its nature and essence, we designate by the term *Gravity;* the general fact or event of bodies falling is denoted by the verbal noun *Gravitation;* and it is a part or consequence of a more universal property, not here entered upon—

that of the mutual *Attraction* of the different bodies in the universe towards each other.

91. Since gravity impresses, or has a tendency to impress, on every particle of bodies, in an instant, a certain velocity with which they would begin to fall, if they were not supported; and since, abstracting the influence of the air, this velocity would be the same for each of the moleculæ of bodies, whatever be their substance, it will not be difficult to attach a just and scientific meaning to that which is commonly called *weight:* it is the effort necessary to prevent a body from falling. But bodies fall in consequence of the action of the force of gravity upon each of their particles, and they can be prevented from falling by a force equal and opposite to the resultant or equivalent of all these actions. Hence, we may readily distinguish between the effect of gravity and that of weight, by adopting the language of *Condorcet,* when he says, "the former is the power of transmitting, or a tendency to transmit into every particle of matter a certain velocity which is absolutely independent on the number of material particles; and the second is the effort which must be exercised to prevent a given mass from obeying the law of gravity. *Weight,* accordingly, *depends on the mass, but gravity has no dependance at all upon it.*"

Every particle of which bodies are composed receiving from gravity equal solicitations towards the centre of the earth, it follows that if the supports of bodies, whether large or minute, were taken away, and they were permitted to fall from equal altitudes, they would arrive at the surface of the earth after equal portions of time: and this is confirmed by experience; for under the exhausted receiver of the Air-pump (where the resistance of the air is removed) the heaviest metals and the lightest feathers, or down, fall in the same time. If, therefore, a body is divided into ever so many parts, each of them left to itself would arrive at the surface of the earth, after the same time as would have been employed by the whole body in descending. All bodies being more or less porous, and possessing different degrees of density, they will contain a greater or less number of equal moleculæ in the same volume or bulk; hence all bodies of equal bulk are not equal in weight. But since the weight is equal to the sum of all the efforts exercised by gravity upon the constituent moleculæ of a body, it is proportional to its *density* or to its *mass.* If p, p', p'', &c. be the several particles of which a body is composed, and M its mass, then will $M = p + p' + p'' +$ &c. and if g represent the force of gravity soliciting each particle, we shall have the weight $= gM = gp + gp' + gp'' +$ &c.

92. When bodies are composed of moleculæ, which are of the same size and substance, and similarly posited throughout, they are said to be *homogeneous*: such are the bodies which we shall consider in this chapter; and in which the mass will manifestly be proportional to the extension or the magnitude, so that the one may be substituted for the other in our investigations. The vertical lines which would be described by bodies if subjected to the free action of gravity, are frequently called *lines of direction.* Since they would, if produced, meet at the centre of the earth, they cannot, strictly speaking, be parallel: but, with respect to any body or any system of bodies connected for mechanical purposes, the whole space occupied by all their particles must be so very minute compared with the magnitude of the earth, that their several lines of direction may be considered as parallel without any danger of sensible error; just as we speak of a moderate portion of the earth's surface as a plane, although it is, in fact, nearly spherical. Hence, then, the actions of gravity upon a body, or system, may be considered as those of parallel forces applied to their various

particles; and, of consequence, the conclusions and theorems which were deduced, arts. 30—33, with regard to such forces, may be adopted in our present investigations relating to the *centre of gravity*. This being admitted, the ensuing particulars are without difficulty inferred.

I. By the definition of the centre of gravity, when *it* is supported the body is in equilibrio; and from the nature of equilibrium it can only be produced singly by the exercise of a force equal and opposite to the resultant of all the other forces acting upon the several particles of the body, that is, since in this case the forces are parallel, by a force equal to the weight of the body applied at the centre of parallel forces: consequently the centre of gravity coincides with *the centre of parallel forces.*

II. Varying the position of the body, will not cause any change in the centre of gravity; since any such mutation will be nothing more than changing the directions of the forces, without their ceasing to be parallel; and if the forces do not continue the same, in consequence of the body being supposed at different distances from the earth, still the forces upon all the moleculæ vary proportionally, and their centre remains unchanged.

III. Let any system be conceived in which no other forces than weights are applied; and let it be imagined of any form or construction whatever, but without any motion. In this case, whatever be the disposition of the bodies of the system, it is clear that if there be an equilibrium, the sum of the resistances of the fixed points or obstacles, estimated in the vertical direction, will be equal to the total weight of the system. But if any motion arises, a part of the force of gravity will be employed in producing it, so that it is only with the surplus that the fixed points can be charged. Therefore, in this case, the sum of the vertical resistances of the fixed points will be less at the first instant of motion than the entire weight of the system: consequently, from those two forces combined, there will result a single force equal to their difference, which will solicit the system downwards. Hence the centre of gravity will necessarily descend with the velocity due to that difference; and hence it follows that *to assure ourselves that several weights applied to any system or machine whatever are in equilibrium, it suffices to prove that if the system be left to itself, its centre of gravity will not descend.*

The immediate and universal consequence of this principle is, that if the centre of gravity of any system is at the lowest point possible, there will necessarily be equilibrium: for, if not, the centre of gravity must descend; yet, how can it *descend* if it be already at the *lowest* point?

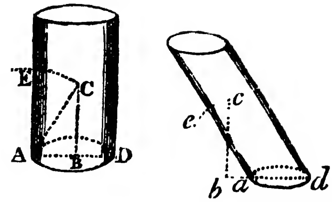
It would not, however, be correct to say, reciprocally, that always when equilibrium obtains the centre of gravity is at the lowest point possible: for that point might be (and in many cases of unstable equilibrium often *is*) in the *highest* possible position; or it might be found at neither the highest nor the lowest point: the exceptions occurring with sufficient frequency in the usual theory of *maxima* and *minima*. But the *principle,* as above stated, has no exception.

IV. When a heavy body is suspended by any other point than its centre of gravity, it will not rest unless that centre is in the same vertical line with the point of suspension: for in all other positions the force which is intended to ensure the equilibrium, will not be directly opposite to the resultant of the parallel forces of gravity upon the several particles of the body, and of course the equilibrium will not be obtained.

V. If a heavy body be sustained by two or more forces, their directions must meet either at the centre of gravity of that body, or in the vertical line which passes through it.

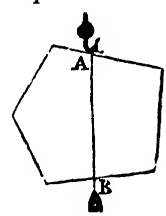
VI. When a body stands upon a plane, if a vertical line passing through the

centre of gravity fall within the base on which the body stands, as in the first of the annexed figures, it will not fall over; but if that vertical line passes without the base, as in the second, the body will fall, unless it be prevented by a prop or a cord. When the vertical line falls upon the extremity of the base, the body *may* stand, but the equilibrium may be disturbed by a very trifling force; and the nearer this line passes to any edge of the base the more easily may the body be thrown over; the nearer it falls to the middle of the base, the more firmly the body stands.

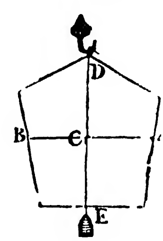


VII. To find the centre of gravity mechanically, it is only requisite to dispose the body successively in two positions of equilibrium, by the aid of two forces in vertical directions, applied in succession to two different points of the body; the point of intersection of these two directions will show the centre required.

This may be thus exemplified. If the body have plane sides, as a piece of board, hang it up by any point, then a plumb-line suspended from the same point will pass through the centre of gravity; therefore mark that line upon it: and after suspending the body by another point, apply the plummet to find another such line, then will their intersection show the centre of gravity. See the marginal figures.



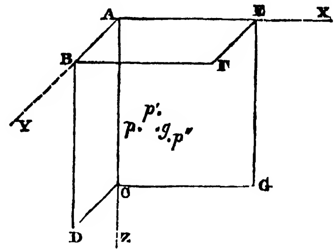
Another method: Lay the body on the edge of a triangular prism, or such like, moving it to and fro till the parts on both sides are in equilibrio, and mark a line upon it close by the edge of the prism: balance it again in another position, and mark the fresh line by the edge of the prism; the vertical line passing through the intersection of these lines, will likewise pass through the centre of gravity. The same thing may be effected by laying the body on a table, till it is just ready to fall off, and then marking a line upon it by the edge of the table: this done in two positions of the body, will in like manner point out the centre of gravity.



93. When a plane or a line can be so drawn as to divide a solid or a plane into two parts equal and similar, or so that its moleculæ shall be disposed two by two, in the same manner, with respect to such plane or such line, we may call the body symmetrical with regard to that plane or axis. And in all such bodies, it is obvious that the sum of the moments of its several moleculæ with relation to such plane or axis, will be nothing: for, if we take two particles disposed in the same manner but on different sides, their moments will be equal but with contrary signs; and, consequently, their sum will be equal to *zero*: and the same may be shown of every other pair of moleculæ, similarly situated: whence, as (by hyp.) there are none but what are similarly situated, the resultant of the system will be in such plane, or line, and, of consequence, its centre of gravity will be there also. The same reasoning may be extended to the centre of figure or of magnitude, that is, the point with respect to which a whole body shall be symmetrical. Hence we conclude that the centre of gravity of a right line, or of a parallelogram, prism, or cylinder, is in its middle point; as is also that of a circle, or of its circumference, or of a sphere, or of a regular polygon; that the centre of gravity of a triangle is somewhere in a line drawn from any angle to

the middle of the opposite side; that of an ellipse, a parabola, a cone, a conoid, a spheroid, &c. somewhere in its axis. And the same of all symmetrical figures whatever.

94. Prop. To deduce the general theorems which are used in finding the centre of gravity of any proposed body.



The determination of the centre of gravity, being reduced to that of the centre of parallel forces, we may here adopt the ordinary theorem. From which it will follow, that if p, p', p'', &c. be equal material particles, and g the point through which the resultant R of the graviating forces upon these particles always passes; and ABCD be a vertical plane, on which perpendiculars from p, p', p'', and g are let fall, then will the sum of the products of the forces upon p, p', p'', into their respective distances from ABCD, be equal to the product of the resultant R into its distance, where the force R would be equal to those upon $p + p' + p''$. The same would likewise obviously hold with respect to perpendiculars upon the other plane AECG: and since the same will also obtain with relation to any vertical plane, although the position of p, p', and p'' be changed, provided they retain their *relative* situations, it will of course obtain when the position of the system is so varied that AEBF becomes a vertical plane: consequently the equality of the products may be affirmed with regard to all the three planes at the same time, and if the distances from the several planes be referred to the rectangular co-ordinates AX, AY, AZ, we may readily appropriate the equations to our present purpose. Denote, as before, the force of gravity by g, the distances referred to AX by d, d', d'', &c. the distances referred to AY, by D, D′, D″, &c. and those referred to Z by $\delta, \delta', \delta''$, &c. the distances from the centre of parallel forces to the same axis being denoted by XY and Z: then we shall have

$$\mathrm{RX} = g\,p\,d + g\,p'\,d' + g\,p''\,d' + \&c.$$
$$\mathrm{RY} = g\,p\,\mathrm{D} + g\,p'\,\mathrm{D}' + g\,p''\,\mathrm{D}'' + \&c.$$
$$\mathrm{RZ} = g\,p\,\delta + g\,g'\,\delta' + g\,p''\,\delta'' + \&c.$$

But $\mathrm{R} = gp + gp' + gp'' + \&c.$ and $\mathrm{M} = p + p' + p'' + \&c.$ whence

$$(\mathrm{I.}) \ldots \left\{\begin{aligned} \mathrm{X} &= \frac{pd + p'd' + p''d'' + \&c.}{\mathrm{M}} \\ \mathrm{Y} &= \frac{p\mathrm{D} + p'\mathrm{D}' + p''\mathrm{D}'' + \&c.}{\mathrm{M}} \\ \mathrm{Z} &= \frac{p\delta + p'\delta' + p''\delta'' + \&c.}{\mathrm{M}} \end{aligned}\right.$$

Here, if we adopt the language of fluxions, and put x, y, and z, for the variable distances from A upon AX, AY, and AZ, respectively, we may express these equations in the following form, which will render it more useful in many investigations.

$$(\mathrm{II.}) \ldots \left\{\begin{aligned} \mathrm{X} &= \frac{\text{fluent of } x\,\dot{\mathrm{M}}}{\text{fluent of } \dot{\mathrm{M}}} = \frac{\text{fluent } x\,\dot{\mathrm{M}}}{\mathrm{M}}. \\ \mathrm{Y} &= \frac{\text{fluent of } y\,\dot{\mathrm{M}}}{\text{fluent of } \dot{\mathrm{M}}} = \frac{\text{fluent } y\,\dot{\mathrm{M}}}{\mathrm{M}}. \\ \mathrm{Z} &= \frac{\text{fluent of } z\,\dot{\mathrm{M}}}{\text{fluent of } \dot{\mathrm{M}}} = \frac{\text{fluent of } z\,\dot{\mathrm{M}}}{\mathrm{M}}. \end{aligned}\right.$$

As these values together determine only one point, we see that a body has but one centre of gravity; of which the three equations determine the three co-ordinates, and of consequence the distances of the centre from three planes respectively perpendicular to each other.

These results being entirely independent of g, that is, of the force of gravity, some philosophers have preferred the term *centre of inertia* to that of *centre of gravity:* other philosophers, on account of other properties, prefer different terms.

When it is required to find the centre of gravity of any line whatever, it is considered as composed of a series of material heavy particles contiguous to each other, and connected by a law which is expressed by the equation of the curve, with respect to any two rectangular co-ordinates x and y. In this case the centre of gravity will manifestly be in the same plane as the proposed line, so that the plane YAX may contain the centre of gravity, whence $Z = 0$, and the value of y being deduced, from the equation of the curve in terms of y, the centre of gravity may be determined by these two equations:

$$(\text{III.}) \ldots X = \frac{\text{flu. } x\dot{M}}{M} \ldots Y = \frac{\text{flu. } y\,\dot{M}}{M}$$

If the curve have two legs symmetrical with relation to any axe, then we may reckon the vertex of that axe the origin of the co-ordinates, and y being $= 0$, we shall only require $X = \frac{\text{flu. } x\,\dot{M}}{M}$; but in this case, the fluxion of M the curve being $= \sqrt{\dot{x}^2 + \dot{y}^2}$, we have also,

$$(\text{IV.}) \ldots X = \frac{\text{flu. } x\sqrt{\dot{x}^2 + \dot{y}^2}}{M}$$

If the figure is a plane, its centre of gravity will be in the same plane, and of course we may take $Z = 0$: and because $\dot{M} = y\,\dot{x}$, our equations become,

$$(\text{V.}) \ldots \begin{cases} X = \dfrac{\text{flu. } x\,y\,\dot{x}}{\text{flu. } y\,\dot{x}} = \dfrac{\text{flu. } x\,y\,\dot{x}}{M}. \\[2ex] Y = \dfrac{\text{flu. } y^2\,\dot{x}}{\text{flu. } y\dot{x}} = \dfrac{\text{flu. } y^2\,\dot{x}}{M}. \end{cases}$$

Here again, if the plane be symmetrical with respect to the axis, the equation for X will alone be wanted.

When the figure is the superficies of a body generated by the rotation of any line about an axis, then will $Y = 0$, and $Z = 0$: and putting $\pi = 3{\cdot}14159$, &c. $2\,\pi\,y$ will denote the circumference of the generating circle, and $2\,\pi\,y\,\dot{M}$ the fluxion of the surface, wherefore

$$(\text{VI.}) \ldots\ldots X = \frac{\text{flu. } 2\,\pi\,y\,x\,\dot{M}}{\text{flu. } 2\,\pi\,y\,\dot{M}} = \frac{\text{flu. } y\,x\,\dot{M}}{\text{flu. } y\,\dot{M}}.$$

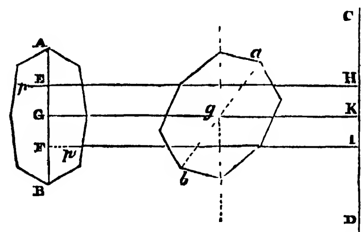
When the figure is a solid of revolution, the centre of gravity being upon its axis, we have $Y = 0$, $Z = 0$: and $\pi\,y^2$ denoting the area of the circle whose radius is y, and $\pi\,y^2\,\dot{x} = \dot{M}$ the fluxion of the solid, we readily find,

$$(\text{VII.}) \ldots X = \frac{\text{flu. } \pi\,y^2 x\,\dot{x}}{\text{flu. } \pi\,y^2\dot{x}} = \frac{\text{flu. } y^2\,x\,\dot{x}}{\text{flu. } y^2\,\dot{x}}.$$

Cor. When $X = 0$, $Y = 0$, $Z = 0$; that is, when the centre of gravity is at the origin of the co-ordinates, the equations (1.) will give $p\,d + p'\,d' + p''\,d'' +$ &c. $= 0$, or $d + d' + d'' +$ &c. $= 0$; in like manner, $D + D' + D'' +$ &c. $= 0$, and $\delta + \delta' + \delta'' +$ &c. $= 0$: and the same will hold with respect to any other co-ordinates whose origin is the centre of gravity; that is, the sum of perpendiculars from all the particles affected with contrary signs as they lie on different sides of either axis is then equal to *zero:* and consequently, *if on any*

plane passing through the centre of gravity of a body, perpendiculars be let fall from each of its moleculæ, the sum of all the perpendicular distances on one side of the plane will be equal to the sum of all those on the other side.

95. PROP. The position, distance, and motion of the centre of gravity of any body is a medium of the positions, distances, and motions of all the particles in the body.



First, the *distance* of the centre of gravity of any body from a given plane, is an average of the distances of each of its constituent particles from the same plane: For, let AB be a body whose centre of gravity is G, and CD any plane, from which the distances are to be estimated: then is the distance of any particle as p, beyond the plane AB (drawn through the body parallel to CD), equal to $p\text{E} + \text{EH} = p\text{E} + \text{GK}$; and if n be the number of particles p on the same side of AB, then will the sum of all the distances pH, be equal to the sum of all the perpendiculars pE, added to all the distances EH or GK, that is $n \cdot p\text{H} = (n \cdot \text{GK}) + (n \cdot p\text{E})$ Again, let p' be a particle of the body between the parallel planes AB, CD, its distance from the plane CD will be equal to $\text{FI} - \text{F}p' = \text{GK} - \text{F}p'$; and, if n' be the number of all the particles p' of the body between AB and CD, we shall have the sum of all their distances, that is, $n' \cdot \text{P'I} = (n' \cdot \text{GK}) - (n' \cdot \text{F}p')$: Hence $(n \cdot p\text{H}) + (n' \cdot p'\text{I}) = (n + n') \cdot \text{GK} + (n \cdot p\text{E}) - (n \cdot \text{F}p')$. But by the corollary to the preceding proposition, the sum of all the perpendiculars, on one side of the plane AB, is equal to the sum of all those on the other: consequently $n \cdot p\text{E} - n' \cdot \text{F}p' = 0$, and $n \cdot p\text{H} + n' \cdot p'\text{I} = (n + n')\,\text{GK}$; that is, GK is the mean of all the distances from every particle of the body to the plane CD. And the like may be shown of any other plane, or, of the body in any other position.

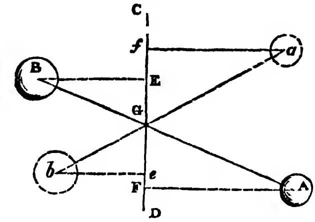
Secondly, the *motion* of the centre of gravity is an average of the motions of the several particles of which the body is composed: For, suppose the body to have moved towards the plane CD, its centre of gravity having passed from G to g, along the right line GK, and the body itself now situated as in ab. Here, it might be shown, in the same manner as in the first case, that $(n + n')\,g\text{K}$, is the sum of the perpendiculars from all the particles in the body to the plane CD: consequently, $(n + n')\,\text{GK} - (n + n')\,g\text{K} = (n + n')\,\text{G}g$, is the sum of the approaches of all the particles towards CD, and Gg, being the $\frac{1}{n + n'}$ of this sum is evidently their mean. And if the motion of G were along a curvilinear path, the same conclusion would be deduced, if we conceive the curve to be separated into its infinitely small elements, and the motions with respect to each determined. The conclusion will likewise be the same although the body may have turned round some centre or axis.

This property of the centre of gravity has occasioned it to be called, by some authors, *the centre of position;* by others, the *centre of mean distances.* The celebrated French mathematician *Carnot,* in his *Geometrie de Position,* makes use of the latter term.

96. PROP. The common centre of gravity, or of position, of two bodies divides the right line drawn between the respective centres of the two bodies in the inverse ratio of their masses.

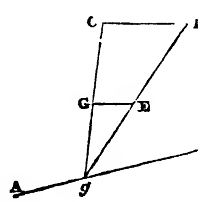
Let A and B be two bodies whose centres of gravity are united by the inflexible

line AB, then, if AG : GB :: B : A, G will be the common centre of gravity of those two bodies, that is, if G be supported, those two bodies actuated by the force of gravity will be in equilibrio in any position: for, through G let the vertical line CD be drawn, on which let fall the perpendiculars AF, BE, from the centres of gravity of the two bodies, then, because of the similar triangles BEG, AFG, we have AG : GB :: AF : BE :: B : A, whence A.AF = B.BE. But A.AF, is equal to the sum of the products of all the particles in A into their respective distances from CD, by the last prop. and, in like manner B . BE, is equal to the sum of the products of the particles in B into their respective distances; therefore g . A . AF = g . B . BE; that is, the sums of the moments of the forces of gravity upon A and B, with respect to CD are equal. If A and B be removed to any other position as a, b, the point G remaining fixed, it will appear in like manner that g . a . af = g . b . be: so that G is the centre of the forces of gravity with respect to A and B, that is, it is their common centre of gravity. In a manner but very little different, G may be shown to be their common centre of position: and the two bodies, if considered as united by their centres of position at G, will then, as well as when their centres are separated by AB, have the sum of the perpendiculars from the several particles on one side of any plane passing through G, equal to the sum of all the perpendiculars on the other side of it.



Cor. The centre of gravity of three or more bodies may, hence, be found, by considering the first and second as a single body equal to their sum and placed in their common centre of gravity, determining the centre of gravity of this imaginary body, and a third. These three again being conceived united at their common centre, we may proceed, in like manner, to a fourth; and so on, *ad libitum*.

97. Prop. *If the particles or bodies of any system be moving uniformly and rectilineally, with any velocities and directions whatever, the centre of gravity is either at rest, or moves uniformly in a right line.* For, let one of the bodies as C move uniformly from C to D: then, g being the centre of gravity of the remaining bodies, join Dg, and take gE to ED, as the mass D, to the sum of the other masses; then, is GE obviously parallel to CD, and CD : GE :: A + B &c. : C, CE being in this case the path of the common centre. And thus may the motion of the centre of gravity be found, which would be produced by the uniform rectilinear motion of each body in the system. Then, because each corresponding motion of the centre of gravity is uniform and rectilinear, the result of the whole will be either a uniform rectilinear motion; or no motion at all.

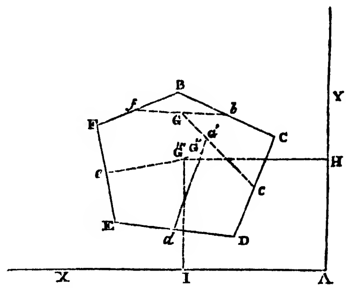


If a rotatory motion be communicated to a body and it be then left to move freely, the axis of rotation will pass through the centre of gravity: for, that centre, either remaining at rest or moving uniformly forward in a right line, has no rotation.

Here too it may be remarked, that *a force applied at the centre of gravity of a body, cannot produce a rotatory motion.* For every particle resists, by its inertia, the communication of motion, and in a direction opposite to that in which the force applied tends to communicate the motion; the resisting forces, therefore,

act in parallel lines, in the same manner as the gravitating forces: consequently, since the latter balance each other on the centre of gravity, the former will do so likewise.

98. Prop. To find the centre of gravity of the perimeter of any right-lined figure.



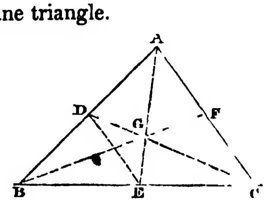
If the figure be a regular polygon, the centre of gravity of its perimeter will be the centre of its circumscribing or inscribed circle. But if it be irregular, we conceive the particles of each line to be all placed at their respective centres of gravity, that is, at the middle of each line (93), and proceed thus. Join the middle points of any two of the sides, as FB, BC, by the line fb: make fG : Gb ∷ BC : BF, or fG : fb ∷ BC : BC + BF, and G will be the common centre of gravity of the two sides BC, BF. Then join Gc, c being the middle point of a third side CD; make GG′ : cG′ ∷ CD : BC + BF, and G′ will be the centre of gravity of the three sides, FB, BC, CD. In like manner, join G′ and the middle point d of a fourth line, and find the new centre of gravity G″: and so on, for all the sides of the figure.

Or, drawing through any point A in the same plane, the rectangular co-ordinates AX, AY, and denoting the lines FB, BC, CD, &c. by p, p', p'', &c. the distances of their middle points from AX, by d, d', d'', &c. and the distances of the said middle points from AY, by D, D′, D″, &c. we shall have (108. I.)

$$X = AI = \frac{pd + p'd' + p''d'' + \&c.}{p + p' + p'' + \&c.} : \; Y = AH = \frac{pD + p'D' + p''D'' + \&c.}{p + p' + p'' + \&c.}.$$

Then, drawing HG‴, IG‴, parallel to AX and AY, their intersection G‴ will be the centre of gravity required.

99. Prop. To find the centre of gravity of a plane triangle.



Let ABC be any triangle: draw AE from one of its angles to the middle of its opposite side, then will AE divide every line which can be drawn in the triangle parallel to BC into two equal parts; consequently the surface of the triangle is symmetrically disposed with respect to AE, and the centre of gravity will be found in that line (93). For a like reason, if from any other angle, as C, we draw CD to the middle of its opposite side, the centre of gravity of the triangle will be somewhere upon that line: it will, therefore, be at G the intersection of those lines.

Now, since the points D, and E, divide the sides BA, BC, of the triangle proportionally, the line DE which joins them must be parallel to the third side AC: hence the triangles BDE, BAC are similar, and so are the triangles GDE, GCA. Consequently, GD : GC ∷ GE : GA ∷ DE : AC ∷ BD : BA ∷ 1 : 2.

Therefore AG $= \frac{2}{3}$AE, and CG $= \frac{2}{3}$CD, also BG $= \frac{2}{3}$BF.

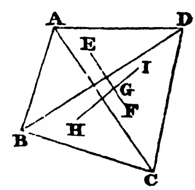
Cor. If AB, BC, CA, be denoted by a, b, and c, and AG, BG, CG, by m, n, and r, we have, by Geom. theor. 38, the three following equations:

$$\begin{cases} AB^2 + AC^2 = 2\,BE^2 + 2\,AE^2, \; i.\,e.\; a^2 + c^2 = \frac{1}{2}b^2 + \frac{9}{2}m^2 \\ AB^2 + BC^2 = 2\,CF^2 + 2\,BF^2, \quad a^2 + b^2 = \frac{1}{2}c^2 + \frac{9}{2}n^2 \\ AC^2 + BC^2 = 2\,AD^2 + 2\,CD^2, \quad c^2 + b^2 = \frac{1}{2}a^2 + \frac{9}{2}r^2. \end{cases}$$

Adding these together, and clearing them of fractions, there results $a^2 + b^2 + c^2 = 3m^2 + 3n^2 + 3r^2$; which gives this curious theorem: *In any plane triangle the sum of the squares of the three sides, is equal to thrice the sum of the squares of the distances of each of its angles from the centre of gravity.*

Cor. 2. From the three equations in the preceding corollary, we readily find $m = \frac{1}{3}\sqrt{2a^2 + 2c^2 - b^2}$, $n = \frac{1}{3}\sqrt{2a^2 + 2b^2 - c^2}$, and $r = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$: by means of which the distance of the centre of gravity from either angle of any given triangle may be readily calculated.

100. Prop. To find the centre of gravity of a trapezium.



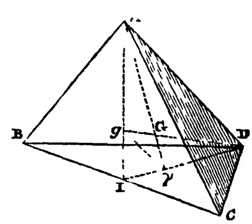
Let ABC be any trapezium. Divide it into two triangles by the diagonal AC: find their centres of gravity H and I, by the last proposition; then join IH, and make IG, to GH, as the triangle ABC, to the triangle ADC, and G will be the centre of gravity of the trapezium.

Or, divide the trapezium into two other triangles by the diagonal BD; find their centres of gravity E, and F, and draw EF. Then the centre of gravity of the trapezium must be in the line HI; and it must likewise be in the line EF; consequently it must be in G their point of intersection.

Where two sides of the quadrilateral are parallel to each other, the centre of gravity may be determined by a neat geometrical process, which is proposed for the student's exercise.

Note.—To find the centre of gravity of the area of any rectilinear figure, divide it into triangles, and find their respective centres of gravity, by art. 99, then conceiving each of the triangles collected into their respective centres of gravity, their common centre of gravity may be found by the theory of (equa. I. 94.)

101. Prop. To find the centre of gravity of any triangular pyramid.



Let ABCD be a triangular pyramid: on one of its faces, as BCD, draw DI from the angle D to the middle of its opposite side BC; set off $I\gamma = \frac{1}{3}ID$, and γ will (99.) be the centre of gravity of that face. From A the vertex of the pyramid, draw $A\gamma$, and it will pass through the centres of gravity of every section of the pyramid parallel to BCD: consequently, it will pass through the centre of gravity of the pyramid. Again, on the face ABC, draw AI from A to the middle of the opposite side; set off $Ig = \frac{1}{3}IA$, and join gD: the centre of gravity of the pyramid will, it is obvious, be found upon this line also. But the two lines $A\gamma$, Dg, being both in the plane of the triangle ADI, must intersect each other, and G their point of intersection must necessarily be the centre of gravity of the pyramid.

Now, if we conceive $g\gamma$ drawn, it will be parallel to AD, since $Ig : IA :: I\gamma : ID$: hence the triangles $Gg\gamma$, GDA, are similar, and,

$$Gg : GD :: G\gamma : GA :: g\gamma : AD :: 1 : 3.$$

Therefore, $Gg = \frac{1}{3}GD = \frac{1}{4}gD$; $G\gamma = \frac{1}{3}AG = \frac{1}{4}A\gamma$, &c.

Cor. 1. If AB be put $= a$, $AC = b$, $AD = c$, $BC = d$, $BD = e$, $CD = f$; and the distances from the angles to the centre of gravity, $AG = m$, $BG = n$, $CG = r$, $DG = s$; then, from Geom. theorem 38, $AI^2 = \frac{AB^2 + AC^2 - 2BI^2}{2}$

$=\frac{a^2+b^2-\frac{1}{2}d^2}{2}$; and, by the same, $DI^2=\frac{e^2+f^2-\frac{1}{2}d^2}{2}$: again, since AG $=\frac{3}{4}A\gamma$, we have $AG^2=\frac{9}{16}A\gamma^2=\frac{9}{16}\left(\frac{AI^2 \,.\, D\gamma+AD^2 \,.\, I\gamma-ID \,.\, I\gamma \,.\, \gamma D}{ID}\right)$. (Vol. i. page 410.) Substituting in this expression the literal values of AI, $D\gamma$, &c. and reducing, we at length obtain $AG^2=\frac{1}{16}\overline{(3a^2+3b^2+3c^2-}$

A like equation being in the same manner deducible for BG^2, CG^2, DG^2, we have the following general theorem: *In any triangular pyramid the distance of any one of the angles of the pyramid, from the centre of gravity, is equal to one-fourth of the square root of the difference of thrice the sum of the squares of the three edges meeting at that angle, and the sum of the squares of the other three edges.*

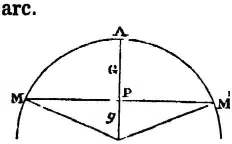
Cor. 1. If the sides of the base are equal, as $BD=BC=CD$, we have $AG^2=\frac{3}{16}(AB^2+AC^2+AD^2-BC^2)$.

Cor. 2. If, moreover, $AB=AC=AD$, then $AG^2=\frac{3}{16}(3AB^2-BC^2)$.

Cor. 3. If all the edges of the pyramid are equal, we have for the regular tetraedron, $AG^2=BG^2=CG^2=DG^2=\frac{3}{8}AB^2$, whence $AG=\frac{1}{4}AB\sqrt{6}$.

Cor. 4. A pyramid whose base is any polygon, will have its centre of gravity upon the line drawn from the vertex to the centre of gravity of the base, and at the distance of $\frac{3}{4}$ of its length from its vertex. Which will be sufficiently obvious if it be considered, that any such pyramid is composed of triangular pyramids, whose centres of gravity all lie in one plane parallel to the base; consequently their common centre of gravity must be in the same plane, and must likewise be in the line drawn from the vertex to the common centre of gravity of all the triangles which constitute the base.

102. Prop. To find the centre of gravity of a circular arc.



From the middle point of the proposed arc MAM' conceive the line AC drawn through C the centre of the circle, on which let AP be denoted by x, the variable ordinate PM by y, and the radius AC of the circle by r, the half arc being denoted by z. Then accounting A the origin of the co-ordinates, since the curve is symmetrical with respect to AC, we need only make use of the equation (94. IV.) which, substituting z for M will become, $X=\frac{\int x\sqrt{\dot{x}^2+\dot{y}^2}}{z}$ *. Now the equation expressing the relation between the rectangular co-ordinates of a circle is $y^2=2rx-x^2$; by means of which we have $X=\frac{r}{z}\int\frac{x\dot{x}}{\sqrt{(ax-x^2)}}$. The fluent of this is $X=\frac{r}{z}(z-y)=r-\frac{ry}{z}$: which needs no correction, because when $y=0$, $z=0$. Hence then, G being the centre of gravity, we have $AG=r-\frac{ry}{z}=AC-\frac{ry}{z}$. Consequently $CG=AC-AG=\frac{ry}{z}$: that is, *the distance of the centre of gravity of a circular arc from its centre, is a fourth proportional to the arc, the radius, and the chord of the arc.*

Cor. 1. When the arc is a semicircle, the chord is double the radius, and CG $=\frac{2r}{3 \cdot 141593}=\frac{r}{1 \cdot 57079}=\cdot 63662\,r$.

* Here, as well as in various parts of this work, we make the character $\int$ denote the fluent of the expression which stands after it.

Cor. 2. When $x = 2r$, $y = 0$, and consequently $CG = \frac{ry}{z} = 0$: that is, the centre of gravity coincides with the centre of the circle; as is sufficiently obvious independent of the fluxional process.

103. Prop. To find the centre of gravity of a circular segment.

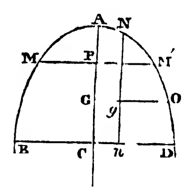
Let MAM′P in the last figure be the segment proposed, and let the parts be denoted as before. Here we take the first of the equations (94. V.), that is $X = \frac{\text{flu. } xy\dot{x}}{M}$, where M denotes the area of APM. Now, since $y = \sqrt{2rx - x^2}$, we have $XM = \int xy\dot{x} = \int x\dot{x}\sqrt{2rx - x^2} = -\frac{(2rx - x^2)^{\frac{3}{2}}}{3} + rM$: this divided by M gives $X = AG = r - \frac{PM^3}{3M} = CA - \frac{PM^3}{3\text{ area APM}}$. Consequently $CG = \frac{PM^3}{3\text{ area APM}} = \frac{MM^3}{12\text{ area seg.}}$.

Cor. When the segment becomes a semicircle, we have $M'M = 2r$, and $CG = \frac{r}{2{\cdot}356194} = {\cdot}42441\, r$.

104. Prop. To find the centre of gravity of any parabola.

Here the general equation is $y = x^n \div a^{n-1}$. And by substitution . . .

$$\frac{\text{flu. } xy\dot{x}}{\text{flu. } y\dot{x}} = \frac{\int x^{n+1}\dot{x}}{\int x^n \dot{x}} = \frac{n+1}{n+2} \times x = X.$$



Cor. 1. If $n = \frac{1}{2}$, which is the case in the common or Apollonian parabola, $\frac{n+1}{n+2}x = \frac{3}{5}x$. That is, $AG = \frac{3}{5} AC$.

Cor. 2. If $n = 1$, the figure becomes a triangle, and then $AG = \frac{2}{3}x$: which agrees with art. 99.

105. Prop. To find the centre of gravity of any semiparabola.

In this case the distance on the absciss, or the value of X is determined by the foregoing problem; we have now to find Gg; in order to which we take the second equation $Y = \frac{\text{flu. } y^2\dot{x}}{M}$. By substituting for y^2 in the numerator of this expression, its value in the equation of the curve, we have $YM = \int \frac{x^{2n}\dot{x}}{a^{2n-2}}$, which, by the rules given for the determination of fluents is $= y^2\left(\frac{h}{2} - \frac{nx}{2n+1}\right)$, where $h = AC$ the height of the parabola. And this, when $x = h$, becomes $y \times \frac{h}{4n+2}$: wherefore, dividing by $M = \frac{h \times CD}{n+1}$ we obtain $Y = \frac{(n+1)\,CD}{4n+2} = Gg$.

Cor. 1. If $n = \frac{1}{2}$, as in the common parabola, $Gg = \frac{3}{8}$ CD.

Cor. 2. In the triangle, where $n = 1$, $Gg = \frac{1}{3}$ CD.

106. Prop. To find the centre of gravity of the segment of a spheroid.

Here the figure may be divided into two parts symmetrical with respect to the axis: so that if we account the vertex A the origin of the co-ordinates, we shall

need the equation for X. Let the fixed axis of the spheroid be called a, the revolving axis c, then the equation of the curve is $y^2 = \frac{c^2}{a^2} \times (ax - x^2)$: consequently $\frac{\text{flu.}\, y^2 x\dot{x}}{\text{flu.}\, y^2 \dot{x}} = \frac{\int (ax - xx) \cdot x\dot{x}}{\int (ax - xx) \cdot \dot{x}} = \frac{\frac{1}{3}ax^3 - \frac{1}{4}x^4}{\frac{1}{2}ax^2 - \frac{1}{3}x^3} = \frac{4a - 3x}{6a - 4x}\, x = \text{AG}.$

Cor. 1. When the segment becomes a hemispheroid, $x = \frac{1}{2}a$, and $\frac{4a - 3x}{6a - 4x}$ $x = \frac{5}{8}x$ for the distance of the centre of gravity from the vertex, therefore $\frac{3}{8}\, x$ is its distance from the centre of the base.

Cor. 2. When $c = a$, the spheroid becomes a sphere, and as the theorem is independent of c, it is alike applicable to both solids, and to their corresponding segments.

Cor. 3. Since the equation to the hyperbola is $y^2 = \frac{c^2}{a^2}\,(ax + x^2)$ which differs from that to the ellipsis only in the sign of the second term, we get by a similar process $\frac{4a + 3x}{6a + 4x}\, x$, for the distance of the centre of gravity from the vertex of a hyperboloid.

SCHOLIUM.

107. To save the trouble of a distinct investigation in some instances which often occur, we may state in addition to the preceding propositions, a few other known results.

1. In a circular sector, the distance from the centre of the circle is $\frac{2cr}{3a}$; where a denotes the arc, c its chord, and r the radius.

2. The centres of gravity of the surface of a cylinder, of a cone, and of a conic frustum, are respectively at the same distances from the origin as are the centres of gravity, of the parallelogram, triangle, and trapezoid, which are vertical sections of the respective solids.

3. The centre of gravity of the surface of a spheric segment, is at the middle of its versed sine or height.

4. The centre of gravity of the convex surface of a spherical zone, is in the middle of that portion of the axis of the sphere which is intercepted by the two bases of the zone.

5. In a cone, as well as any other pyramid, the distance from the vertex is $\frac{3}{4}$ of the axis.

6. In a conic frustum, the distance on the axis from the centre of the less end, is $\frac{1}{4}h \cdot \frac{3R^2 + 2Rr + r^2}{R^2 + Rr + r^2}$: where h denotes the height, and R and r, the radii of the greater and less ends.

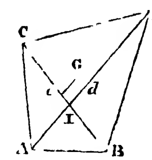
7. The same theorem will serve for the frustum of any regular pyramid, taking R and r, for the sides of the two ends.

8. In the paraboloid, the distance from the vertex is $\frac{2}{3}$ of the axis.

9. In the frustum of the paraboloid, the distance on the axis from the centre of the less end, is $\frac{1}{3}h \cdot \frac{2R^2 + r^2}{R^2 + r^2}$: where h denotes the height, R and r the radii of the greater and less ends.

10. In a spheric sector $\frac{1}{8}(2r + 3x)$ is the distance from A (fig. art. 102). to C.G, when r is the radius and $x = AP$.

PROBLEMS FOR EXERCISE.



1. Let AD and BC, the diagonals of any trapezium, intersect in I. On IC the greater segment of BC, set off $Ic = \frac{1}{3}(IC - IB)$; and on ID set off $Id = \frac{1}{3}(ID - IA)$. Complete the parallelogram $IcdG$; then is G the centre of gravity of the trapezium. .Required a demonstration of this.

2. Find, geometrically, the position of the common centre of gravity of the three squares described upon the sides of a right-angled triangle; as in Euc. vi. 47.

3. Seven equal bodies are placed with their centres of gravity in seven of the angles of a cube. Required the distance of their common centre of gravity from the remaining angle.

4. Find, algebraically, the centre of gravity of the frustum of a pyramid.

5. Let a sphere whose diameter is 4 inches, and a cone whose altitude is 8 inches, and diameter of its base 3 inches, be fastened upon a thin wire which shall pass through the centre of the globe and the axis of the cone; let the vertex of the cone be toward the sphere, and let its distance from the sphere's surface be 12 inches. Required the place of their common centre of gravity.

6. Demonstrate 1st, That the surface produced by a plane line or curve by revolving about an axis in the plane of that curve, is equal to the product of the generating line or curve into the path described by the centre of gravity.

And 2dly, That the solid produced by the revolution of a plane figure about an axis posited in the plane of that figure, is equal to the product of the generating surface into the circumstance described by the centre of gravity.

7. If four bodies whose weights are 3, 4, 5, and 6 lbs. are placed at the successive angles of a square whose side is 12 inches; required the position of their common centre of gravity, both by construction and calculation.

8. Two cones of equal base, and one double the altitude of the other, are placed with their bases in contact. Required the position of their common centre of gravity.

9. Suppose the same two cones to be placed with their vertices in contact, and their axes in one right line; required the position of their common centre of gravity.

10. The diameter of a sphere, that of the base of a cone, and the height of a cone, are equal to each other. The prolongation of the cone's axis passes through the centre of the sphere, and the vertex of the cone touches the surface of the sphere: where is the common centre of gravity posited?

11. Suppose the same sphere to touch the centre of the same cone's base; where will then be their common centre of gravity?

12. Two equal bodies move from the vertex of an isosceles triangle, with an equal and uniform velocity along the two sides. Compare their velocity with that of their common centre of gravity; the vertical angle of the triangle being 120°.

13. From a given rectangle ABCD of uniform thickness, to cut off a triangle CDO, so that the remainder ABCO, when suspended at O, shall hang with AB vertical, and the sides AO, BC, horizontal, both geometrically and algebraically.

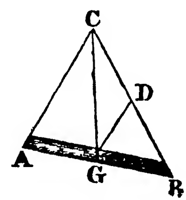
14. When a weight W and power P in equilibrio upon an inclined plane, are put into motion, either by P being made to ascend or to descend; show that the common centre of gravity of W and P (fig. to the Inclined Plane) neither ascends nor descends, but moves horizontally.

ON THE EQUILIBRIUM OF DIFFERENT COMPOUND STRUCTURES, AS ROOFS, ARCHES, DOMES, &c.

108. In all these cases, gravity is the force to be equilibrated, and the whole inquiry resolves itself into a determination of the forces which in any structure are opposed to its operation, and the most efficient and economical method of so preventing the operation of gravity from destroying the structure, in any proposed case. The following investigations and examples will give the student a clear idea of most of the cases that usually occur in practice.

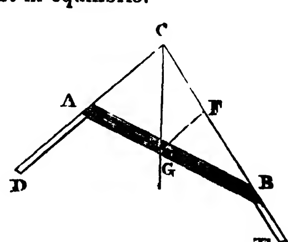
I. ON BEAMS AND CORDS.

109. Prop. I.—To determine the position of the beam AB, hanging by one cord ACB, having its ends fastened at A and B, and sliding freely over a tack or pulley fixed at C.



G being the centre of gravity of the beam, CG will be perpendicular to the horizon, as in the last problem. Now as the cord ACB moves freely about the point C, the tension * of the cord is the same in every part, or the same both in AC and BC. Draw GD parallel to AC: then the sides of the triangle CGD are proportional to the three forces, the weight and the tensions of the string; that is, CD and DG are as the forces or tensions in CB and CA. But these tensions are equal; therefore CD = DG, and consequently the opposite angles DCG and DGC are also equal: but the angle DGC is = the alternate angle ACG; therefore the angle ACG = BCG; hence the line CG bisects the vertical angle ACB, and consequently AC : CB :: AG : GB.

110. Prop. II.—To determine the position of the two posts AD and BE, supporting the beam AB, so that the beam may rest in equilibrio.



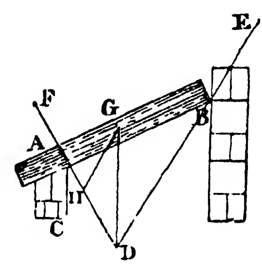
Through the centre of gravity G of the beam, draw CG perp. to the horizon; from any point C in which draw CAD, CBE through the extremities of the beam; then AD and BE will be the positions of the two posts or props required, so as AB may be sustained in equilibrio; because the three forces sustaining any body in such a state, must be all directed to the same point C.

* The *tension* of a cord is the force which acts at one end of it when the other is fixed, or it is equivalent to that force: thus in the case of the equilibrium of powers applied to a physical point, if we regard that point as fixed, the tension of each cord is precisely the force applied at each cord to move the point: but if the equilibrium does not obtain, as when, for example, a cord has two unequal powers acting at its extremities, the tension is the least of the two forces, for the tension will obviously be the same as if the one of the extremities were fixed, and the least of the two forces acted solely at the other end.

Corol. If GF be drawn parallel to CD; then the quantities of the three forces balancing the beam, will be proportional to the three sides of the triangle CGF, viz. CG as the weight of the beam, CF as the thrust or pressure in BE, and FG as the thrust or pressure in AD.

Scholium.—The equilibrium may be equally maintained by the two posts or props AD, BE, as by the two cords AC, BC, or by two planes at A and B perp. to those cords.—It does not always happen that the centre of gravity is at the lowest place to which it can get, to make an equilibrium; for here when the beam AB is supported by the posts DA, EB, the centre of gravity is at the highest it can get; and being in that position, it is not disposed to move one way more than another; and therefore it is as truly in equilibrio, as if the centre were at the lowest point. It is true, this is only a tottering equilibrium, and any the least force will destroy it; and then, if the beam and posts be moveable about the angles A, B, D, E, which is all along supposed, the beam will descend till it is below the points D, E, and gain such a position as that G will be at the lowest point, coming there to an equilibrium again. In planes, the centre of gravity G may be either at its highest or lowest point. And there are cases, when that centre is neither at its highest nor lowest point.

111. Prop III. Supposing the beam AB hanging by a pin at B, and lying on the wall AC; it is required to determine the forces or pressures at the points A and B, and their directions.



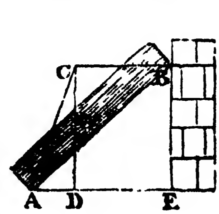
Draw AD perp. to AB and through G, the centre of gravity of the beam, draw GD perp. to the horizon, to meet AD in D; and join BD. Then the weight of the beam, and the two forces or pressures at A and B, will be in the directions of, and proportional to, the three sides of the triangle GDH, having drawn GH parallel to BD; viz. the weight of the beam as GD, the pressure at A as HD, and the pressure at B as GH, and in these directions.

For, the action of the beam is in the direction GD; and the action of the wall at A, is in the perp. AD; conseq. the stress on the pin at B must be in the direction BD, because all the three forces sustaining a body in equilibrio, must tend to the same point, as D.

Corol. 1. If the beam were supported by a pin at A, and laid upon the wall at B; the like construction must be made at B, as has been done at A, and then the forces and their directions will be obtained.

Corol. 2. It is all the same thing, whether the beam is sustained by the pin B and the wall AC, or by two cords BE, AF, acting in the directions DB, DA, and with the forces HG, HD.

112. Prop. IV. To determine the quantities and directions of the forces, exerted by a heavy beam AB, at its two extremities and its centre of gravity, bearing against a perpendicular wall at its upper end B.



From B draw BC perp. to the face of the wall BE, which will be the direction of the force at B; also through G, the centre of gravity, draw CGD perp. to the horizontal line AE; then CD is the direction of the weight of the beam; and because these two forces meet in the point C, the third force or push A, must be in CA, directly from C; so that

the three forces are in the directions CD, BC, CA, or in the directions CD, DA, CA; and, these last three forming a triangle, the three forces are not only in those directions, but are also proportional to these three lines; viz. the weight in or on the beam, as the line CD; the push against the wall at B, as the horizontal line AD; and the thrust at the bottom, as the line AC.

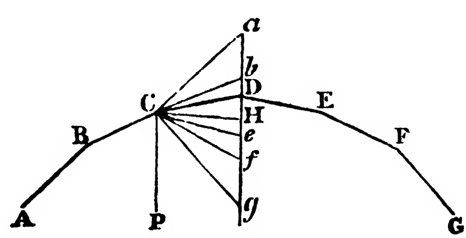
This, when the beam is prismatic, may be conveniently expressed by an equation. Let l = length of the beam, a = its inclination to the horizon, and W = its weight.

$$\text{Pressure at B} = \frac{W}{2\tan. a} = \tfrac{1}{2}W\cot. a.$$

$$\text{thrust at A} = \frac{W\sqrt{\sin.^2 a + \frac{1}{4}\cos.^2 a}}{\sin. a}.$$

Some of these propositions will be found useful in different cases of carpentry, especially in adapting the framing of the roofs of buildings, so as to be nearest in equilibrio in all their parts. And the last problem, in particular, will be very useful in determining the push or thrust of any arch against its piers or abutments, and thence to assign their thickness necessary to resist that push. The following problem will also be of great use in adjusting the form of a mansard roof, or of an arch, and the thickness of every part, so as to be truly balanced in a state of just equilibrium.

113. Prop. V.—Let there be any number of lines, or bars, or beams, AB, BC, CD, DE, &c., all in the same vertical plane, connected together and freely moveable about the joints or angles A, B, C, D, E, &c., and kept in equilibrio by weights laid on the angles: it is required to assign the proportion of those weights; as also the force or push in the direction of the said lines; and the horizontal thrust at every angle.



Through any point, as D, draw a vertical line $aDHg$, &c.; to which, from any point, as C, draw lines in the direction of, or parallel to, the given lines or beams, viz. Ca parallel to AB, and Cb parallel to BC, and Ce to DE, and Cf to EF, and Cg to FG, &c.; also CH parallel to the horizon, or perpendicular to the vertical line adg, in which also all these parallels terminate.

Then will all those lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, and of all the three kinds, viz. vertical, horizontal, and oblique. That is,

the oblique forces in direction of the bars	AB,	BC,	CD,	DE,	EF,	FG,	
are proportional to their parallels . .	Ca,	Cb,	CD,	Ce,	Cf,	Cg;	and
the vertical weights on the angles . .	B,	C,	D,	E,	F,	&c.	
are as the parts of the vertical . . .	ab,	bD,	De,	ef,	fg,		

and the weight of the whole frame ABCDEFG, is proportional to the sum of all the verticals, or to ag; also the horizontal thrust at every angle, is every where the same constant quantity, and is expressed by the constant horizontal line CH.

Dem. All these proportions of the forces evidently follow immediately from the general well-known property, in Statics, that when any forces balance and keep each other in equilibrio, they are respectively in proportion as the lines drawn parallel to their directions, and terminating each other.

Thus, the point or angle B is kept in equilibrio by three forces, viz. the weight laid and acting vertically downward on that point, and by the two oblique forces or thrusts of the two beams AB, CB, and in these directions. But C*a* is parallel to AB, and C*b* to BC, and *ab* to the vertical weight; these three forces are therefore proportional to the three lines *ab*, C*a*, C*b*.

In like manner, the angle C is kept in its position by the weight laid and acting vertically on it, and by the two oblique forces or thrusts in the direction of the bars BC, CD: consequently these three forces are proportional to the three lines *b*D, C*b*, CD, which are parallel to them.

Also, the three forces keeping the point D in its position, are proportional to their three parallel lines, D*e*, CD, C*e*. And the three forces balancing the angle E, are proportional to their three parallel lines *ef*, C*e*, C*f*. And the three forces balancing the angle F, are proportional to their three parallel lines *fg*, C*f*, C*g*. And so on continually, the oblique forces or thrust in the directions of the bars or beams, being always proportional to the parts of the lines parallel to them, intercepted by the common vertical line; while the vertical forces or weights, acting or laid on the angles, are proportional to the parts of this vertical line intercepted by the two lines parallel to the lines of the corresponding angles.

Again, with regard to the horizontal force or thrust: since the line DC represents, or is proportional to the force in the direction DC, arising from the weight or pressure on the angle D; and since the oblique force DC is equivalent to, and resolves into, the two DH, HC, and in those directions, by the resolution of forces, viz. the vertical force DH, and the horizontal force HC; it follows, that the horizontal force or thrust at the angle D, is proportional to the line CH; and the part of the vertical force or weight on the angle D, which produces the oblique force DC, is proportional to the part of the vertical line DH.

In like manner, the oblique force C*b*, acting at C, in the direction CB, resolves into the two *b*H, HC; therefore the horizontal force or thrust at the angle C, is expressed by the line CH, the very same as it was before for the angle D; and the vertical pressure at C, arising from the weights on both D and C, is denoted by the vertical line *b*H.

Also, the oblique force *a*C, acting at the angle B, in the direction BA, resolves into the two *a*H, HC; therefore again the horizontal thrust at the angle B, is represented by the line CH, the very same as it was at the points C and D; and the vertical pressure at B, arising from the weights on B, C, and D, is expressed by the part of the vertical line *a*H.

Thus also, the oblique force C*e*, in direction DE, resolves into the two CH, H*e*, being the same horizontal force, with the vertical H*e*; and the oblique force C*f*, in direction EF, resolves into the two CH, H*f*; and the oblique force C*g*, in direction FG, resolves into the two CH, H*g*; and so on continually, the horizontal force at every point being expressed by the same constant line CH; and the vertical pressures on the angles by the parts of the verticals, viz. *a*H the whole vertical pressure at B, from the weights on the angle B, C, D; and *b*H the whole pressure on C from the weights on C and D; and DH the part of the weight on D causing the oblique force DC; and H*e* the other part of the weight on D causing the oblique pressure DE; and H*f* the whole vertical pressure at E from the weights on D and E; and H*g* the whole vertical pressure on F arising from the weights laid on D, E, and F. And so on.

So that, on the whole, *a*H denotes the whole weight on the points from D to A; and H*g* the whole weight on the points from D to G; and *ag* the whole weight on all the points on both sides; while *ab*, *b*D, D*e*, *ef*, *fg* expresses the several particular weights, laid on the angles B, C, D, E, F.

Also, the horizontal thrust is every where the same constant quantity, and is denoted by the line CH.

Lastly, the several oblique forces or thrusts, in the directions AB, BC, CD, DE, EF, FG, are expressed by, or are proportional to, their corresponding parallel lines, C*a*, C*b*, CD, C*e*, C*f*, C*g*.

Corol. 1. It is obvious, and remarkable, that the *lengths* of the bars AB, BC, &c. do not affect or alter the proportions of any of these loads or thrusts; since all the lines C*a*, C*b*, *ab*, &c. remain the same, whatever be the lengths of AB, BC, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given the position of DC, and the weights or loads laid on the angles D, C, B; set these on the vertical, DH, D*b*, *ba*, then C*b*, '*a* give the directions or positions of CB, BA, as well as the quantity or proportion CH of the constant horizontal thrust.

Corol. 2. If CH be made radius; then it is evident that H*a* is the tangent, and C*a* the secant of the elevation of C*a* or AB above the horizon; also H*b* is the tangent and C*b* the secant of the elevation of C*b* or CB; also HD and CD the tangent and secant of the elevation of CD; also H*e* and C*e* the tangent and secant of the elevation of C*e* or DE; also H*f* and C*f* the tangent and secant of the elevation of EF; and so on; also the parts of the vertical *ab*, *b*D, *ef*, *fg*, denoting the weights laid on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts, in the directions of the bars, are to one another, directly in proportion as the secants of their angles of elevation above the horizontal directions; or, which is the same thing, reciprocally proportional to the cosines of the same elevations, or reciprocally proportional to the sines of the vertical angles, *a*, *b*, D, *e*, *f*, *g*, &c. made by the vertical line with the several directions of the bars; because the secants of any angles are always reciprocally in proportion as their cosines.

2. The weight or load laid on each angle, is directly proportional to the *difference* between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle, is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical *ag*, is to the line CH, so is the weight of the whole assemblage of bars, to the horizontal thrust. Other properties also, concerning the weights and the thrusts, might be pointed out, but they are less simple and elegant than the above, and are therefore omitted; the following only excepted, which are inserted here on account of their usefulness.

Corol. 3. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it. Thus, in the triangle *b*CD, in which the side *b*D is proportional to the weight laid on the angle C, because the sides of any triangle are to one another as the sines of their opposite angles, therefore as sin. D : C*b* :: sin. *b*CD : *b*D; that is, *b*D is proportional to $Cb \cdot \frac{\sin. bCD}{\sin. D}$; but the sine of angle D is the cosine of the elevation DCH, and the cosine of any angle is reciprocally proportional to the secant, therefore *b*D is as C*b*. sin. *b*CD . sec. DCH; and C*b* being as the secant of the angle *b*CH of the elevation of *b*C or BC above the horizon, therefore *b*D is

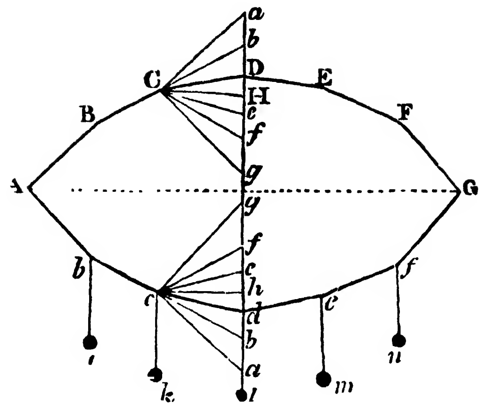
as sin. bCD . sec. bCH . sec. DCH : and the sine of bCD being the same as the sine of its supplement BCD; therefore the weight on the angle C, which is as bD, is as the sin. BCD . sec. DCH . sec. bCH, that is, as the continual product of the sine of that angle, and the secants of the elevations of its two sides above the horizon.

Corol. 4. Further, it easily appears also, that the same weight on any angle C, is directly proportional to the sine of that angle BCD, and inversely proportional to the sines of the two parts BCP, DCP, into which the same angle is divided by the vertical line CP. For the secants of angles are reciprocally proportional to their cosines or the sines of their complements: but BCP = CbH, is the complement of the elevation bCH, and DCP is the complement of the elevation DCH; therefore sec. bCH . sec. DCH is reciprocally as sin. BCP . sin. DCP; also the sine of bCD is = the sine of its supplement BCD; consequently the weight on the angle C, which is proportional to sin. bCD . sec. bCH . sec. DCH, is also proportional to $\frac{\text{sin BCD}}{\text{sin. BCP . sin. DCP}}$, when the whole frame or series of angles is balanced, or kept in equilibrio, by the weights on the angles; the same as in the preceding proposition.

Scholium. The foregoing proposition is very fruitful in its practical consequences, and contains the whole theory of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears, too, that the horizontal thrust, which is constant or uniformly the same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

The solution of the foregoing proposition depends on this consideration, viz. that an assemblage of bars or beams, being connected together by joints at their extremities, and freely moveable about them, may be placed in such a vertical position, as to be exactly balanced, or kept in equilibrio, by their mutual thrusts and pressures at the joints; and that the effect will be the same if the bars themselves be considered as without weight, and the angles be pressed down by laying on them weights which shall be equal to the vertical pressures at the same angles, produced by the bars in the case when they are considered as endued with their own natural weights. And as we have found that the bars may be of any length, without affecting the general properties and proportions of the thrusts and pressures, therefore by supposing them to become short, like arch stones, it is plain that we shall then have the same principles and properties accommodated to a real arch of equilibration, or one that supports itself in a perfect balance.

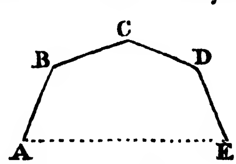
114. Prop. VI. If the whole figure in the last problem be inverted, or turned round the horizontal line AG as an axis, till it be completely reversed, or in the same vertical plane below the first position, each angle D, d, &c. being in the same plumb line; and if weights i, k, l, m, n, which are respectively equal to the weights laid on the angles B, C, D, E, F, of the first figure, be now suspended by threads from the corresponding angles b, c, d, e, f, of the lower figure; it is required to show that those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical or horizontal or oblique, will be exactly equal to the corresponding forces of weight or pressure or thrust in the analogous directions of the first figure.



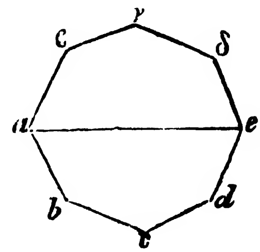
This necessarily happens, from the equality of the weights, and the similarity of the positions, and actions of the whole in both cases. Thus, from the equality of the corresponding weights, at the like angles, the ratios of the weights, *ab*, *bd*, *dh*, *he*, &c. in the lower figure, are the very same as those, *ab*, *b*D, DH, H*e*, &c. in the upper figure; and from the equality of the constant horizontal forces CH, *ch*, and the similarity of the positions, the corresponding vertical lines, denoting the weights, are equal, namely, $ab = ab$, $b\text{D} = bd$, $\text{DH} = dh$, &c. The same may be said of the oblique lines also, *ca*, *cb*, &c. which being parallel to the beams A*b*, *bc*, &c. will denote the tensions of these, in the direction of their length, the same as the oblique thrusts or pushes in the upper figures. Thus, all the corresponding weights and actions, and positions, in the two situations, being exactly equal and similar, changing only drawing and tension for pushing and thrusting, the balance and equilibrium of the upper figure is still preserved the same in the hanging festoon or lower one.

Scholium.—The same figure, it is evident, will also arise, if the same weights, *i*, *k*, *l*, *m*, *n*, be suspended at like distances A*b*, *bc*, &c. on a thread, or cord, or chain, &c. having in itself little or no weight. For the equality of the weights, and their directions and distances, will put the whole line, when they come to equilibrium, into the same festoon shape of figure. So that, whatever properties are inferred in the corollaries to the foregoing prob. will equally apply to the festoon or lower figure hanging in equilibrio.

This is a most useful principle in all cases of equilibriums, especially to the mere practical mechanist, and enables him in an experimental way to resolve problems, which the best mathematicians have found it no easy matter to effect by mere computation. For thus, in a simple and easy way he obtains the shape of an equilibrated arch or bridge; and thus also he readily obtains the positions of the rafters in the frame of an equilibrated curb or mansard roof; a single instance of which may serve to show the extent and uses to which it may be applied. Thus, if it should be required to make a curb frame roof having a given width AE, and consisting of four rafters AB, BC, CD, DE, which shall either be equal or in any given proportion to each other. There can be no doubt but that the best form of the roof will be that which puts all its parts in equilibrio, so that there may be no unbalanced parts which may require the aid of ties or stays to keep the frame in its position. Here the



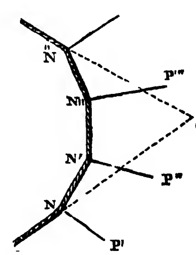
mechanic has nothing to do, but to take four like but small pieces, that are either equal or in the same given proportions as those proposed, and connect them closely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend this from two pins, *a*, *e*, fixed in a horizontal line, and the chain of the pieces will arrange itself in such a festoon or form, *abcde*, that all its parts will come to rest in equilibrio. Then, by inverting the figure, it will exhibit the form and frame of a curb roof $a\beta\gamma\delta e$, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual pulls or tensions of the hanging festoon *a b c d e*. By varying the distance *ae*, of the points of suspension, moving them nearer to, or farther off, the chain will take different forms; then the frame ABCDE may be made similar to that form which has the most pleasing or convenient shape, found above as a model.



Indeed this principle is exceeding fruitful in its practical consequences. It is easy to perceive that it contains the whole theory of the construction of arches: for each stone of an arch may be considered as one of the rafters or beams in the foregoing frames, since the whole is sustained by the mere principle of equilibration, and the method in its application, will afford some elegant and simple solutions of the most difficult cases of this important problem *.

115. Prop. VII. To determine the conditions of equilibrium in the funicular polygon when many forces are acting at different points of the cord, but in the same plane.

Let PNN′N″, &c. be the polygon proposed, being kept in equilibrio by the powers P′, P″, . . . P^n, acting in the directions PN, P′N, P″N′, &c. And call t, t', t'', &c. the respective tensions of the parts of the cord PN, NN′, N′N″, &c. Now, since the equilibrium obtains in the system, it must necessarily have place in each portion of the polygon separately. Hence P, P′, and t', must be in equilibrio about the node N; and t'' must be the resultant of the component forces P, P′; the force which acts on the point N′ in direction N′N, is therefore equivalent to the two forces P, P′, acting simultaneously at N; and the node N″ is acted upon as though it were solicited by the four forces P, P′, P″, t'', in directions respectively parallel to PN, P′N, P″N′, N″N′. In like manner it may be shown that the node N″ is kept in equilibrio in the same way as it would be, if subjected to the simultaneous action of the powers P, P′, P″, P‴, t''', in directions parallel to PN, P′N, P″N′, &c. And so on throughout. Hence it follows, that *when a funicular polygon is sustained in equilibrio by any number of forces whatever, if we transport these powers parallel to their respective directions, so as all to exert their energies upon one point, it will be kept in equilibrio by their combined action.*

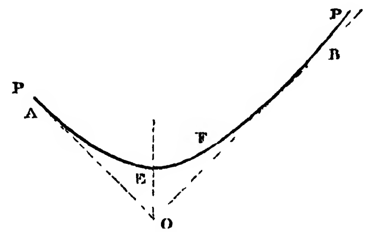


Corol. 1. If we neglect the two extreme forces P, P^n, and reason upon the

* The skilful preceptor will supply various numerical, and some geometrical, examples in illustration of these, and the subjoined propositions on the stability of structures.

others as in the proposition, it is clear we may conceive all of them applied at the same point (provided they make respectively the same directions with any assumed axis AX), without at all altering the magnitude of their resultant: and hence as that resultant is destroyed by the two powers P, P″, it must necessarily pass through the point of concourse O of the directions PN, P″N‴.

Corol. 2. If, therefore, a cord AEB, fixed at two points A and B, have all its points solicited by any forces whatever in the same plane, it will assume a plane curvature: and the point of intersection O of two tangents will fall upon the direction of the resultant of all the forces applied to the various points of the cord. And if we transfer these forces parallel to their directions, so as to apply them all at the point O, their resultant being resolved into two others acting according to the directions AO and OB, we shall thence obtain the effort exerted upon each of the fixed points.



Corol. 3. The case of the last corol. applies obviously to gravity; for, on the one hand, this force exercises its action on all the points of the cord, and on the other, these efforts may be assimilated to the weight distributed throughout the length of the heavy cord. Hence, the curve thus formed, and known by the names of *funicular curve, chainette*, or *catenary*, is a plane curve.

Corol. 4. In the catenary the total effort exerted on the fixed points A and B, is the whole weight of the cord: if, therefore, at the point of concourse O of the two tangents to the curve at A and B, a weight equal to that of the cord were sustained by two threads AO, BO, void of gravity, the points A and B would be acted upon in the same manner as they are by the action of gravity upon the cord AEB; viz. the powers P, P, necessary to retain either the heavy cord AEB, or the equal weight at O, would be the same in both cases. As the resultant is the weight of the cord, if we erect upon O an indefinite perpendicular OE, it will pass through the centre of gravity, and the forces exerted upon A and B will be proportional to the sines of the angles BOE and AOE: thus, if W be the entire weight of the cord, we have

$$W : P : P' :: \sin. AOB \,.\, \sin. EOB : \sin. AOE.$$

Corol. 5. The same will obtain, wherever the points A and B are found in the curve, since the state of equilibrium allows us to consider as fixed any two points in the curve. If, therefore, we consider the point F as fixed instead of the point B, the portion AEF of the curve will not change its form: nor will there be any change in the tension P exerted at A.

Corol. 6. If several weights W, W′, W″, &c. hang upon a cord PC′ CP′, the weight of which is inconsiderable with respect to either of those weights, the pressure upon any angle C of the funicular polygon, will be $\propto \dfrac{\sin. dCe}{\sin. eCc \,.\, \sin. cCo}$, the line C*e* being a continuation of the vertical W″C, as in cor. 4, prop. V.

Corol. 7. If the number of weights hanging from the cord be increased, and the distances on the cord of the points from which the weights hang be increased indefinitely, or if instead of the weights we conceive pieces of heavy cord, or of chains, to be hung from different points of the cord, our funicular polygon will then become a curve; being indeed a species of catenary. The angle *eCd* will then become the angle of contact formed by the tangent and curve, whose sine is equal to the measure of the angle; and hence, because the

angle of contact is as the curvature of the arch, or reciprocally as the radius of curvature, the weight hanging at any point C, will be reciprocally as the radius of curvature at that point, and the square of the sine of the angle made by the curve (or its tangent) and the vertical.

Corol. 8. Lastly, a heavy cord cannot by any force be stretched into a right line, except it be in a vertical position: for, the weight of the cord may be considered as a force applied at its centre of gravity; and then, the cord AEB being retained by the two forces P, P′, if W be its weight, we have

$$\text{(cor. 4.) } P : W :: \sin. EOB : \sin. AOB;$$

where, it is obvious, the more the cord is stretched, the greater the angle AOB becomes, and the more nearly the angle EOB approaches to a right angle: so that the cord can only be stretched straight horizontally, when this analogy obtains, $P : W :: 1 : 0$, that is, when P is infinite. Thus, however small the weight is, it will cause the cord to be curved, unless it be placed vertically: which is, indeed, a circumstance experienced daily.

II. ON ARCHES AND PIERS.

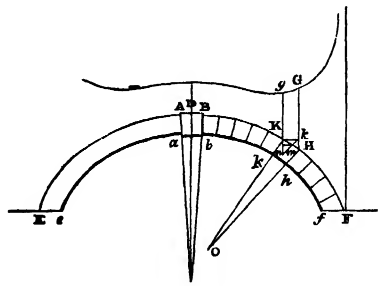
116. The construction of arches is one of the most important and difficult branches of Architecture, particularly when considered in relation to the erection of bridges over broad and rapid rivers: it commonly imposes the double task of blending the handsome forms and the decorations of the ordinary architecture with the firmness and durability which ought always to be found in works destined not merely for the accommodation of the public, but in many cases for its safety. The theory of arches, when carried to the extent its importance and utility demands, would itself fill a volume; but all that we shall attempt in this place will be a concise view of the leading particulars, according to the most simple and obvious theory.

117. Defs. By an arch we here mean an artful disposition of several stones, or bricks, or other suitable materials, the under part of which is in a bow-like form, their weight producing a mutual pressure, so that they not only support each other but may be made to carry the most enormous weights. Other particulars relating to an arch, the defining of which is necessary here, may be soon learnt by turning to the fig. to prop. XIV. Thus AS′ESB is the ponderating arch; A or B the *spring* of the arch; D its *crown;* AB its *span;* CD its *height* or versed sine, or rise; ADB the *intrados,* or the lower surface of the arch (often called the arch); S′ES the *extrados,* being in bridges the superior surface, or the roadway; F′, F, the *flanks* or *hances;* the spaces above these are called the *spandrels;* the portions of wedges which lie in a course contiguous to the intrados are called *voussoirs,* or arch stones; that which is at D is called the *keystone;* the walls or masses PQST, P′Q′S′T′, built to support the arches, and from which they spring as their bases, are either called *piers* or *abutments; piers* when they stand between two neighbouring arches, *abutments* when they support the arches which are contiguous to the shore: the part of the pier from which an arch springs is called the *impost;* the curve formed by the upper sides of the voussoirs the *archivolt;* and the lines FS, F′S′, about the flanks, in which a break is most likely to take place, are called *joints of fracture.* The other terms we shall use will need no explanation.

When we reflect upon the immense quantity of heavy materials suspended in the air in a large arch, and compare it with the small cohesion which the firmest cement can give to such an edifice, we shall be convinced that its parts are not kept together by the force of the cement; the stability of the whole is the result of the just balance and equilibration of all its parts. The principles of this equilibration we shall now exhibit, premising, that they are founded

upon the hypothesis of the ponderating matter pressing upon the voussoirs in the vertical direction, and that we here pay no regard to any small pressure in other directions which may be occasioned by filling up the spandrels with rubbles, &c.

118. Prop. VIII. The intrados being given to find the height of the extrados above any given point in it.



Let Hh Kk be one of the voussoirs, and gH be the materials resting upon it; let D be the crown of the arch, $ABba$ the key-stone; and let the voussoirs be produced to meet in C and O respectively.

The horizontal thrust is supported by the abutments: and hence the only force tending to destroy the arch when the abutments are secure is the vertical pressure of the wedge and its superincumbent mass. If all these pressures be equal, there is no tendency in one voussoir more than in another to fall down: and as the arch cannot be destroyed but by some one of these voussoirs falling, there will be an equilibrium when all these vertical forces are equal.

But the force tending to push any voussoir downwards is the weight of the mass upon that voussoir. Let its breadth Hh be b; the inclination of its face to the horizon be θ; the forces HO, Oh of the entire wedge formed by supposing the voussoir completed be R; and GH the height of the roadway above the voussoir be A.

Then $gH = GH \,.\, Km = A \,.\, b \cos.\theta$; the effect of this perpendicular to HK is $gH \cos.\theta\ GH \,.\, hm = A \,.\, b \cos.\theta \,.\, \cos.\theta = A \,.\, b \cos.^2\theta$.

The tendency of this to drive the wedge inwards is $\frac{R}{b} \,.\, A \,.\, b \cos.^2\theta = R \,.\, A \cos.^2\theta$.

The portion of this force which cuts perpendicularly to the horizon will be

$$R \,.\, A \,.\, \cos.^2\theta \,.\, \cos.\theta = R \,.\, A \,.\, \cos.^3\theta \quad . \quad . \quad . \quad . \quad (1.)$$

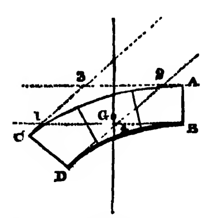
Let r, a, 0, be the values of R, h, θ, at the crown of the arch; then at this point the downward pressure is

$$ra \cos.^3 0^\circ = ra \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

And by the foregoing reasoning the pressures (1), (2) are equal. Hence

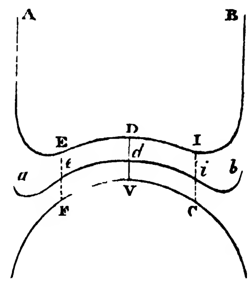
$$R \,.\, A \,.\, \cos.^3\theta = ra, \text{ or } A = \frac{ra}{R} \sec.^3\theta.$$

Now, if the points H, h approach indefinitely close to one another, R and r are the radii of curvature at H and D; and hence if we consider the arch as being one continued curve, we may take as the value of R, the value of the radius of curvature at the middle point of the voussoir.



It is, however, also necessary to the equilibrium that the vertical drawn through the centre of gravity G of the part A C D B should cut the parallelogram 1 2 3 4, made by perpendiculars to AB and CD drawn from their extremities: for otherwise there would be no point in the vertical through G (at some part of which the weight must be supposed to act), at which the directions of the perpendicular pressures could meet, and no three forces can maintain equilibrium unless their directions pass through one point.

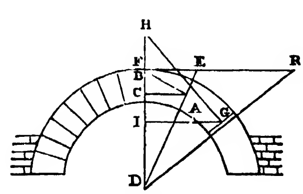
119. PROP. IX. If AEDIB be the extrados and PVP′ the intrados of an equilibrated arch, and if any number of vertical lines EF, DV, IC, &c. be drawn from the one curve to the other, and these lines be divided in a given ratio in the points a, e, d, i, b, then the curve drawn through these points of division will also be a proper extrados, the mass contained between it and the intrados being duly balanced, as well as that comprised between AEDIB and PVCP′.



For, since the whole is kept in equilibrio by the vertical pressures of the superincumbent mass on the intrados, the points F, C, V, &c. are sustained in equilibrio by the pressures of the parts EF, DV, IC, &c. bearing upon them: if, therefore, these lines be divided in e, d, i, &c. so that EF : eF :: DV : dV :: IC : iC, &c. all in the same constant ratio, then $aedib$ being considered as the extrados, the arch will still be in equilibrium; because the load on the intrados being every where lessened in a constant ratio, its tendency to break the arch will be every where in a constant ratio to its preceding tendency to cause a rupture, and the equilibrium can be no more destroyed in the one case than in the other. And the same kind of reasoning would apply if $aedib$ were above AEDIB, or if the weight and density of the materials between the extrados and intrados should be changed *throughout* in any constant ratio.

Corol. Hence we may in many cases give the extrados a pretty regular and practicable form, by diminishing the thickness over the crown: and hence appears one great advantage of iron as a material for bridges, since its requisite thickness at the crown is much less than that of stone.

120. PROP. X. When an arch is composed of blocks acting on each other without friction, the weight of the arch must increase at each step as the portion of the vertical tangent cut off by lines drawn from a given point in a direction parallel to that of the joints.



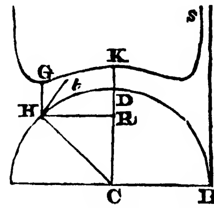
The thrust in the direction AB, by which the block A is supported, must be to its weight as AB to BC, or as DE to EF, and to the horizontal thrust, as AB to AC, or DE to DF: and for the same reason the weight of any other part FG must be to the horizontal thrust as HI to IG, or as FK to FD: but the horizontal thrust is equal throughout the arch, being propagated from the abutments, since the weight of the blocks, acting in a vertical direction, can neither increase nor diminish it; and it may therefore always be represented by the line DF, while FE, EK, represent the weight of the arch and of its parts; and it will be equal to the weight of a portion of the length of the radius DF and of the depth of the block AC, as is obvious from considering the effect of the upper block acting as a wedge.

PRACTICAL SCHOLIUM.

121. The radii of curvature for the different curves are always determinable by the method of fluxions; by means of the fluxional value $r = \dfrac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, or $r = \dfrac{\dot{z}^3}{\dot{y}\ddot{x}}$, when $\dot{y}$ is constant: they are here supposed known.

EXAMPLES.

1. Suppose for example, it were required to find the requisite thickness over any point of a circular arc, to ensure equilibration, the thickness a = DK, at the crown of the arch being given.



Here, rad. of curve, at H, = rad. of curv. at D, that is $R = r$.

and sec. RHt = sec. arc. DH.

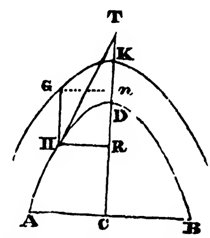
consequently $GH = \sec.^3 DH \times \frac{ar}{R} = \sec.^3 DH \cdot a$.

Hence we have a convenient logarithmic expression for computation; viz.

$$\text{Log. DK} + 3 \text{ log. sec. DH} = \text{log. GH}.$$

In this example, the curve of equilibration, GKs, runs up to an infinite height over B, the springing of a semicircular arch. But over a portion of 30° or 35° on each side the vertex, as DH, the curve KG of the extrados accords very well with what would be required for a roadway.

2. Determine the requisite thickness for equilibration over any point of a parabola.



Here if $DR = x$, $RH = y$, $R = \frac{(4x + p)^{\frac{3}{2}}}{2\sqrt{p}}$; which, at the vertex, where x vanishes becomes $r = \frac{1}{2}p$:

Also, in the general case,

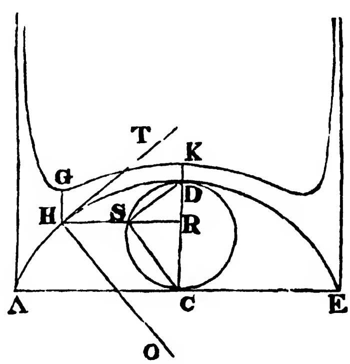
$$RT = 2x,\ RH = y = \sqrt{px},\ TH = \sqrt{HR^2 + RT^2} = \sqrt{px + 4x^2}$$

$$\text{sec. } THR = \frac{TH}{HR} = \sqrt{\frac{px + 4x^2}{px}} = \sqrt{\frac{p + 4x}{p}},$$

$$\therefore GH = \sec.^3 THR \cdot \frac{r}{R} a$$

$$= \frac{(p + 4x)^{\frac{3}{2}}}{p^{\frac{3}{2}}} \cdot \tfrac{1}{2}p \cdot \frac{2\sqrt{p}}{(p + 4x)^{\frac{3}{2}}} a = a = KD.$$

So that the extrados is a parabola equal to the intrados, and every where vertically equidistant from it.



3. To determine the requisite thickness over any point of a cycloidal arch.

Here, putting $DK = a$, $DR = x$, $DC = d$; we have, from the known properties of the cycloid, the tangent HT parallel to the corresponding chord SD, or angle $THR = \angle DSR$. $SD = \sqrt{dx}$, $SR = \sqrt{dx - x^2}$;

$$R \text{ (parallel to SC)} = HO = 2SC = 2\sqrt{d^2 - dx};\ r = 2CD = 2d;$$

$$\text{sec. } THR = \frac{SD}{SR} = \sqrt{\frac{dx}{dx - x^2}} = \sqrt{\frac{d}{d - x}}.$$

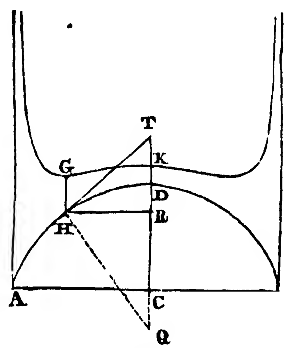
$$GH = \sec.^3 \, THR \, . \, \frac{r}{R} \, . \, a$$

$$= \frac{d^{\frac{3}{2}}}{(d-x)^{\frac{3}{2}}} \, . \, 2d \, . \, \frac{a}{2(d^2 - dx)^{\frac{1}{2}}},$$

$$= \frac{d^{\frac{3}{2}}}{(d-x)^{\frac{3}{2}}} \, . \, \frac{2da}{2d^{\frac{1}{2}}(d-x)^{\frac{1}{2}}},$$

$$= \frac{d^{\frac{4}{2}}}{(d-x)^{\frac{4}{2}}} \, a = \frac{ad^2}{(d-x)^2} = \frac{DK \, . \, CD^2}{CR^2}.$$

By computing the value of GH for several corresponding values of DR, and CR, and thence constructing the extrados by points, it will, as in the figure, appear analogous to that for the circle, but rather flatter till it approach the extremities of the arch, where the curve runs off to infinity, as in the case for the circle.



4. To determine the requisite thickness over any point of an elliptical arch.

Here, taking x, y, and a, as before, take $AC = t$, $DC = c$, $HQ = \pi$, being perpendicular to the tangent HT. Then, by the property of the ellipse,

$$DC^2 : AC^2 :: CR : QR,$$

or, $c^2 : t^2 :: c - x : \frac{t^2}{c^2}(c - x) = QR.$

Also, $\sec. \, THR = \sec. \, HQR = \frac{HQ}{QR} = \pi \div \frac{t^2}{c^2}(c - x) = \frac{\pi c^2}{t^2 (c - x)}.$

Radius of curvature at $H = R = \frac{4\pi^3}{p^2}$, p being the parameter to $CD = \frac{2t^2}{c}$:

$\therefore R = \frac{4\pi^3 c^2}{4t^4} = \frac{\pi^3 c^2}{t^4}$; and r (rad. curv. at D) $= \frac{t^2}{c}$.

Whence, lastly, $GH = \sec.^3 \, THR \, . \, \frac{r}{R} \, . \, a,$

$$= \frac{\pi^3 c^6}{t^6 (c - x)^3} \, . \, \frac{t^2}{c} \, . \, \frac{t^4}{\pi^3 c^2} \, a,$$

$$= \frac{c^3}{(c - x)^3} \, a = \frac{DK \, . \, DC^3}{CR^3}:$$

as before, a convenient expression for logarithmic operation.

Here, again, computing values of GH for several assumed values of CR, the curve of the extrados may thence be constructed, and, like that for the cycloid, it will be found rather flatter than that for the circle, but still analogous to it.

5. For the Catenary. (See the fig. to Exam. 2.)

Here, put $DR = x$, $GR = y$, $DG = z$, $t =$ tension at the vertex D when the chain hangs from A and B. Then, by the nature of the curve, $z^2 = 2tx + x^2$, subtang. $TR = \frac{zy}{t}$. Rad. curv. at $G = \frac{t^2 + z^2}{t} = R$, and therefore at D where z vanishes $r = t$.

$$HT = \sqrt{HR^2 + RT^2} = \sqrt{y^2 + \frac{z^2y^2}{t^2}},$$

$$\text{sec. THR} = \frac{TH}{HR} = \sqrt{y^2 + \frac{z^2y^2}{t^2}} \div y = \sqrt{1 + \frac{z^2}{t^2}} = \sqrt{\frac{t^2 + z^2}{t^2}}.$$

$$\therefore GH = \text{sec.}^3\text{THR} \,.\, \frac{r}{R}\, a = \frac{(t^2 + z^2)^{\frac{3}{2}}}{t^3} \,.\, t \,.\, \frac{t}{t^2 + z^2} \,.\, a$$

$$= \frac{(t^2 + z^2)^{\frac{1}{2}} ta}{t^3} = \frac{(t^2 + z^2)^{\frac{1}{2}} a}{t} \text{ [substituting for } z^2 \text{ its value.]}$$

$$= \frac{(t^2 + 2tx + x^2)^{\frac{1}{2}} a}{t} = \frac{a(t + x)}{t} = a + \frac{ax}{t}.$$

Corol. If $a = t$, or the thickness at the crown equal to a line whose weight expresses the tension,

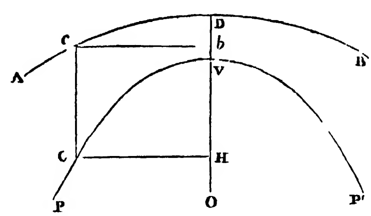
then $GH = a + x = KD + DR$.

Corol. 2. If $a \gtrless t$, the exterior curve will proceed $\left\{\begin{matrix}\text{upwards} \\ \text{downwards}\end{matrix}\right\}$ both ways from K.

Corol 3. If DK, the thickness at the crown, be very small compared with t, then will the thickness over H be nearly the same throughout: thus, suppose $a = \frac{1}{10000}\, t$, then $GH = a + \frac{tx}{10000t} = a + \frac{x}{10000} = a$ very nearly. Consequently, a heavy flexible cord or chain, left to adjust itself into a hanging catenary, and inverted, would support itself upon props perpendicular to the tangents at A and B.

Or $Kn = a + x - (a + \frac{ax}{t}) = x - \frac{ax}{t} = (t - a)\,\frac{x}{t}$; which when a vanishes becomes $= x$, or $nR = GH = a$.

122. Prop. XI. Having given the extrados of an equilibrated arch, to find the intrados.



Let ADB be the extrados proposed to which the intrados PVP′ is to be adapted; OV being the common axis of both curves: from c and C, corresponding points equi-distant from the axis, draw the ordinates ch, CH. Put DV (the thickness of the arch at the crown) $= a$, $Dh = \dot{X}$, $VH = x$, the equal ordinates $ch, = CH = y$, and the arch $VC = z$. Then $Cc \propto \frac{\ddot{y}x - \ddot{x}y}{\dot{y}^3} = \frac{\ddot{y}x - \ddot{x}y}{\dot{y}^3}$. C, where C is a constant quantity found by taking the actual value of Cc in V the vertex of the curve. But it is manifest that $Cc = DV + VH - Dh = a + x - \dot{X}$: consequently $a + x - \dot{X} = C$. $\frac{\ddot{y}x - \ddot{x}y}{\dot{y}^3} = \frac{C}{\dot{y}} \times \text{flux. of } \frac{\dot{x}}{\dot{y}}$. If, then, we substitute the true value of $\dot{X}$ in terms of y (which is given because the form of the extrados is known), the equations thence resulting will contain only x and y with their first and second fluxions, and known quantities; and from this the real relation of x and y must be struck out by such means as seem most naturally to apply to any proposed instance.

This, however, in many cases will be a matter of considerable difficulty: we shall here, therefore, solely trace the process in the most useful instance, which, happily, admits of a comparatively simple investigation. We advert to the case in which *the extrados is a straight horizontal line,* which shall be now considered.

Retaining the same notation, we have $Dh = \dot{X} = 0$, and consequently $a + x = \frac{C}{\dot{y}} \times$ flux. of $\frac{\dot{x}}{\dot{y}}$. Assume $\dot{y} = \frac{\dot{x}}{u}$, whence $u = \frac{\dot{x}}{\dot{y}}$, and $\frac{C}{\dot{y}} \times$ flux. of $\frac{\dot{x}}{\dot{y}} = \frac{Cu\dot{u}}{\dot{x}}$ that is, $a + x = \frac{Cu\dot{u}}{\dot{x}}$, and of course $a\dot{x} + x\dot{x} = Cu\dot{u}$: taking the fluents of this we have $x^2 + 2ax = Cu^2$, and $u = \sqrt{\frac{x^2 + 2ax}{C}}$. But because $\dot{y} = \frac{\dot{x}}{u}$ it is also $= \dot{x} \div \sqrt{\frac{x^2 + 2ax}{C}} = \dot{x}\sqrt{\frac{C}{x^2 + 2ax}}$. The fluent of this expression is $y = \sqrt{C} \times$ hyp. log. $[2x^2 + 2ax + 2\sqrt{(x^2 + 2ax)}]$. Now at the vertex where $x = 0$, we have $y = \sqrt{C} \times$ hyp. log. $2a$, so that the corrected fluent is

$$y = \sqrt{}\left\{C \times \text{hyp. log.} \frac{x + a + \sqrt{(x^2 + 2ax)}}{a}\right\}.$$

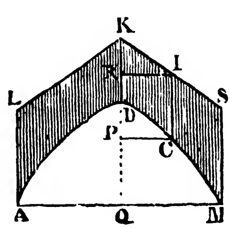
We have yet to ascertain the constant quantity C, in order to which we may proceed thus: when H arrives at O we have $x = vo = h$, and $y = OP = s$; substituting these in the last equation it becomes $h = \sqrt{C} \times$ hyp. log. $\frac{s + a + \sqrt{(ss + 2as)}}{a}$, and consequently $\sqrt{C} = h \div$ hyp. log. $s + a + \frac{1}{a}\sqrt{s^2 + 2as}$. Hence, then, we at length obtain this general value of y, that is,

$$y = h \times \text{hyp. log.} \frac{a + x + \sqrt{(2ax + x}}{a} \div \text{hyp log.} \frac{a + s + \sqrt{(2as + s^2)}}{a}.$$

SCHOLIUM.

123. A useful illustration of the value of this class of investigations presents itself in a powder magazine, of which the roof is constituted of two inclined planes. Of this we annex Dr. Hutton's original solution in his own terms and notation.

Let $a = DK$ the thickness of the arch at the top, $x =$ any absciss DP of the required arch ADCM, $u =$ KR the corresponding absciss of the given exterior line KI, and $y = PC = RI$ their equal ordinates. Then is CI or $w = a + x - u = Q \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $= Q \times \frac{\ddot{x}}{\dot{y}^2}$, supposing $\dot{y}$ a constant quantity, and where Q is some certain quantity to be determined hereafter. But KR or u is $= ty$, if t be put to denote the tangent of the given angle of elevation KIR, to radius 1; and then the equation is $w = a + x - ty = \frac{Q\ddot{x}}{\dot{y}^2}$.



Now, the fluxion of the equation $w = a + x - ty$, is $\dot{w} = \dot{x} - t\dot{y}$, and the 2d fluxion is $\ddot{w} = \ddot{x}$; therefore the foregoing general equation becomes $w = \frac{Q\ddot{w}}{\dot{y}^2}$; and hence $w\dot{w} = \frac{Q\dot{w}\ddot{w}}{\dot{y}^2}$, the fluent of which gives $w^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$: but at D the value of w is $= a$, and $\dot{w} = 0$, the curve at D being parallel to KI; therefore the correct fluent is $w^2 - a^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$. Hence then $\dot{y}^2 =$

$\frac{Q\dot{w}^2}{w^2-a^2}$, or $\dot{y} = \frac{\dot{w}\sqrt{Q}}{\sqrt{(w^2-a^2)}}$; the correct fluent of which gives $y = \sqrt{Q} \times$ hyp. log. of $\frac{w + \sqrt{(w^2 - a^2)}}{a}$.

Now, to determine the value of Q, we are to consider that when the vertical line CI is in the position AL or MS, then $w =$ CI becomes $=$ AL or MS $=$ the given quantity c suppose, and $y =$ AQ or QM $= b$ suppose, in which position the last equation becomes $b = \sqrt{Q} \times$ hyp. log. $\frac{c + \sqrt{(c^2 - a^2)}}{a}$; and hence it is found that the value of the constant quantity $\sqrt{Q}$ is $\frac{b}{\text{h. l. } c + \sqrt{(c^2 - a^2)}}$; which being substituted for it, in the above general value of y, that value becomes

$$y = b \times \frac{\text{log. of } \frac{w + \sqrt{(w^2 - a^2)}}{a}}{\text{log. of } \frac{c + \sqrt{(c^2 - a^2)}}{a}} = b \times \frac{\text{log. of } w + \sqrt{(w^2 - a^2)} - \text{log. } a}{\text{log. of } c + \sqrt{(c^2 - a^2)} - \text{log. } a};$$ from which equation the value of the ordinate PC may always be found, to every given value of the vertical CI.

But if, on the other hand, PC be given, to find CI, which will be the more convenient way, it may be found in the following manner. Put A $=$ log. of a, and C $= \frac{1}{b} \times$ log. of $\frac{c + \sqrt{(c^2 - a^2)}}{a}$; then the above equation gives $Cy + A =$ log of $w + \sqrt{(w^2 - a^2)}$; again, put $n =$ the number whose log. is $Cy + A$; then $n = w + \sqrt{(w^2 - a^2)}$; and hence $w = \frac{a^2 + n^2}{2n} =$ CI.

Now, for an example in numbers, in a real case of this nature, let the foregoing figure represent a transverse vertical section of a magazine arch balanced in all its parts, in which the span or width AM is 20 feet, the pitch or height DQ is 10 feet, thickness at the crown DK $= 7$ feet, and the angle of the ridge LKS $112^\circ\ 37'$, or the half of it LKD $= 56^\circ\ 18'\frac{1}{2}$, the complement of which, or the elevation KIR, is $33^\circ\ 41'\frac{1}{2}$, the tangent of which is $= \frac{2}{3}$, which will therefore be the value of t in the foregoing investigation. The values of the other letters will be as follows, viz. DK $= a = 7$; AQ $= b = 10$; DQ $= h = 10$; AL $= c = 10\frac{1}{3} = \frac{31}{3}$; A $=$ log. of 7 $=$ ·8450980; C $= \frac{1}{b} \times$ log. of $\frac{c + \sqrt{(c^2 - a^2)}}{a} = \frac{1}{10}$ log. of $\frac{31 + \sqrt{520}}{21} = \frac{1}{10}$ log. of 2·56207 $=$ ·0408591; $cy + A =$ ·0408591y $+$ ·8450980 $=$ log. of n. From the general equation then, viz. CI $= w = \frac{a^2 + n^2}{2n} = \frac{a^2}{2n} + \frac{1}{2}n$, by assuming y successively equal to 1, 2, 3, 4, &c., thence finding the corresponding values of $Cy + A$ or ·0408591y $+$ ·8450980, and to these, as common logs, taking out the corresponding natural numbers, which will be the values of n; then the above theorem will give the several values of w or CI, as they are here arranged in the annexed table, from which the figure of the curve is to be constructed, by thus finding so many points in it.

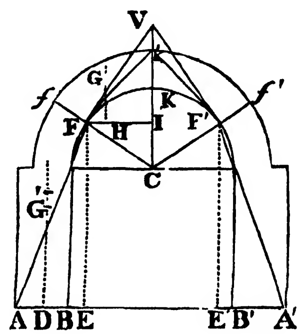
Val. of y or CP.	Val. of w or CI.
1	7·0309
2	7·1243
3	7·2806
4	7·5015
5	7·7888
6	8·1452
7	8·5737
8	9 0781
9	9·6623
10	10·3333

Otherwise. Instead of making n the number of the log. $Cy + A$, if we put $m =$ the natural number of the log. Cy only; then $m = \frac{w + \sqrt{(w^2 - a^2)}}{a}$, and $am - w = \sqrt{(w^2 - a^2)}$, or by squaring, &c., $a^2m^2 - 2amw + w^2 = w^2 - a^2$, and hence $w = \frac{m^2 + 1}{2m} \times a$: to which the numbers being applied, the very same conclusions result as in the foregoing calculation and table.

124. When the operation of cements is taken into the consideration, the conditions to ensure equilibrium are more easily investigated than when the gravitating tendency of the superincumbent matter is alone regarded. If the cohesive energy of the cement were insuperable, the arch might then be considered as one mass, which would be every where secure, whatever its form might be, provided

the piers or abutments were sufficiently strong to resist the horizontal thrust. And, although this property cannot safely be imputed to any cement (strong as many cements are known to be), yet, in a structure, whose component parts are united with a very powerful cement, the matter above an arch will not yield, as when the whole is formed of simple wedges, or as when it would give way in vertical columns, but by the separation of the entire mass into three, or at most, into four pieces: that is, either into the two piers, and the whole mass between them, or into the two piers, and the including mass splitting into two at its crown. It may be advisable, therefore, to investigate the conditions of equilibrium for both these classes of dislocations.

125. Prop. XII. Suppose that the arch $Fff'F'$ tend to fall vertically in one mass, by thrusting out the piers at the joints of fracture, Ff, $F'f'$; it is required to investigate the equations by which the equilibrium may be determined.



Let 2A denote the whole weight of the arch lying between Ff, and $F'f'$, G the centre of gravity of one half of that arch, the centre of gravity of the whole lying on CV; let P be the weight of one of the piers, reckoned as high as Ff, and G′ the place of its centre of gravity.

Now, FV, F′V, being respectively perpendicular to Ff, $F'f'$, the weight 2A may be understood to act from V, in the directions VF, VF′, and pressing upon the two joints Ff, $F'f'$. The horizontal thrust which it exerts on F, will be $= A \tan. FVI = A \cot. FCI = A \cdot \frac{CI}{FI}$; and at the same time the vertical effort will $= A$.

Now, the first of these forces tends to thrust out the solid AF horizontally, an effort which is resisted by friction; and since it is known that, *cœteris paribus*, the friction varies as the pressure, that is, here, as the weight, we shall have for the resisting force, $f \cdot A + f \cdot P$. Equating this with the above expression, $A \cdot \frac{CI}{FI}$, we obtain for the first equation of equilibrium

$$f \cdot P = A \left(\frac{CI}{FI} - f\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

Moreover, the horizontal thrust that tends to overturn the pier AF about the angle A, must be regarded as acting at the arm of lever FE, and, therefore, as exerting altogether the energy, $A \cdot \frac{CI}{FI} \cdot FE$. This is counteracted by the vertical stress A, operating at the horizontal distance AE, and by the weight P, acting at the distance AD; DG′ being the vertical line passing through the centre of gravity, G′, of the pier. Hence we have

$$A \cdot \frac{CI}{FI} \cdot FE = A \cdot AE + P \cdot AD;$$

and, after a little reduction, there results for the second equation of equilibrium:

$$P \cdot \frac{AD}{FE} = A \left(\frac{CI}{FI} - \frac{AE}{FE}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

126. Prop. XIII. Suppose that each of the two halves kF, kF', of the arch,

tend to turn about the vertex k, removing the points F, and F′: it is required to investigate the conditions of equilibrium in that case.

Referring the weight, A, of the semi-arch from its centre of gravity to the direction of the vertical joint kK, its energy is represented by $A \cdot \frac{FH}{FI}$; and the resulting horizontal thrust at A is, evidently, $A \cdot \frac{FH}{FI} \cdot \frac{FI}{kI} = A \cdot \frac{FH}{kI}$. The vertical stress is $= P + A$; and therefore the friction is represented by $f \cdot P + f \cdot A$. Equating this with the above value of the horizontal thrust, that the pier AF may not move horizontally, we have

$$f \cdot P = A\left(\frac{FH}{kI} - f\right) \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

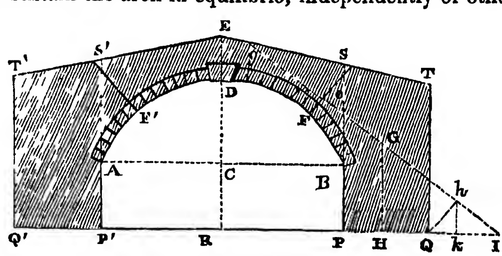
Then, considering the arch and piers as a polygon capable of moving about the angles A, F, k, F′, A′, we must, in order to equilibrium, balance the joint action of P and the semi-arch A at the point F, with the horizontal thrust beforementioned, acting at the arm of lever EF. Thus we shall have $P \cdot AD + A \cdot AE = A \cdot \frac{FH}{kI} \cdot EF$: from which, after due reduction, there results

$$P \cdot \frac{AD}{EF} = A\left(\frac{FH}{kI} - \frac{AE}{EF}\right) \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

Corol. Hence it will be easy to examine the stability of any arch whose parts are cemented as in the hypotheses of these two propositions. Assume different points such as F, in the arch, for which let the numerical values of the equations (I.) and (II.) be computed. To ensure stability, the first members of those equations, which represent the resistance to motion, must exceed the second members; the weakest points will be those in which the excess of the first above the second member is the least.

If the dimensions of the arch were given, and the thickness of the pier required, the same equations would serve for its determination *.

127. Prop. XIV. To determine the magnitude of the piers, or abutments, that they may sustain the arch in equilibrio, independently of other arches.



In order to give a solution to this problem we must assume some particulars as having been determined by adequate experiments and admeasurement: for we do not consider the piers as prisms standing upon their bases and resisting the pressure of the arches, though upon such an hypothesis it would be easy to lay down rules for the determination of the centres of gravity both of the arch and the piers; but, since the stones, &c. of the wall above the voussoirs are

* The principles adopted in the two last propositions are due to De la Hire, and Coulomb, respectively. For a more comprehensive view of this interesting subject, the student may consult *Hutton's Tracts*, vol. i., the *Appendix to Bossut's Mechanics, and Berard's Treatise on the Statics of Vaults and Domes.*

bonded in with those of the pier, the pier will by these means become augmented, and the weight of the arch diminished. We, therefore, regard the piers as extending to the *joints of fracture* (art. 117), and that portion of the arch which is comprised between those joints as a ponderating body resting in a state of equilibrium upon those joints as upon two inclined planes. Let, then, FS, F′S′, be the joints of fracture, G the centre of gravity of the pier QTSFBP, g that of the half arch DEFS, and let go the perpendicular from g upon FS be produced till it meets the horizontal line Q′Q in I: draw GH perpendicular to QQ′ and Qh perpendicular to gI: and let the mass of the semi-arch DESF be represented by A, that of the pier by P, and the force of gravity by g: the weight of the former will then be gA, and of the latter gP. The magnitude of the pier is generally computed on the supposition that the pressure of the arch has a tendency to make the pier turn upon Q as a centre or fulcrum; and this hypothesis is often consistent with fact: but when the height CD is small compared with the span, the weight of the arch has a strong tendency to make the pier slide along the horizontal line PI; we shall, therefore, state the conditions of equilibrium on this supposition also. First, supposing the pier solely capable of turning upon Q as a centre of rotation: then will the case be the same as if the body DESF whose weight is gA, by pressing upon the face ES, tended to move the mass FSTQP upon the fulcrum Q. But the weight gA is to its pressure upon FS, as sine of angle included between ED and FS, to sine of angle ERQ, that is $\sin. I : \text{rad.} :: gA : \frac{gA}{\sin. I}$ = pressure of half arch upon the joint of fracture. Now g being the centre of gravity of the half arch, the pressure it occasions is exerted in the direction gI: and G being the centre of gravity of the pier, the force resulting from its weight acts in the vertical direction GH; therefore in the case of equilibrium, we must, by the nature of the lever, have, pressure on SF $\times$ Qh = weight of pier $\times$ QH, that is, $\frac{gA}{\sin. I} . Qh = gP . QH$, whence we readily obtain

$$\text{(I.)} \quad . \quad . \quad . \quad . \quad \frac{A}{P} = \frac{QH}{Qh} . \sin. I.$$

This equation comprises the conditions of the equilibrium of rotation about the point Q; and we may find by its means any one of the five quantities it contains, when the other four are given.

When the arch springs vertically from rectangular piers, whose height and breadth are H and B respectively, the preceding theorem reduces to

$$\text{(i.)} \quad . \quad . \quad . \quad . \quad HA \cot. I = SB + \tfrac{1}{2}HB^2.$$

In the second case, in which we suppose the pier may slide along in the horizontal direction, let f be a force which is exerted horizontally in opposition to the motion of translation: then fP acting in the direction Ik must counterbalance $\frac{gA}{\sin. I}$ acting along hI. Here hI being to Ik as radius to cos. I, we shall have, $\text{rad.} : \cos. I :: \frac{gA}{\sin. I} : fP$; whence $gA . \frac{\sin. I}{\cos. I} = fP$, and for an equation including the conditions of the equilibrium of translation we have

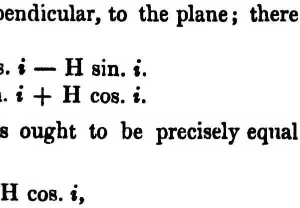
$$\text{(II.)} \quad . \quad . \quad . \quad . \quad . \quad \frac{A}{P} = \frac{f}{g} . \frac{\sin. I}{\cos. I} = \frac{f}{g} \tan. I.$$

As to the position of the joints of rupture, and of the centres of gravity of the semi-arch and pier, they may in most cases be determined with tolerable accuracy, thus: having drawn on pasteboard the arch and proposed pier, upon a

pretty large scale, and described the voussoirs of the arch, of the intended thickness, draw from the middle of the key voussoir a tangent to the intrados, and produce it till it again meets the middle of a voussoir, as at F, from which point draw FS perpendicular to the intrados; it will be nearly the position of a joint of fracture. Next, cut the pasteboard through at the several outlines, and find by some of the methods described in art. 92, the centres of gravity of the two parts DESF, STQPF. With regard to the ratio of A to P, it may always be found pretty nearly, either by weighing or measuring the pieces of pasteboard which represent them; and the distances QH, Qh, and angle I, will be ascertained by the construction. If, when these values of A, P, &c. are introduced into the equations, the first members are less than the second, the piers will be large enough to ensure the equilibrium: if otherwise, some of these particulars must be changed until that takes place.

This mode of considering the subject suggests, that to diminish the thrust of the arch, or increase the stability of the pier, the commencement of the flanks ought to be loaded; and that the thickness of the voussoirs near the key ought to be lessened considerably: in short, to make the arch, instead of having a uniform thickness throughout its whole extent, to be very thick at its origin, and at the key to be no thicker than is necessary to resist the pressure of the flanks: for by such a procedure a part of the force which tends to move the pier is thrown upon that which resists being overturned, and the latter will gain a great advantage in point of stability.

128. Prop. XV. Pressure of Earth against Walls.



Lemma. A weight W, being placed on a plane, inclined to the vertical in angle i, to find a horizontal force, H, sufficient to sustain it, so that it shall not run down the plane, taking friction into the account.

Each of the forces, W, H, being resolved into two, the one parallel, the other perpendicular, to the plane; there will result,

parallel to the plane, a force $= \text{W} \cos. i - \text{H} \sin. i$.
perp. to the plane, a force $= \text{W} \sin. i + \text{H} \cos. i$.

In order to an equilibrium, the first of these forces ought to be precisely equal to the friction down the plane.

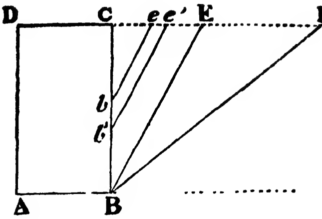
That is, $\text{W} \cos. i - \text{H} \sin. i = f\text{W} \sin. i + f\text{H} \cos. i$,
whence $f\text{H} \cos. i + \text{H} \sin. i = -f\text{W} \sin. i + \text{W} \cos. i$,

$$\text{and H} = \text{W}\,\frac{\cos. i - f \sin. i}{\sin. i + f \cos. i} = \text{W}\,\frac{1 - f \tan. i}{\tan. i + f}.$$

Corol. Hence, if instead of a horizontal force, the weight W were sustained by a wall, or by any obstacle whatever, the horizontal effort exerted by the weight against the obstacle would be $\text{W} \,.\, \dfrac{1 - f \tan. i}{\tan. i + f}$.

129. Prop. XVI. To determine the horizontal stress of the terrace whose vertical section is BCEF, against the wall whose section is ABCD, and the momentum of the pressure to overturn the wall about the angle A.

Considering, first, the stress of a triangle CBE, whose sloping side BE makes the angle i with the vertical; let be, $b'e'$, be each parallel to BE, limiting the



elementary trapezoid $bb'e'e$. Let $BC = a$, $Cb = x$, $bb' = \dot{x}$; then area of $bb'e'e = x\dot{x}$ tan. i; and if s be the specific gravity of the earth, the weight of the portion $bb'e'e$ will be $= sx\dot{x}$ tan. i. Therefore the horizontal effort, against the line bb', will be

$$= sx\dot{x} \text{ tan. } i \, . \, \frac{1 - f \text{ tan. } i}{\text{tan. } i + f} = sx\dot{x} \, \frac{1 - f \text{ tan. } i}{1 + f \text{ cot. } i}$$

$$= sx\dot{x} \text{ M}; \text{ putting } \frac{1 - f \text{ tan. } i}{1 + f \text{ tan. } i} = \text{M}.$$

The fluent of $sx\dot{x}$M, when $x = a$, gives $\frac{1}{2}a^2s$M, for the whole horizontal thrust of the triangle CBE.

Referring the momentum of the thrust of the elementary portion $bb'e'e$, to the length of lever bB $= a - x$, we have for that momentum Ms $(a - x)x\dot{x}$. The fluent of this when $x = a$, is $= \frac{1}{6}a^3s$M.

130. It remains to determine the angle i.

Now, it is evident that $\frac{1 - f \text{ tan. } i}{1 + f \text{ cot. } i} =$ M, vanishes, and consequently, both the horizontal thrust and its momentum vanish, whether tan. $i = 0$, or $= \frac{1}{f}$. Between these two values, therefore, there is one which gives both the greatest thrust and the greatest momentum. This value is found by making

$$\phi\text{M} = 0, \text{ that is, } \phi \, \frac{1 - f \text{ tan. } i}{1 + f \text{ cot. } i} = 0. \quad \text{Put tan. } i = z,$$

$$\text{then } -f\dot{z}\,(1 + \frac{f}{z}) + \frac{f\dot{z}}{z^2}(1 - fz) = 0;$$

$$\text{or } \dot{z} + \frac{f\dot{z}}{z} = \frac{\dot{z} - fz\dot{z}}{z^2},$$

$$1 + \frac{f}{z} = \frac{1 - fz}{z^2} \quad . \; . \quad z^2 + fz = 1 - fz,$$

$$z^2 + 2fz = 1 \quad . \; . \quad z^2 + 2fz + f^2 = 1 + f^2,$$

$$z + f = \sqrt{(1 + f)^2} \quad . \; . \; . \quad z = -f + \sqrt{1 + f^2},$$

that is,

$$\text{tan. } i = -f + \sqrt{1 + f^2}.$$

Substituting this value of tan. i for it in the above expression for M, we have for the horizontal thrust

$$\tfrac{1}{2}a^2s\left\{-f + \sqrt{(1 + f^2)}\right\}^2 = \tfrac{1}{2}a^2s \text{ tan.}^2 \text{ I}.$$

while the momentum of the stress is found to be

$$\tfrac{1}{6}a^3s\left\{-f + \sqrt{(1 + f^2)^2}\right\} = \tfrac{1}{6}a^3s \text{ tan.}^2 \text{ I},$$

which was to be found.

131. The angle which has for its tangent $\frac{1}{f}$ is the angle of the slope, which the earth would, of itself, naturally take, if it were not sustained by any wall.

For a body has a tendency to descend along a plane (inclination to vertical $= i$) with a force $= g$ cos. i, and it presses the plane with a force $= g$ sin. i.

Wherefore the friction $= fg$ sin. i; and since it counterbalances the force with which the body endeavours to descend, we have

$$fg \sin. i = g \cos. i \ \therefore \frac{\sin. i}{\cos. i} = \tan. i = \frac{1}{f};$$

$$\text{also } f = \cot. i.$$

Farther, the angle whose tangent is $-f + \sqrt{1+f^2}$ is half the angle whose tangent is $\frac{1}{f}$.

$$\text{For } \tan. i = \frac{2 \tan. \frac{1}{2}i}{1 - \tan.^2 \frac{1}{2}i}. \quad \text{(Equa. 17, pa. 403, vol. i.)}$$

$$\text{Or, } \frac{2[-f+\sqrt{1+f^2}]}{1-[-f+\sqrt{1+f^2}]^2} = \frac{-2f + 2\sqrt{1+f^2}}{-2f^2 + 2f\sqrt{1+f^2}} = \frac{1}{f}.$$

Let, therefore, BF be the slope which loose earth would, of itself, naturally assume: then, the line BE which determines the triangle of earth that exerts the greatest horizontal stress against the vertical wall bisects the angle CBF.

132. SCHOLIUM.

Sandy and loose earth takes a natural declivity of 60° from the vertical; stronger earth will take a declivity of 53°. Therefore, for a terrace of loose earth we have $i = 30°$; for another of strong and close earth $i = 26\frac{1}{2}°$.

Hence, for the former kind, where tan. $30° = \frac{1}{3}\sqrt{3}$, the value of the stress is $\frac{1}{6}a^2s$, and that of the momentum of the stress $\frac{1}{18}a^3s$.

For the latter kind, where tan. $26\frac{1}{2}° = \frac{1}{2}$ nearly, the stress $= \frac{1}{8}a^2s$, its momentum $= \frac{1}{24}a^3s$

132. The horizontal stress and momentum being thus known, it is easy to proportion to them the resistance of the wall ABCD.

Let $b =$ AB, while BC $= a$, and let s be the spec. grav. of the wall. For brick, $s = 2000$, for strong earth, $s = 1428$. Then the momentum of the resistance referred to the point AB, being $\frac{1}{2}ab^2S$; we shall have

$$\tfrac{1}{2}ab^2S = \tfrac{1}{24}a^3S \text{ (for strong earth)}$$

$$\therefore b = a\sqrt{\frac{s}{12S}} = \cdot 288675 \times a\sqrt{\frac{s}{S}}.$$

Thus, if $a = 39\cdot37$ feet, s and S as above, we shall find $b = 9\cdot6033$ feet.

Exam. 2. Supposing the earth of the same kind as in the above example, s to S, as 4 to 5, and the height of the wall and bank each 12 feet; required the thickness of the wall, being rectangular. Ans. 2·986 feet.

Note.—The preceding investigation proceeds upon the principles assumed by *Coulomb* and *Prony*. They who wish to go thoroughly into this subject, and have not opportunity to make experiments, may advantageously consult *Traité Expérimental, Analytique et Pratique de la Poussée des Terres, &c. par M. Mayniel.*

DYNAMICS.

134. This branch of Mechanics (as already defined in art. 5.) relates to the circumstances of bodies in actual motion: that is, of bodies acted upon forces, constant or variable, which do not at every instant of time neutralise each other.

In this science then, we have to consider the forces as they vary in their in-

tensity with the *time* during which they act; the law of their action according to their *mode* of transmission to the body acted upon; the *spaces* which bodies so acted upon describe (either linear or angular); the *velocities* of those bodies at any given period of their motion; and the paths themselves along which the bodies, under these circumstances, pass.

135. Definitions and Principles.

A body is said to be in *uniform motion* when the spaces which it passes over in equal successive portions of time are equal. And in this case tho space is, obviously, as the time and the velocity conjointly.

A *force* is said to be *constant* when it generates equal increments of velocity in equal successive times.

The *unit of time* is generally taken one *second;* and the *unit of space* one *foot.* Hence in Dynamics, time is usually estimated in seconds and space in feet.

The *measure of force* is the velocity which it would communicate, if it continued constant, to a given body in a second of time.

When there is only a certain quantity of force acting on a body it will generate a less velocity in given time than it would if the body were less; or, in other words, the velocities produced by equal forces are *inversely* as the bodies on which they act.

In these researches, the several quantities are generally denoted by their initial letters: viz. the force by f, time by t, space by s, velocity by v, and the mass of the body by b.

136. Prop. I. If a body be acted on by a constant force during any given time, the relation between the force, time, and velocity, is expressed by

$$v = ft \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

For the velocity generated at each second is the same and denoted by f; hence in t seconds it is $f\,t$.

137. Prop. II. The space described in t seconds by a body subjected to a constant force is denoted by

$$S = \tfrac{1}{2}vt = \tfrac{1}{2}ft^2.$$

For, the whole space being described by velocities which are in arithmetical progression, it is the same as would be described by their mean velocity in the same time. Hence it is

$$S = \frac{0+v}{2}t = \tfrac{1}{2}vt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

And since $v = f\,t$, it is also

$$S = \tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Here, also, S varies as v^2 (4), because v varies as t.

These equations are sufficient for resolving all problems relative to *constant* forces.

138. Prop. III. If there be a second body b' subjected to the same force as in the preceding propositions we suppose b was, then

$$S' = \tfrac{1}{2} \cdot \frac{b}{b'} ft^2 = \tfrac{1}{2} \cdot \frac{b}{b'} vt \quad . \quad . \quad . \quad . \quad (5.)$$

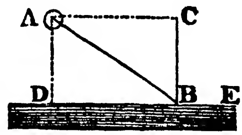
For the velocity generated by the given force is inversely as the body moved, and hence the conclusion follows.

ON THE COLLISION OF SPHERICAL BODIES.

139. Prop. I. If a spherical body strike or act obliquely on a plain surface, the force or energy of the stroke, or action, is as the sine of the angle of incidence.

Or, the force on the surface is to the same if it had acted perpendicularly, as the sine of incidence is to radius.

Let AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB to AD or BC, drawing AD and BC perpendicular, and AC parallel to DE.



For, by art 34, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as sin. BAC, or sin. ABD, to radius.

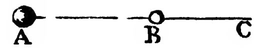
Corol. 1. If a body act on another, in any direction, and by any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts. For the force in AB acts on DE only by the force CB, and in that direction.

Corol. 2. If the plane DE be not absolutely fixed, it will move, after the stroke, in the direction perpendicular to its surface. For it is in that direction that the force is exerted.

140. Prop. If one body A, strike another body B, which is either at rest or moving towards the body A, or moving from it, but with a less velocity than that of A; then the momenta, or quantities of motion, of the two bodies, estimated in any one direction, will be the very same after the stroke that they were before it.

For, because action and re action are always equal, and in contrary directions, art. 24, whatever momentum the one body gains one way by the stroke, the other must just lose as much in the same direction; and therefore the quantity of motion in that direction, resulting from the motions of both the bodies, remains still the same as it was before the stroke *.

* If bodies were exposed to no resistance, the result of a single impulsion or of a simple impact, would be uniform motion. The result of a mere continuance of action must be an acceleration of the motion, or a retardation, according as the action is in the same or the opposite direction to that of the first motion. It is equally obvious that the effect of the continued action of a transverse force must be a continual deflection, that is, a curvilinear motion. In inquiries relative to collision, we assume that action and reaction are equal, because otherwise the surplus action or reaction would be action or reaction *against nothing*, which is absurd: and we assume $b\ v$ as the true measure of moving force, or momentum, because that measure alone consists with the well known universal fact now stated: namely, that the relative motions of bodies, resulting from their mutual actions, are not affected by any common motion, or the action of any equal and parallel force on both bodies: for this universal fact imports, that when two bodies are moving with equal velocities in the same direction, a force applied to one of them, so as to increase its velocity, gives it the same motion relative to the other, as if both bodies had been at rest. Here it is plain, that the space described by the body in consequence



141. Thus, for illustration, if A with a momentum of 10, strike B at rest, and communicate to it a momentum of 4, in the direction AB. Then A will have only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up still the same momentum between them as before, namely 10.

142. If B were in motion before the stroke, with a momentum of 5, in the same direction, and receive from A an additional momentum of 2. Then the motion of A after the stroke will be 8, and that of B, 7; which between them make 15, the same as 10 and 5, the motions before the stroke.

143. Lastly, if the bodies move in opposite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the stroke, the whole motion from both, in the direction of AB, is 10 — 5 or 5. But, after the stroke, the motion of A is 4 in the direction AB, and the motion of B is 6 — 5 or 1 in the same direction AB; therefore the sum 4 + 1, or 5, is still the same motion from both, as it was before.

144. Prop. The motion of bodies included in a given space, is the same with regard to each other, whether that space be at rest, or move uniformly in a right line.

For, if any force be equally impressed both on the body and the line on which it moves, this will cause no change in the motion of the body along the right line. For the same reason, the motions of all the other bodies, in their several directions, will still remain the same. Consequently their motions among themselves will continue the same, whether the including space be at rest, or be moved uniformly forward. And therefore their mutual actions on one another, must also remain the same in both cases *.

145. Prop. If a hard and fixed plane be struck perpendicularly by either a soft or a hard unelastic body, the body will remain upon it. But if the plane be struck by a perfectly elastic body, it will rebound from it again with the same velocity with which it struck the plane.

For, since the parts which are struck, of the elastic body, suddenly yield and give way by the force of the blow, and as suddenly restore themselves again with a force equal to the force which impressed them, by the definition of elastic bodies; the intensity of the action of that restoring force on the plane, will be equal to the force or momentum with which the body struck the plane. And, as action and re-action are equal and contrary, the plane will act with the same force on the body, and so cause it to rebound or move back again with the same velocity as it had before the stroke.

But hard or soft bodies, being devoid of elasticity, by the definition, having no restoring force to throw them off again, they must necessarily remain upon the plane struck.

146. *Corol.* 1. The effect of the blow of the elastic body, on the plane, is double to that of the unelastic one, the velocity and mass being equal in each.

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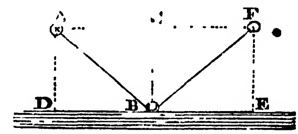
of the primitive force, and of the force now added, is the sum of the spaces which each of them would generate in a body at rest. Therefore the forces are proportional to the *velocities* or changes of motion which they produce, and not to the *squares* of those velocities, as was asserted by Leïbnitz. This measure of forces, or the fact that a force makes the same change on any velocity whatever, and the independence of the relative motions on any motion that is the same on all the bodies of a system, are counterparts of each other, and the corresponding laws of the communication of motion and force may, therefore, be assumed without hesitation.

* See also the preceding note.

For the force of the blow from the unelastic body, is as its mass and velocity, which is only destroyed by the resistance of the plane. But in the elastic body, that force is not only destroyed and sustained by the plane; but another also equal to it is sustained by the plane, in consequence of the restoring force, and by virtue of which the body is thrown back again with an equal velocity. And therefore the intensity of the blow is doubled.

147. *Corol.* 2. Hence unelastic bodies lose, by their collision, only half the motion lost by elastic bodies; their mass and velocities being equal.—For the latter communicate double the motion of the former.

148. Prop. If an elastic body A impinge on a firm plane DE at the point B, it will rebound from it in an angle equal to that in which it struck it; or the angle of incidence will be equal to the angle of reflection; namely, the angle ABD equal to the angle FBE.



Let AB express the force of the body A in the direction AB; which let be resolved into the two AC, CB, parallel and perpendicular to the plane —Take BE and CF equal to AC, and draw BF. Now action and re-action being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on or diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF, by composition. But, because AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually similar and equal; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflection.

149. Prop. To determine the motion of non-elastic bodies, when they strike each other directly, or in the same line of direction.

Let the non-elastic body B, moving with the velocity V in the direction B*b*, and the body *b* with the velocity *v*, strike each other. Then, because the momentum of any moving body is as the mass into the velocity, $BV = M$ is the momentum of the body B, and $bv = m$ the momentum of the body *b*, which let be the less powerful of the two motions. Then, by art. 140, the bodies will both move together as one mass in the direction BC after the stroke, whether before the stroke the body *b* moved towards C or towards B. Now, according as that motion of *b* was from or towards B, that is, whether the motions were in the same or contrary ways, the momentum after the stroke, in direction BC, will be the sum or difference of the momentums before the stroke; namely, the momentum in direction BC will be

$BV + bv$, if the bodies moved the same way, or
$BV - bv$, if they moved contrary ways, and
BV only, if the body *b* were at rest.

Then divide each momentum by the common mass of matter $B + b$, and the quotient will be the common velocity after the stroke in the direction BC; namely, the common velocity will be, in the first case,

$$\frac{BV + bv}{B + b}, \text{ in the 2d } \frac{BV - bv}{B + b}, \text{ and in the 3d } \frac{BV}{B + B}.$$

$$\textit{Corol. } V - \frac{BV + bv}{B + b} = \frac{V - v}{B + b} \times b, \text{ the veloc. lost by B.}$$

150. For example, if the bodies, or weights, B and *b*, be as 5 to 3, and their

velocities, V and v, as 6 to 4, or as 3 to 2, before the stroke; then 15 and 6 will be as their momentums, and 8 the sum of their weights; consequently, after the stroke, the common velocity will be as

$$\frac{15+6}{8} = \frac{21}{8} \text{ or } 2\tfrac{5}{8} \text{ in the first case,}$$

$$\frac{15-6}{8} = \frac{9}{8} \text{ or } 1\tfrac{1}{8} \text{ in the second, and}$$

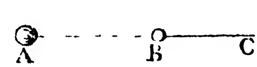
$$\frac{15}{8} \quad . \quad . \quad . \text{ or } 1\tfrac{7}{8} \text{ in the third.}$$

151. Prop If two perfectly elastic bodies impinge on one another, their relative velocity will be the same both before and after the impulse; that is, they will recede from each other with the same velocity with which they approached and met.

For the compressing force is as the intensity of the stroke; which, in given bodies, is as the relative velocity with which they meet or strike. But perfectly elastic bodies restore themselves to their former figure, by the same force by which they were compressed; that is, the restoring force is equal to the compressing force, or to the force with which the bodies approach each other before the impulse. But the bodies are impelled from each other by this restoring force; and therefore this force, acting on the same bodies, will produce a relative velocity equal to that which they had before: or it will make the bodies recede from each other with the same velocity with which they before approached, or so as to be equally distant from one another at equal times before and after the impact.

152. *Remark.* It is not meant by this proposition, that each body will have the same velocity after the impulse as it had before; for that will be varied according to the relation of the masses of the two bodies; but that the velocity of the one will be, after the stroke, so much increased, and the other decreased, as to have the same difference as before, in one and the same direction. So, if the elastic body B move with a velocity V, and overtake the elastic body b moving the same way with the velocity v; then their relative velocity, or that with which they strike, is $V - v$, and it is with this same velocity that they separate from each other after the stroke. But if they meet each other, or the body b move contrary to the body B; then they meet and strike with the velocity $V + v$, and it is with the same velocity that they separate and recede from each other after the stroke. But whether they move forward or backward after the impulse, and with what particular velocities, are circumstances that depend on the various masses and velocities of the bodies before the stroke, and which make the subject of the next proposition. It may further be remarked, that the sums of the two velocities, of the bodies, before and after the stroke, are equal to each other. Thus, V, v being the velocities before the impact, if x and y be the corresponding ones after it; since $V - v = y - x$, therefore $V + x = v + y$.

153. Prop. To determine the motions of elastic bodies after striking each other directly.



Let the elastic body B move in the direction BC, with the velocity V; and let the velocity of the other body b be v in the same line; which latter velocity v will be positive if b move the same way as B, but negative if b move in the opposite direction to B. Then their relative velocity in the direction BC is $V - v$; also the momenta before the stroke are BV and bv, the sum of which is $BV + bv$ in the direction BC.

Again, put x for the velocity of B, and y for that of b, in the same direction BC, after the stroke; then their relative velocity is $y - x$, and the sum of their momenta $Bx + by$ in the same direction.

But the momenta before and after the collision estimated in the same direction, are equal, by art. 140, as also the relative velocities, by the last prop. Whence arise these two equations:

$$\text{viz. } BV + bv = Bx + by,$$
$$\text{and } V - v = y - x;$$

the resolution of which equations gives

$$x = \frac{(B - b)V + 2bv}{B + b}, \text{ the velocity of B,}$$
$$y = \frac{-(B - b)v + 2BV}{B + b}, \text{ the velocity of } b.$$

Or, $x = V - \frac{2b}{B + b}(V - v)$, and $y = v + \frac{2b}{B + b}(V - v)$.

So that the velocity lost by B is $\frac{2b}{B + b}(V - v)$,

and the velocity gained by b is $\frac{2b}{B + b}(V - v)$;

which two velocities are in the ratio of b to B, or reciprocally as the two bodies themselves.

Corol. 1. The velocity lost by B drawn into B, and the velocity gained by b drawn into b, give each of them $\frac{2Bb}{B + b}(V - v)$, for the momentum gained by the one and lost by the other, by the stroke; which increment and decrement being equal, they cancel one another, and leave the same momentum $BV + bv$ after the impact, as it was before it.

Corol. 2. Hence also, $BV^2 + bv^2 = Bx^2 + by^2$, or the sum of the vires vivarum is always preserved the same, both before and after the impact. For, since

$$BV + bv = Bx + by,$$
$$\text{or } BV - Bx = by - bv,$$
$$\text{and } V + x = y + v, \text{ these two equas. multiplied,}$$
$$\text{give } BV^2 - Bx^2 = by^2 - bv^2,$$
$$\text{or } BV^2 + bv^2 = Bx^2 + by^2,$$

the equation of the so called living forces.

Corol. 3. But if v be negative, or the body b moved in the contrary direction before collision, or towards B; then, changing the sign of v, the same theorems become

$$x = \frac{(B - b)V - 2bv}{B + b}, \text{ the velocity of B,}$$
$$y = \frac{(B - b)v + 2BV}{B + b}, \text{ the veloc. of } b, \text{ in the direction BC.}$$

And if b were at rest before the impact, making its velocity $v = 0$, the same theorems give

$$x = \frac{B - b}{B + b}V, \text{ and } y = \frac{2B}{B + b}V, \text{ the velocities in this case.}$$

And, in this case, if the two bodies B and b be equal to each other; then $B - b = 0$, and $\frac{2B}{B + b} = \frac{2B}{2B} = 1$; which give $x = 0$, and $y = V$; that is, the body B will stand still, and the other body b will move on with the whole

velocity of the former; a thing which we sometimes see happen in playing at billiards; and which would happen much oftener if the balls were perfectly elastic.

Scholium.

154. If the bodies be elastic only in a partial degree, the sum of the momenta will still be the same, both before and after collision, but the velocities after will be less than in the case of perfect elasticity, in the ratio of the imperfection. Hence, with the same notation as before, the two equations will now be $BV + bv = Bx + by$,

$$\text{and } V - v = \frac{m}{n}(y - x),$$

where m to n denotes the ratio of perfect to imperfect elasticity. And the resolution of these two equations, gives the following values of x and y, viz.

$$x = V - \frac{m+n}{m} \cdot \frac{b}{B+b}(V - v),$$

$$y = v + \frac{m+n}{m} \cdot \frac{B}{B+b}(V - v),$$

for the velocities of the two bodies after impact in the case of imperfect elasticity: and these would become the same as the former if n were $= m$.

Hence, if the two bodies B and b be equal, then

$$x = V - \frac{m+n}{2m}(V - v), \text{ and } y = v + \frac{m+n}{2m}(V - v),$$

where the velocity lost by B is just equal to that gained by b. And if in this case b was at rest before the impact, or $v = 0$, then the resulting motions would be

$$x = \frac{m-n}{2m}V, \text{ and } y = \frac{m+n}{2m}V,$$

which are in the ratio of $m - n$ to $m + n$.

Also, if $m = n$, or the bodies perfectly elastic, then $x = 0$, and $y = V$; or B would be at rest, and b go on with the first motion of B.

Further, in this case also, the velocity of B before the impact, is to that of b after it, as V to $\frac{m+n}{2m}V$, or as $2m$ to $m + n$. But, if the bodies be now supposed to vibrate in circles, as pendulums, in which case the chords (C and c) of the arcs described are known to be proportional to the velocities; then it will be $2m : m + n :: C : c$; hence $m : n :: C : 2c - C$. So that, by measuring these chords, of the arcs thus experimentally described, the ratio of m to n, or the degree of elasticity in the bodies, may be determined.

155. Prop. The greatest velocity which can be generated by the propagation of motion through a row of contiguous perfectly elastic bodies, will be when those bodies are in geometrical progression.

First, take three bodies, A, X, and C: then (art. 153) the velocity communicated from A to X $= \frac{2Aa}{A+X}$, a being the velocity of A: and when the body X impinges upon C at rest with this velocity, the vel. communicated to C will

$$= \frac{2X}{X+C} \cdot \frac{2Aa}{A+X} = \frac{4AaX}{(A+X)(X+C)}$$

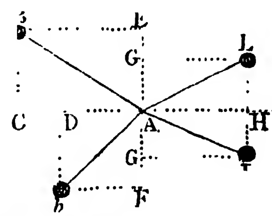
$$= \frac{4Aa}{\left(\frac{A}{X} + 1\right)(X+C)} = \frac{4Aa}{A + X + \frac{AC}{X} + C}.$$

This fraction is evidently a max. when its denominator is a min.; that is, since

A and C are given, when $X^2 = AC$, or when X is a mean proportional between A and C.

For the same reason the velocity communicated from the second body through C the third, to a fourth, D, will be greatest when C is a mean proportional between the second and fourth. Like reasoning will evidently hold for a series of perfectly elastic bodies. Further, if the number of bodies in the geometrical progression be increased without limit, the quantity of motion communicated to the last, from a given quantity of motion in the first, however small, may also be increased without limit.

156. Prop. If bodies strike one another obliquely, it is proposed to determine their motions after the stroke.



Let the two bodies B, *b*, move in the oblique directions BA, *b*A, and strike each other at A, with velocities which are in proportion to the lines BA, *b*A; to find their motions after the impact. Let CAH represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars BC, *b*D, and complete the rectangles CE, DF. Then the motion in *b*A is resolved into the two BC, CA; and the motion in *b*A is resolved into the two *b*D, DA; of which the antecedents BC, *b*D, are the velocities with which they directly meet, and the consequents CA, DA, are parallel; therefore by these the bodies do not impinge on each other, and consequently the motions, according to these directions, will not be changed by the impulse; so that the velocities with which the bodies meet, are as BC and *b*D, or their equals EA and FA. The motions therefore of the bodies B, *b*, directly striking each other with the velocities EA, FA, will be determined by art. 149 or 153, according as the bodies are elastic or non-elastic; which being done, let AG be the velocity, so determined, of one of them, as A; and since there remains also in the body a force of moving in the direction parallel to BE, with a velocity as BE, make AH equal to BE, and complete the rectangle GH: then the two motions in AH and AG, or HI, are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the stroke. And after the same manner is the motion of the other body *b* determined after the impact.

If the elasticity of the bodies be imperfect in any given degree, then the quantity of the corresponding lines must be diminished in the same proportion. For the full consideration of this branch of the inquiry the student is referred to the *Treatises of Mechanics* by *Gregory* and *Bridge.*

Problems for Exercise on Collision.

Exam. 1. A cannon ball weighing 12lbs. moving with a velocity of 1200 feet per second, *meets* another of 18lbs. moving with a velocity of 1000 feet per second. Required the velocity of each after impact, supposing both to be non-elastic.

Exam. 2. B and *b* are as 3 to 2, and the velocity of B is to that of *b* as 5 to 4. They are perfectly hard, and move before impact in the same direction; what are the velocities lost by B and gained by *b*?

Exam. 3. B and *b* are perfectly elastic, and move in opposite directions. B is triple of *b*, but *b*'s velocity is double that of B. How do those bodies move after impact?

Exam. 4. A body whose elasticity is to perfect elasticity as 15 to 16, falls from the height of 100 feet upon a perfectly hard horizontal plane. It then rebounds and falls again, and so on, always in a vertical direction. It is required to find the whole space described by the body before its motion ceases, as well as the entire time of its motion.

Exam. 5. Investigate what must be the force of elasticity, so that the sums of the products formed by multiplying each body into *any* assumed power, n, of its velocity, may not be altered by the impact of the two bodies.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

157. Prop. The properties of motion delivered in arts. 136—138, and their obvious deductions, for constant forces, are true in the motions of bodies freely descending by their own gravity; namely, that the velocities are as the times, and the spaces as the squares of the times, or as the squares of the velocities.

For, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth; and since this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently all the motions freely produced, by gravity, are as above specified, by that proposition, &c.

SCHOLIUM.

158. Now it has been found, by numberless experiments, that gravity is a force of such a nature, that all bodies, whether light or heavy, fall vertically through equal spaces in the same time, abstracting from the resistance of the air; as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also found that the velocities acquired by descending, are in the exact proportion of the times of descent: and further, that the spaces descended are proportional to the squares of the times, and therefore to the squares of the velocities. Hence then it follows, that the weights or gravities, of bodies near the surface of the earth, are proportional to the quantities of matter contained in them; and that the spaces, times, and velocities, generated by gravity, have the relations contained in the three general propositions before laid down. Further, as it is found, by accurate experiments, that a body in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the first second of time, and consequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet, therefore, if $\frac{1}{2}g$ denote $16\frac{1}{12}$ feet, the space fallen through in one second of time, or g the velocity generated in that time; then, because the velocities are directly proportional to the times, and the spaces to the squares of the times; it will therefore be,

$$\text{as } 1'' : t'' :: g : gt = v \text{ the velocity,}$$
$$\text{and } 1^2 : t^2 :: \tfrac{1}{2}g : \tfrac{1}{2}gt^2 = s \text{ the space.}$$

So that, for the descents of gravity, we have these general equations, namely,

$$s = \tfrac{1}{2}gt^2 = \frac{v^2}{2g} = \tfrac{1}{2}tv.$$
$$v = gt = \frac{2s}{t} = \sqrt{2gs}.$$
$$t = \frac{v}{g} = \frac{2s}{v} = \sqrt{\frac{2s}{g}}.$$
$$g = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s}.$$

Hence, because the times are as the velocities, and the spaces as the squares of either, therefore,

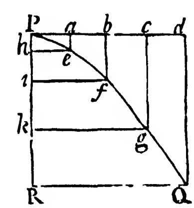
if the times be as the numbers.	1, 2, 3, 4, 5, &c.
the velocities will also be as	1, 2, 3, 4, 5, &c.
and the spaces as their squares	1, 4, 9, 16, 25, &c.
and the space for each time as	1, 3, 5, 7, 9, &c.

namely, as the series of the odd numbers, which are the differences of the squares denoting the whole spaces. So that if the first series of natural numbers be seconds of time, namely,

the times in seconds,	1″,	2″,	3″,	4″,	&c.
the velocities in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c.
the spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c.
and the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

of which spaces the common difference is $32\frac{1}{6}$ feet, the natural and obvious measure of g, the force of gravity.

159. These relations, of the *times*, *velocities*, and *spaces*, may be represented by the abscisses and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and Pa, Pb, Pc, &c. parallel to RQ, represent the times from the beginning, or the velocities, then ae, bf, cg, &c. parallel to the axis PR will represent the spaces described by a falling body in those times; for, in a parabola, the abscisses Ph, Pi, Pk, &c. or ae, bf, cg, &c. which are the spaces described, are as the squares of the ordinates he, if, kg, &c. or Pa, Pb, Pc, &c. which represent the times or velocities.



160. And because the laws for the destruction of motion are the same as those for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1*st*, A body thrown directly upward, with any velocity, will lose equal velocities in equal times.

2*d*, If a body be projected upward, with the velocity it acquired in any time by descending freely, it will lose all its velocity in an equal time, and will ascend just to the same height from which it fell, and will describe equal spaces in equal times, in rising and falling, but in an inverse order; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3*d*, If bodies be projected upward, with any velocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

In solving problems, when a body, instead of being permitted to fall freely, is projected vertically upwards or downwards with a given velocity, it will assist the comprehension of what takes place, to ascertain what results from the original

projection, and what from the force of gravity. Thus, if a body be projected with a velocity v it will, in the time t, describe the space tv (art. 132) apart from the operation of gravity or any other force. Blending this with the preceding expression for the space described by a falling body, we have

$$S = tv \mp \tfrac{1}{2}gt^2,$$

in which the *lower* sign must be employed when the projection is vertically *downwards*, the *upper* when the projection is vertically *upwards*.

EXERCISES ON RISING AND FALLING BODIES.

1. Find the space descended vertically by a body in 7 seconds of time, and the velocity acquired. Ans. $788\frac{1}{12}$, space; $225\frac{1}{6}$, velocity.

2. Required the time of generating a velocity of 100 feet per second, and the whole space descended. Ans. $3''\frac{21}{193}$, time; $155\frac{85}{193}f$. space.

3. Find the time of descending 400 feet, and the velocity at the end of that time. Ans. $4''\frac{75}{77}$, time; $160\frac{33}{77}$, velocity.

4. If a body fall freely for 5″, how far will it descend during the last second of its motion?

5. If an arrow be propelled vertically upwards from a bow with a velocity of $96\frac{1}{2}$ feet per second, how high will it rise, and how long will it be before it returns again to the ground?

6. If a ball be projected vertically *downwards* with a velocity of 100 feet per second, how far will it have descended in three seconds?

7. If a ball be projected vertically *upwards* with a velocity of 100 feet per second, how far will it have arisen in three seconds?

8. If a ball be projected vertically upwards with a velocity of 44 feet per second, will it be above or below the point of projection in four seconds, the force of gravity tending all the time to draw it downwards?

9. A drop of rain falls through $176\frac{11}{12}$ feet in the last second; how high is the cloud from which it descended?

10. A body falling freely was observed to pass through half its descent in the last second; how far did it fall, and how long was it in falling?

11. Two weights, one of 5lbs. the other of 3lbs., hang freely over a pulley: after motion is allowed to commence, how far will the larger weight descend, or the smaller arise, in four seconds?

[The theorem for operation is $S = \dfrac{W - w}{W + w} \cdot \tfrac{1}{2}gt^2$; which the student is required to investigate.]

12. Two equal weights are balanced over a pulley. A pound weight being added to one of them, and motion in consequence taking place, the preponderating weight descended through $16\frac{1}{12}$ feet in four seconds. Required the measure of the two equal weights?

DESCENT OF BODIES.

Additional Exercises.

1. A body has been falling for 5 seconds; compare the spaces described in the *third* and *fifth* seconds of its descent. Ans. $80\frac{5}{12}$ and $144\frac{3}{4}$, or as 5 : 9.

2. A body has fallen through 579 feet; what was the space described by it in the *last* second? Ans. $176\frac{11}{12}$ feet.

3. A body has fallen for 10$\frac{1}{2}$ seconds; what was the space described by it in the *last* second of its fall? Ans. 321$\frac{2}{3}$ feet.

4. What was the space described in the last 2 seconds by a body which had fallen from the top of a tower 300 feet high? Ans. 212·29 feet.

5. A body has been falling for 9$\frac{1}{2}$ seconds; what space did it describe in the last second but 3 of its fall? Ans. 193 feet.

6. With what velocity must a body be projected downwards from a height of 150 feet, that it may reach the bottom in 2 seconds? Ans. 42$\frac{5}{6}$ feet.

7. Suppose a body to have descended for 3$\frac{1}{2}$ seconds, and then to move uniformly for 2$\frac{1}{2}$ seconds with the velocity it had acquired; what space will it describe? Ans. 478$\frac{1}{2}$ feet.

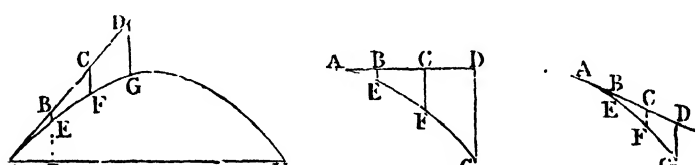
8. Suppose a body to have fallen through 50 feet, and at that instant another body begins to fall just 100 feet below it; how far will the latter body fall before it is overtaken by the former? Ans. 50 feet.

9. In an actual experiment, where two weights of 5 and 3lbs. connected with a cord, hung over a pulley, the heavier weight descended 50 feet in four seconds. Required the measure of the resistance and inertia from the expression $\frac{W - w}{W + w + r} \cdot \frac{1}{2}gt^2 = S$, where r is the unknown quantity.

Ans. $r = 2·2933$lbs.

PROJECTILES IN EMPTY SPACE.

161. Prop. If a body be projected in free space, either parallel to the horizon, or in an oblique direction, by the force of gunpowder, or any other impulse; it will, by this motion, in conjunction with the action of gravity, describe the curve line of a parabola.



Let the body be projected from the point A, in the direction AD, with any uniform velocity; then, in any equal portions of time, it would describe the equal spaces AB, BC, CD, &c. in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c. in the direction of gravity, or perpendicular to the horizon, and equal to the spaces through which the body would descend by its gravity in the same time in which it would uniformly pass over the corresponding spaces AB, AC, AD, &c. by the projectile motion. Then, since by these two motions the body is carried over the space AB, in the same time as over the space BE, and the space AC in the same time as the space CF, and the space AD in the same time as the space DG, &c.; therefore, by the composition of motions, at the end of those times, the body will be found respectively in the points E, F, G, &c ; and consequently the real path of the projectile will be the curve line AEFG, &c. But the spaces AB, AC, AD, &c. described by uniform motion, are as the times of description; and the spaces BE, CF, DG, &c. described in the same times by the accelerating force of gravity, are as the squares of the times; consequently the perpendicular descents are as the squares of the spaces in AD, that is, BE, CF, DG, &c. are

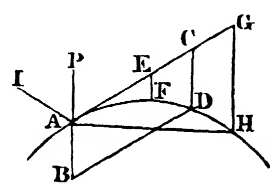
respectively proportional to AB^2, AC^2, AD^2, &c.; which is the property of the parabola by theor. 7, Con. Sect. Therefore the path of the projectile is the parabolic line AEFG, &c. to which AD is a tangent at the point A.

62. *Corol.* 1. The horizontal velocity of a projectile, is always the same constant quantity, in every point of the curve; because the horizontal motion is in a constant ratio to the motion in AD, which is the uniform projectile motion. And the projectile velocity is in proportion to the constant horizontal velocity, as radius to the cosine of the angle DAH, or angle of elevation or depression of the piece above or below the horizontal line AH.

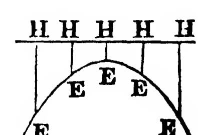
163. *Corol.* 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the secant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is constant, and AI is to AB, as radius to the secant of the angle A; therefore the motion at A, in AB, is everywhere as the secant of the angle A.

164. *Corol.* 3. The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the first uniform projectile velocity at A, or point of contact of a tangent, as 2GD is to AD. For, the times in AD and DG being equal, and the velocity acquired by freely descending through DG, being such as would carry the body uniformly over twice DG in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space AD is to the space 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

165. Prop. The velocity in the direction of the curve, at any point of it, as A, is equal to that which is generated by gravity in freely descending through a space which is equal to one-fourth of the parameter of the diameter of the parabola at that point.



Let PA or AB be the height due to the velocity of the projectile at any point A in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD = AB or AP, the height due to the velocity in the curve at A; and CD is also the height due to the perpendicular velocity at D. which must be equal to the former; but by the last corol. the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as these velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{2}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB, by corol. to theor 8 of the parabola.



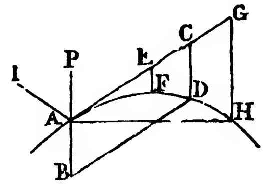
166. *Corol.* 1. Hence, and from cor. 2, theor. 9 of the parabola, it appears that if from the directrix of the parabola which is the path of the projectile, several lines HE be drawn perpendicular to the directrix, or parallel to the axis; then the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.

167. *Corol.* 2. If a body, after falling through the height PA (last fig. but one), which is equal to AB, and when it arrives at A, have its course changed, by reflection from an elastic plane AI, or otherwise, into any direction AC, without altering the velocity; and if AC be taken = 2AP or 2AB, and the

parallelogram be completed; then the body will describe the parabola passing through the point D.

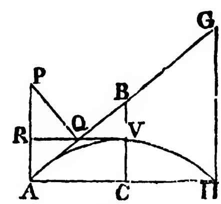
168. *Corol.* 3. Because AC = 2AB or 2CD or 2AP, therefore $AC^2 = 2AP \times 2CD$ or AP . 4CD; and, because all the perpendiculars EF, CD, GH, are as AE^2, AC^2, AG^2; therefore also $AP \cdot 4EF = AE^2$, and $AP \cdot 4GH = AG^2$, &c.; and because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is AP : AE :: AE : 4EF,
and AP : AC :: AC : 4CD,
and AP : AG :: AG : 4GH,
and so on.



169. PROP. Having given the direction, and the impetus, or altitude due to the first velocity of a projectile; to determine the greatest height to which it will rise, and the random or horizontal range.

Let AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. On AG let fall the perpendicular PQ, and on AP the perpendicular QR; so shall AR be equal to the greatest altitude CV, and 4QR equal to the horizontal range AH. Or, having drawn PQ perp. to AG, take AG = 4AQ, and draw GH perp. to AH; then AH is the range.

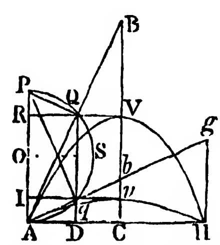


For, by the last corollary, AP AG AG 4GH;
and, by similar triangles, AP AG AQ GH,
or . . . AP AG 4AQ 4GH;
therefore AG = 4AQ; and, by similar triangles, AH = 4QR.

Also, if V be the vertex of the parabola, then AB or $\frac{1}{2}$AG = 2AQ, or AQ = QB; consequently AR = BV, which is = CV by the property of the parabola.

170. *Corol.* 1. Because the angle Q is a right angle, which is the angle in a semicircle, therefore if, on AP as a diameter, a semicircle be described, it will pass through the point Q.

171. *Corol.* 2. If the horizontal range and the projectile velocity be given, the direction of the piece so as to hit the object H, will be thus easily found: Take AD = $\frac{1}{4}$ AH, draw DQ perpendicular to AH, meeting the semicircle, described on the diameter AP, in Q and *q*; then AQ or A*q* will be the direction of the piece. And hence it appears, that there are two directions AB, A*b*, which, with the same projectile velocity, give the very same horizontal range AH. And these two directions make equal angles *q*AD, QAP, with AH and AP, because the arc PQ = the arc A*q*.



172. *Corol.* 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take AD = $\frac{1}{4}$AH, and erect the perpendicular DQ, meeting AB in Q; so shall DQ be equal to the greatest altitude CV. Also, erect AP perpendicular to AH, and QP to AQ; so shall AP be the height due to the velocity.

173. *Corol.* 4. When the body is projected with the same velocity, but in different directions: the horizontal ranges AH will be as the sines of double the

angles of elevation. Or, which is the same, as the rectangle of the sine and cosine of elevation. For AD or RQ, which is $\frac{1}{4}$AH, is the sine of the arc AQ, which measures double the angle QAD of elevation.

And when the direction is the same, but the velocities different; the horizontal ranges are as the square of the velocities, or as the height AP, which is as the square of the velocity; for the sine AD or RQ or $\frac{1}{4}$AH is as the radius or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

174. *Corol.* 5. The greatest range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greatest sine. Or the radius OS, which is $\frac{1}{4}$ of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of 45°, is just double the altitude AP which is due to the velocity, or equal to 4VC. Consequently, in that case, C is the focus of the parabola, and AH its parameter. Also the ranges are equal, at angles equally above and below 45°.

175. *Corol.* 6. When the elevation is 15°, its double, or 30°, has its sine equal to half the radius; consequently then the range will be equal to AP, or half the greatest range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

176. *Corol.* 7. The greatest altitude CV, being equal to AR, is as the versed sine of double the angle of elevation, and also as AP or the square of the velocity. Or as the square of the sine of elevation, and the square of the velocity; for the square of the sine is as the versed sine of the double angle.

177. *Corol.* 8. The time of flight of the projectile, which is equal to the time of a body falling freely through GH or 4CV, four times the altitude, is therefore as the square root of the altitude, or as the projectile velocity and sine of the elevation.

SCHOLIUM.

178. From the last proposition and its corollaries, may be deduced the following set of theorems, for finding all the circumstances of projectiles on horizontal planes, having any two of them given. Thus, let e denote the elevation; r the horizontal range; t the time of flight; v the projectile velocity; h the greatest height of the projectile; $g = 32\frac{1}{6}$ feet, and a the impetus, or the altitude due to the velocity v. Then,

$$r = 2a \sin. 2e = \frac{v^2}{g} \sin. 2e = \tfrac{1}{2}gt^2 \cot. e = 4h \cot. e = tv \cos. e.$$

$$v = \sqrt{2ag} = \sqrt{gr \operatorname{cosec.} 2e} = \tfrac{1}{2}gt \operatorname{cosec.} e = 2 \operatorname{cosec.} e \sqrt{\tfrac{1}{2}gh}.$$

$$t = \frac{v \,.\, \sin. e}{\frac{1}{2}g}{}^{*} = 2 \sin. e \sqrt{\frac{a}{\frac{1}{2}g}} = \sqrt{\frac{r \tan. e}{\frac{1}{2}g}} = 2\sqrt{\frac{h}{\frac{1}{2}g}}.$$

$$h = a \sin.^2 e = \tfrac{1}{2}a \operatorname{versin.} 2e = \tfrac{1}{4}r \tan. e = \tfrac{1}{8}gt^2.$$

* This time, with 30° elevation, is just equal to the time of perpendicular ascent, with the same velocity v.

$$\text{At } 45°,\ v = \sqrt{32\tfrac{1}{6}\, r} = 2{\cdot}828\sqrt{\tfrac{1}{2}gh} = \frac{r}{t} \,.\, 1{\cdot}4142.$$

$$\text{At } 45° \text{ also } h = \tfrac{1}{4}r = 11{\cdot}3432\sqrt{h} = 5{\cdot}6716\sqrt{r}.$$

Sometimes P is employed instead of e for the angle of elevation, and I for *impetus* instead of a *altitude*, or h height due to the velocity. It is well to recollect, and perhaps occasionally to use these changes in the notation.

$$r = \frac{\text{cos. } e \text{ sin. } (e+i)}{\text{cos.}^2 i} \cdot 4a = \frac{2 \text{ cos. } e \text{ sin. } (e+i)}{\text{cos.}^2 i} \cdot \frac{v^2}{g} = \frac{g \text{ cos. } e}{2 \text{ sin. } (e+i)} \cdot t^2 = \frac{4 \text{ cos. } e}{\text{sin. } (e+i)} \cdot h.$$

$$h = \frac{\text{sin.}^2 (e+i)}{\text{cos.}^2 i} a = \frac{\text{sin.}^2 (e+i)}{\text{cos.}^2 i} \cdot \frac{v^2}{2g} = \frac{\text{sin. } (e+i)}{4 \text{ cos. } i} \cdot r = \frac{g}{8} \cdot t^2$$

$$v = \sqrt{2ag} = \text{cos. } i \sqrt{\frac{gr}{2 \text{ cos. } e \text{ sin. } (e+i)}} = \frac{g \text{ cos. } e}{2 \text{ sin. } (e+i)} t = \frac{2 \text{ cos. } i}{\text{sin. } (e+i)} \sqrt{\tfrac{1}{2} gh}.$$

$$t = \frac{2 \text{ sin. } (e+i)}{\text{cos. } i} \sqrt{\frac{a}{\frac{1}{2}g}} = \frac{\text{sin. } (e+i)}{\text{cos. } i} \cdot \frac{v}{\frac{1}{2}g} = \sqrt{\frac{r \text{ sin. } (e+i)}{\frac{1}{2}g \text{ cos. } i}} = 2 \sqrt{\frac{h}{\frac{1}{2}g}}.$$

190. Geometrical constructions of the principal cases in projectiles in a non-resisting medium, flow readily from the properties of the parabola; and in many cases those constructions suggest simple modes of computation. The following problems will serve by way of exercise.

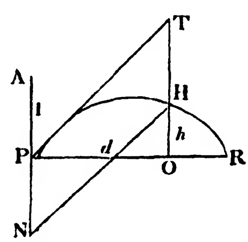
1. Given the impetus and elevation; to find, by construction, the range, on a horizontal plane, the greatest height, and thence the time of flight.

2. Given the impetus, and the range, on a horizontal plane; to find, by construction, the elevation, and the greatest height.

3. Given the elevation, and the range on a horizontal plane; to find, by construction, the impetus, the greatest height, and thence, by computation, the time.

4. Given the impetus, the point and direction of projection, to find the place where the ball will fall upon any plane given in position.

5. Given the impetus and the point of projection, to find the elevation necessary to hit any given point; and to show the limits of possibility. Both construction and mode of computation are required *.



* The main properties of projectiles in a non-resisting medium, may be deduced analytically, as below.

By the nature of falling bodies, if AP be the impetus:

$$\sqrt{AP} : \sqrt{PN} :: PT : 2PN,$$
$$\text{or } \sqrt{AP} : \sqrt{TH} :: PT : 2TH,$$
$$AP : TH :: PT^2 : 4TH^2,$$
$$AP : I :: PT^2 : 4TH,$$
$$\therefore PT^2 = 4AP \cdot TH.$$
$$PO^2 + OT^2 = 4AP\,(TO - HO).$$

If $t = \text{tan. } TPO$, then $TO = td$, and the last equation becomes

$$d^2 + d^2 t^2 = 4I\,(dt - h) = 4Idt - 4Ih \text{ (A)},$$
$$d^2 t^2 - 4Idt = -d^2 - 4Ih,$$
$$t^2 - \frac{4I}{d} t = \frac{-d^2 - 4Ih}{d^2},$$
$$t^2 - \frac{4I}{d} t + \frac{4I^2}{d^2} \cdot = \frac{4I^2 - d^2 - 4Ih}{d^2},$$
$$t - \frac{2I}{d} = \pm \frac{1}{d} \sqrt{(4I - d^2 - 4Ih)},$$
$$t = \frac{2I}{d} \pm \frac{1}{d} \sqrt{(4I^2 - d^2 - 4Ih)}.$$

Thus an equation is obtained for tangent elevation from the nature of the inquiry, without ascertaining the nature of the curve.

When OH is below PR, h changes its sign.

If H fall upon PR, h vanishes.

Either I, h, or d, may be easily found from equa. (A).

$$\text{Thus } I = \frac{t^2 d^2 + d^2}{4\,(td - h)} = \frac{\text{sec.}^2 P d^2}{4\,(td - h)},$$
$$h = \frac{4Idt - t^2 d^2 - d^2}{4I},$$
$$d = \frac{2It}{\text{sec.}^2 P} \pm \frac{2t}{\text{sec.}^2 P} \sqrt{I\left(I - h - \frac{h}{t^2}\right)},$$

v and time become known from I and t, &c.

191. SOME PRACTICAL APPROXIMATE RULES IN GUNNERY.

I. *To find the Velocity of any Shot or Shell.*

Rule.—Divide triple the weight of the charge of powder by the weight of the shot, both in lbs. Extract the square root of the quotient. Multiply that root

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Ex. 1. Suppose PO = 1600 feet, OH = 50, impetus 1000. Required the elevation so that the point H may be struck.

Ex. 2. Suppose PO = 1600 feet, OH = 50 feet, below PR, impetus 1000. Required the elevation.

Ex. 3. Suppose PO = 1600, the impetus 1000, and O and H to coincide. Required the elevation.

That the student may trace the relative advantages of different analytical processes, we subjoin an obvious method of deducing the equation of the curve in reference to rectangular co-ordinates x and y, with a few deductions and examples. In this case PO $= x$, HO $= y$; let v be the projectile velocity, T any time, P, as above, the angle TPO; then

(1.) $a = v\mathrm{T} \cos. \mathrm{P}$

(2.) $y = (\mathrm{OT} - \mathrm{TH}) = \mathrm{T} \sin. \mathrm{P} - \frac{1}{2}gt^2$, and, eliminating T,

(3.) $y = x \tan. \mathrm{P} - \dfrac{gx^2}{2v^2 \cos.^2 \mathrm{P}}$, the equation to the curve.

Cor. 1. If as all along $\mathrm{I} = \dfrac{v^2}{2g}$,

(4.) $y = x \tan. \mathrm{P} - \dfrac{x^2}{4\,\mathrm{I} \cos.^2 \mathrm{P}}$.

Cor. 2. To find where the curve meets the horizontal plane, we must put $y = 0$

$$\text{then } x \tan. \mathrm{P} - \frac{x^2}{4\,\mathrm{I} \cos.^2 \mathrm{P}} = 0$$

$$\therefore x = 4\,\mathrm{I} \tan. \mathrm{P} \cos.^2 \mathrm{P}.$$
$$= 4\,\mathrm{I} \sin. \mathrm{P} \cos. \mathrm{P}.$$
$$= 2\,\mathrm{I} \sin. 2\mathrm{P}, \text{ which agrees with art. 173.}$$

Cor. 3. If v does not enter the conditions of the problem, we have, by eliminating v,

(5.) $$y = x \tan. \mathrm{P} - \tfrac{1}{2}gt^2$$

Cor. 4. To find the angle which the curve makes with the horizon at any point. Let ϕ be this angle, $\tan. \phi = \dfrac{\dot{y}}{\dot{x}}$, and differentiating the value of y,

$$\tan. \phi = \tan. \mathrm{P} - \frac{x}{2\mathrm{I} \cos.^2 \mathrm{P}}.$$

Ex. 1. Let a body be projected from the top of a tower horizontally with a velocity acquired in falling down its height; at what distance from the base will it strike the horizon?

$$y = x \tan. \mathrm{P} - \frac{gx^2}{2v^2 \cos.^2 \mathrm{P}}.$$

Here if a = altitude of tower, $y = -a$, $\mathrm{P} = 0$, and $v^2 = 2ga \therefore -a = -\dfrac{x^2}{4a}$, and $x = 2a$

Ex. 2. A ball is projected at an angle of 45°, with a velocity of 50 feet per second; find its horizontal range.

$$y = x \tan. \mathrm{P} - \frac{gx^2}{2v^2 \cos.^2 \mathrm{P}}.$$

Here $\mathrm{P} = 45°$, $v = 50$, $\therefore$ when $y = 0$

$$x = \frac{2500}{g} = 77{\cdot}72 \text{ feet.}$$

Ex. 3. A ball is thrown across a plain 120 feet wide, to strike a mark 30 feet high, the velocity of projection being that acquired down 80 feet, required the angle of projection.

$$y = x \tan. \mathrm{P} - \frac{gx^2}{2v^2 \cos.^2 \mathrm{P}}.$$

Here $y = 30$, $x = 120$, $v^2 = 160\,g$,

$$\therefore 1 = 4 \tan. \mathrm{P} - \frac{3}{2 \cos.^2 \mathrm{P}}.$$

$$\therefore \tan.^2 \mathrm{P} - \tfrac{8}{3} \tan. \mathrm{P} = -\tfrac{5}{3}.$$

$$\text{and } \tan. \mathrm{P} = 1 \text{ or } \tfrac{5}{3}. \text{ and } \mathrm{P} = 45°.$$

by 1600, and the product will be the velocity in feet, or the number of feet the shot passes over per second, nearly.

Or say—As the root of the weight of the shot, is to the root of triple the weight of the powder, so is 1600 feet, to the velocity *.

II. *Given the Range at One Elevation; to find the Range at another Elevation.*

Rule.—As the sine of double the first elevation, is to its range; so is the sine of double another elevation, to its range.

III. *Given the Range for one Charge; to find the Range for another Charge, or the Charge for another Range.*

Rule.—The ranges have the same proportion as the charges; that is, as one range is to its charge, so is any other range to its charge: the elevation of the piece being the same in both cases.

192. *Example* 1. If a ball of 1lb. acquire a velocity of 1600 feet per second, when fired with $5\frac{1}{3}$ ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs.	196	90	48	16	8
Charge of powder in lbs.	9	4	2	1	$\frac{1}{2}$
Ans. The velocities are	594	584	565	693	693

Ex. 2. If a shell be found to range 1000 yards when discharged at an elevation of 45°; how far will it range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet, or 871 yards.

Ex. 3. The range of a shell, at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder? Ans. at 24° 16′, or at 65° 44′.

Ex. 4. With what impetus, velocity, and charge of powder, must a 13-inch shell be fired, at an elevation of 32° 12′, to strike an object at the distance of 3250 feet? Ans. impetus = 1802, veloc. = 340, charge = 2·95lbs.

Ex. 5. A shell being found to range 3500 feet, when discharged at an elevation of 25° 12′; how far then will it range at an elevation of 30° 15′ with the same charge of powder? Ans. 4332 feet.

Ex. 6. If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases? Ans. $6\frac{3}{4}$lb. of powder.

Ex. 7. What will be the time of flight for any given range, at the elevation of 45°, or for the greatest range?
Ans. the time in secs. is $\frac{1}{4}$ the sq. root of the range in feet.

* In a series of experiments carried on at Woolwich, a few years ago, by the Editor of the present edition, in conjunction with the select committee of artillery officers, it has been found that a charge of *a third* of the weight of the ball, gives, at a medium, a velocity of 1600 feet: gunpowder being much improved in its manufacture since the time when Sir Thomas Blomfield and Dr. Hutton made their experiments. Putting B for the weight of the ball, and C for that of the charge, $v = 1600\sqrt{\frac{3C}{B}}$, is now found a good approximative theorem for the initial velocity, and is the expression of the rule in the text.

Remember that $R \propto I \propto v^2 \propto \frac{C}{B}$.

Ex. 8. In what time will a shell range 3250 feet, at an elevation of 32°?
Ans. 11¼ sec. nearly.

Ex. 9. How far will a shot range on a plane which ascends 8° 15′, and another which descends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′?
Ans. 4244 feet on the *ascent*,
and 6745 feet on the *descent*.

Ex. 10. How much powder will throw a 13-inch shell 4244 feet on an inclined plane, which ascends 8° 15′, the elevation of the mortar being 32° 30′?
Ans. 4·92535lb. or 4lb. 15oz. nearly.

Ex. 11. At what elevation must a 13-inch mortar be pointed, to range 6745 feet, on a plane which descends 8° 15′; the charge $4\frac{15}{16}$lb. of powder.
Ans. 32° 18′½.

Ex. 12. Suppose, in Rirochet firing PO = 1200 feet (fig. in last note) OH = 10 feet, OR = 50 feet; required the elevation and the velocity, so that the ball shall just clear H and hit R.

Ex. 13. Given the heights y, y', of a ball above a horizontal plane, at the horizontal distances x, x', to find the initial velocity and the angle of elevation of the ball.

Ex. 14. The greatest horizontal range of a shell is 3990 feet: what is the greatest range with the same velocity on a plane elevated 9°, and on another depressed 9°? and what must be the elevation in each case?

Ex. 15. Given the proportion between the range and greatest height of a ball projected with a given velocity to find the angle of elevation; and for a numerical example, take the range to the greatest height as 10 to 1.

DESCENTS ON INCLINED PLANES.

193. WHEN we treated of the Inclined Plane as a mechanical power, and compared the forces which sustained a weight upon any such plane, we found that the sustaining force acting *parallel* to the plane, was to the absolute weight or gravity of the body sustained, as the height of the plane to its length, (art. 77.) that is to say, taking the initial letters, we found $f : g :: h : l$.

$$\text{whence } f = g : \frac{h}{l}.$$

This, therefore, is the value of the *accelerating force* which urges a body down a plane of height h, length l; and which must be substituted for g in the expressions

$$v = gt,\ s = \tfrac{1}{2}gt^2,\ v = \sqrt{2gs}.$$

Which hence become, in the case of the inclined plane,

$$(1.)\ v = gt\,\frac{h}{l}\ .\ (2.)\ s = \tfrac{1}{2}gt^2\,\frac{h}{l}\ .\ (3.)\ v = \sqrt{\frac{2gsh}{l}}.$$

[In these equations we may for $\frac{h}{l}$ substitute sin. i, i being the inclination of the plane: they will thereby be simplified for *practical* uses.]

If we deduce the value of the *time* from the equation $s = \frac{1}{2}gt^2\frac{h}{l}$, making s (space) $= l$ (length of plane) we shall have

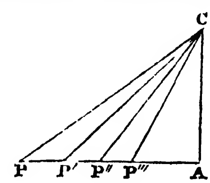
$$(4.)\ t = \sqrt{\frac{2l^2}{gh}} = l\sqrt{\frac{2}{gh}}.$$

Whence we learn that, abstracting from friction and the resistance of the air, the *times of descent* down planes of the same height are as the *lengths*.

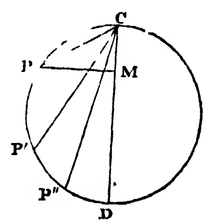
194. Supposing, still, $s = l$, or the body to descend through the whole length of the plane, the equation $v = \sqrt{\frac{2gsh}{l}}$, becomes

$$(5.)\ v = \sqrt{2gh}.$$

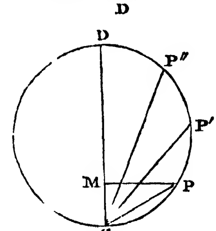
Whence we learn, *that the velocity acquired by a body in falling down an inclined plane, is the same as the velocity acquired by falling vertically through the height* h; *and consequently if bodies descended freely over ever so many inclined planes,* CP, CP′, CP″, &c. *of the same altitude* CA, *the velocity acquired at* P, P′, P″, P‴, &c. IN THE RESPECTIVE DIRECTION CP, *is equal to the vertical velocity at* A, after *falling through* CA.



195. Once more: all the cords CP, CP′, CP″, &c. of the same vertical circle, drawn from the same extremity C of the upright diameter CD, are run over in the same time by bodies falling down them.

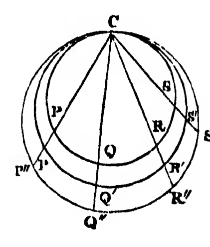


For, putting CD $= d$, calling CP, l, and CM, h, PM being horizontal, the well known property of the circle $CP^2 = CD \cdot CM$, gives $l^2 = dh$. Substituting this value of l^2 for it, in the equation $t = \sqrt{\frac{2l^2}{gh}}$, it becomes $t = \sqrt{\frac{2dh}{gh}} = \sqrt{\frac{2d}{g}}$; a value which is altogether independent of CP, or l, and indicates evidently the time of descent through CD. Therefore the time of descent down either of the chords is $=$ to the time through the vertical diameter.



196. Hence follows this curious property.

If rings starting together run freely down straight wires CP″, CQ″, CR″, CS″, all posited in one vertical plane: at any one moment of time all the rings will be in the circumference of one and the same circle PQRS. At the end of any other equal interval, they will all be in the circumference of another circle P′Q′R′S′, touching the former in C. After any other interval, all in the circumference of another circle P″Q″R″S″, touching both the former in C. And so on.



If the wires, instead of being conceived all in one vertical plane, be imagined directed different ways in space—then the rings that run down them, will at the end of successive intervals, be found simultaneously in the surfaces of so many successive *spheres*, all touching in the point C, the origin of the motions.

SCHOLIUM.

197. We may here introduce some useful formulæ, relative to motions along inclined planes, analogous to those already given for bodies falling freely (art. 158.)

I. Let g, as before, $= 32\frac{1}{6}$ feet, s the space along an inclined plane whose inclination is i, t the time, v the velocity; then

$$1.\ s = \tfrac{1}{2}gt^2 \sin. i = \frac{v^2}{2g \sin. i} = \tfrac{1}{2}tv.$$

$$2.\ v = gt \sin. i = \sqrt{(2gs \sin. i)} = \frac{2s}{t}.$$

$$3.\ t = \sqrt{\frac{2s}{g \sin. i}} = \frac{2s}{v}.$$

II. Suppose V to be the velocity with which a body is projected up or down the plane; then, we have

$$4.\ v = V \mp gt \sin. i.$$

$$5.\ s = Vt \mp \tfrac{1}{2}gt^2 \sin. i = \frac{V^2 \mp v^2}{2g \sin. i}.$$

Making $v = 0$, in equa. 4, and the latter member of equa. 5; the first will give the *time* at which the body will cease to rise, the latter the *space*.

III. If R be a constant resistance to motion on a horizontal plane, then

$$6.\ v = V - Rt.$$

$$7.\ s = Vt - \tfrac{1}{2}Rt^2 = \frac{V^2 - v^2}{2R},$$

where, making $v = 0$, we find when the motion ceases.

198. The first eight of the following problems will serve to exemplify these theorems.

1. How far will a body descend from quiescence in 4 seconds, along an inclined plane whose length is 400 and height 300 feet?

2. What velocity will such a body have acquired when it has reached the bottom A of a plane?

3. Suppose AD = DB, in what time will the body pass over each of those portions?

4. How long would a body be in falling down 100 feet of a plane whose length AB is 150 feet, and height BC 60?

5. If AB = 90, and BC = 25 feet, what velocity would a body acquire in falling through 70 feet?

6. A body is projected up an inclined plane, whose length is 10 times its height, with a velocity of 30 feet per second; in what time will its velocity be destroyed, and it cease to ascend?

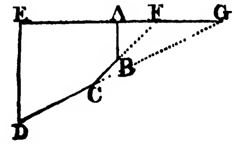
7. Suppose that at the moment a body is projected up AB with the velocity acquired by falling down it, another body begins to fall down it, where will they meet, the length of AB being given?

8. Given AB = 90, BC = 60 feet. And suppose two bodies to be let fall the same moment, one vertically, the other down the plane BA; what distance BD will the latter have moved, when the former has descended to C?

9. Ascertain, geometrically, the position of the right line of quickest descent, from a given point to a given plane.

10. Find, geometrically, the slope of a roof, down which rain may descend quickest.

199. PROP. If a body descend down any number of contiguous planes, AB, BC, CD; it will at last acquire the same velocity, as a body falling perpendicularly through the same height ED, supposing the velocity not altered by changing from one plane to another.



Produce the planes DC, CB, to meet the horizontal line EA produced in F and G. Then, by cor. 1, last art. the velocity at B is the same, whether the body descend through AB or FB. And therefore the velocity at C will be the same, whether the body descend through ABC or through FC, which is also again the same as by descending through GC. Consequently it will have the same velocity at D, by descending through the planes AB, BC, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C: and this again, is the same velocity as by descending through the same perpendicular height ED.

Corol. 1. If the lines ABCD, &c. be supposed indefinitely small, they will form a curve line, which will be the path of the body; from which it appears that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

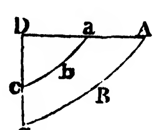
Corol. 2. Hence also, bodies acquire the same velocity by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

Corol. 3. And a body, after its descent through any curve, will acquire a velocity which will carry it to the same height through an equal curve, or through any other curve, either by running up the smooth concave side, or by being retained in the curve by a string, and vibrating like a pendulum: Also, the velocities will be equal, at all equal altitudes; and the ascent and descent will be performed in the same time, if the curves be the same.

PENDULOUS MOTION.

200. PROP. The times in which bodies descend through similar parts of similar curves, ABC, *abc*, placed alike, are as the square roots of their lengths.

That is, the time in AC is to the time in *ac*, as $\sqrt{AC}$ to $\sqrt{ac}$.



For, as the curves are similar, they may be considered as made up of an equal number of corresponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the corresponding small similar parts will also be parallel to each other. But the time of describing each of these pairs of corresponding parallel parts, by art 194, 195, are as the square roots of their lengths, which, by the supposition, are as $\sqrt{AC}$ to $\sqrt{ac}$,

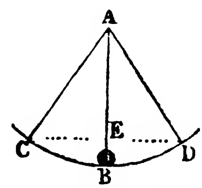
the roots of the whole curves. Therefore, the whole times are in the same ratio of $\surd AC$ to $\surd ac$.

Corol. 1. Because the axes DC, Dc, of similar curves, are as the lengths of the similar parts AC, ac; therefore the times of descent in the curves AC, ac, are as $\surd DC$ to $\surd Dc$, or the square roots of their axes.

Corol. 2. As it is the same thing, whether the bodies run down the smooth concave side of the curves, or be made to describe those curves by vibrating like a pendulum, the lengths being DC, Dc; therefore the times of the vibration of pendulums, in similar arcs of any curves, are as the square roots of the lengths of the pendulums.

SCHOLIUM.

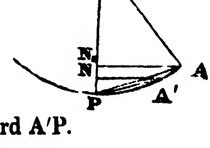
201. Having, in the last corollary, mentioned the pendulum, it may not be improper here to add some remarks concerning it.



A simple pendulum consists of a small ball, or other heavy body B, hung by a fine string or thread, moveable about a centre A, and describing the arc CBD; by which vibration the same motions happen to this heavy body, as would happen to any body descending by its gravity along the spherical superficies CBD, if that superficies were perfectly hard and smooth. If the pendulum be carried to the situation AC, and then let fall, the ball in descending will describe the arc CB; and in the point B it will have that velocity which is acquired by descending through CB, or by a body falling freely through EB. This velocity will be sufficient to cause the ball to ascend through an equal arc BD, to the same height D from whence it fell at C; having there lost all its motion, it will again begin to descend by its own gravity; and in the lowest point B it will acquire the same velocity as before; which will cause it to re-ascend to C: and thus, by ascending and descending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no resistance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never cease. But from these obstructions, though small, it happens, that the velocity of the ball in the point B is a little diminished in every vibration; and consequently it does not return precisely to the same points C or D, but the arcs described continually become shorter and shorter, till at length they are insensible; unless the motion be assisted by a mechanical contrivance, as that in clocks called a maintaining power.

Our present investigations relate to the simple pendulum, above described: the consideration of compound pendulums requires the previous knowledge of the centre of oscillation.

202. Prop. When a pendulum vibrates in a circular arc, the velocities acquired in the lowest point, are as the chords of the semi-arcs described.

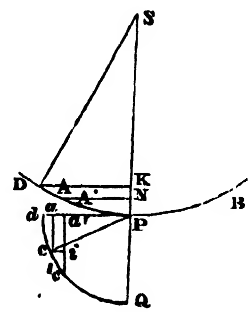


For, the velocity at P of a body that has descended through any arc AP, is equal to the velocity at P of a body that has fallen freely through the versed-sine NP (art. 199, cor. 2.)

Hence, velocity at P after descent through arc AP, is to velocity at P after descent through arc A′P, as $\surd NP$ to $\surd N'P$, that is (Geom. th. 87) as chord AP to chord A′P.

Corol. If, therefore, we would impart to a body a given velocity, v, we have only to compute the height NP, such that NP $= \frac{v^2}{2g} = \frac{v^2}{64\frac{1}{3}\text{ feet}}$, and through the point N draw the horizontal line NA; then AA′P an arc (of any circle passing through P) is one, through which when a body has fallen it will have acquired the proposed velocity. This is extremely useful in experiments on collision.

203. Prop. To investigate the time of vibration of a pendulum of given length, in an indefinitely small arc.



Now, in estimating the time of an oscillation in an indefinitely small circular arc, let it be recollected that the excess of such an arc above its chord, being incomparably less than itself, may be neglected; so that we may consider the square of such an arc (like that of its chord, Geom. th. 37) as equal to the rectangle under the versed sine and the diameter.

Indeed, if instead of indefinitely small arcs we took arcs of 40′ or 50′, and compared the respective differences of their squares and those of their chords, we should find that the error would not exceed the 29000dth part of either result.

Thus, $\text{arc}^2\ 50' - \text{arc}^2\ 40' = 145444^2 - 116355^2 = 261799 \times 29089$,
while $\text{chord}^2\ 50' - \text{chord}^2\ 40' = 145442'^2 - 116354^2 = 261796 \times 29088$.

Let, then, DPB represent such a very short oscillation of a pendulum whose length, l, is SP, S being the point of suspension.

$$\text{Then, versin. KP} = \text{arc}^2\ \text{DP} \div 2l$$
$$\text{versin. NP} = \text{arc}^2\ \text{AP} \div 2l.$$

Their diff. KN $= \frac{\text{DP}^2 - \text{AP}^2}{2l}$; which is the altitude through which a body must fall to acquire the velocity at A. Putting this value of the altitude in the usual expression for falling bodies, $v = \sqrt{(2gs)}$, it becomes $v = \sqrt{(2g \cdot \frac{\text{DP}^2 - \text{AP}^2}{2l})}$ $= \sqrt{\frac{g}{l}} \cdot \sqrt{(\text{DP}^2 - \text{AP}^2)}$. This will be the velocity with which the pendulum will describe an exceedingly minute portion of the arc, such as AA′.

Draw, horizontally, dP = arc DP; with dP as radius describe the quadrantal arc dcc'Q; make da = DA, aa' AA′, and draw ac, $a'c'$, parallel to PQ.

Then, vel. at A

$$= \sqrt{\frac{g}{l}} \cdot \sqrt{(\text{DP}^2 - \text{AP}^2)} = \sqrt{\frac{g}{l}} \cdot \sqrt{(d\text{P}^2 - a\text{P}^2)} = ac\sqrt{\frac{g}{l}}.$$

But, since time of describing a space as AA′ $= aa'$, is inversely as the velocity, or $t = \frac{s}{v}$, we have

$$\text{time through AA}' \text{ (or } aa') = \frac{aa'}{ac}\sqrt{\frac{l}{g}} = \frac{cc'}{c\text{P}}\sqrt{\frac{l}{g}},$$

$$\text{(because, by sim. tri. } \frac{aa'}{ac} = \frac{cc'}{c\text{P}}).$$

The same reasoning applies for every minute successive portion, such as AA′, of the semi-arc described by the pendulum: and when the ball has descended

from D to P, the corresponding arc to dP its equal is the quadrant $dcc'Q$: the expression for the time, therefore, becomes, in that case,

$$t' = \frac{dcQ}{PQ}\sqrt{\frac{l}{g}} = \frac{\text{semicircum.}}{\text{diam.}}\sqrt{\frac{l}{g}} = \tfrac{1}{2}\pi\sqrt{\frac{l}{g}}.$$

The time of ascending through PB = PD is, manifestly, equal to the above: therefore, ultimately, the time of complete oscillation through DPB, is,

$$t = \pi\sqrt{\frac{l}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Consequently, ***the times of oscillation are as the square roots of the lengths of the pendulums,*** the force of gravity remaining the same.

204. For the same reason that we have the above equa. when l is the length of the pendulum, and g the lineal measure of the force of gravity, we have $t' = \pi\sqrt{\frac{l'}{g'}}$, in any other place where g' measures the force of gravity, and l' is the length of the pendulum.

Consequently, in general,

$$t : t' :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l'}{g'}}. \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If the force of gravity be the same, we have

$$t : t' :: \sqrt{l} : \sqrt{l'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

If the same pendulum be actuated by different gravitating forces, we have

$$t : t' :: \sqrt{\frac{1}{g}} : \sqrt{\frac{1}{g'}} :: \sqrt{g'} : \sqrt{g}. \quad . \quad . \quad . \quad (4).$$

When pendulums oscillate in equal times in different places, we have

$$g : g' :: l : l'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Other theorems may readily be deduced.

205. If either g or l be determined by experiment, the equa. 1, for t will give the other. Thus, if $\frac{1}{2}g$, or the space fallen through by a heavy body in 1″ of time, be found, then this theorem will give the length of the seconds pendulum. Or, if the length of the seconds pendulum be observed by experiment, which is the easier way; this theorem will give g. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $\pi\sqrt{\frac{39\frac{1}{8}}{g}} = 1''$: and hence is found $\frac{1}{2}g = \frac{1}{2}\pi^2 l = \frac{1}{2}\pi^2 \times 39\frac{1}{8} = 193{\cdot}07$ inches $= 16\frac{1}{12}$ feet, for the descent of gravity in 1″; which it has also been found to be very exactly, by many accurate experiments. Hence $l = \frac{1}{2}g \times {\cdot}20264$; $\frac{1}{2}g = l \times 4{\cdot}9348$.

SCHOLIUM.

206. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half-seconds pendulum, or a quarter-seconds pendulum; that is, a pendulum to vibrate twice in a second, or four times in a second. Then, since the time of vibration is as the square root of the length,

therefore $1 : \frac{1}{2} :: \sqrt{39\frac{1}{8}} : \sqrt{l}$,

or $1 : \frac{1}{4} :: 39\frac{1}{8} : \frac{39\frac{1}{8}}{4} = 9\frac{3}{4}$ inches nearly, the length of the half-seconds pendulum.

And $1 : \frac{1}{16} :: 39\frac{1}{8} : 2\frac{7}{16}$ inches, the length of the quarter seconds pendulum.

Again, if it were required to find how many vibrations a pendulum of 80

inches long will make in a minute. Here $\sqrt{80} : \sqrt{39\frac{1}{8}} :: 60''$ or $1' : 60 \sqrt{\frac{39\frac{1}{8}}{80}} = 7\frac{1}{2} \sqrt{31\cdot3} = 41\cdot95987$, or almost 42 vibrations in a minute.

207. For military men it is a good practice to have a portable pendulum, made of painted tape with a brass bob at the end, so that the whole, except the bob, may be rolled up within a box, and the whole enclosed in a shagreen case. The tape is marked 200, 190, 180, 170, 160, &c. 80, 75, 70, 65, 60, at points, which being assumed respectively as points of suspension, the pendulum will make 200, 190, &c. down to 60 vibrations in a minute. Such a portable pendulum is highly useful in experiments relative to falling bodies, the velocity of sound, &c.

For the comparison of the times of oscillation in indefinitely small arcs of circles, in finite arcs of circles, and in cycloidal arcs, the student may turn to some subsequent problems relating to forces, &c.

EXERCISES ON THE DOCTRINE OF PENDULUMS.

1. How long will a pendulum 60 inches in length be in making one vibration? Ans. 1·2386 sec.

2. How many vibrations will a pendulum 36 inches long make in a minute? Ans. 62·55.

3. What is the length of the pendulum that vibrates 3 times in a second? Ans. $4\frac{25}{72}$ in.

4. Required the length of a pendulum that makes as many vibrations in a minute as it is inches in length. Ans. 52·03 in.

5. Find the length of a pendulum that will make 3 vibrations while a heavy body falls through a space of 500 feet. Ans. 135 096 in.

6. How often will a pendulum 25 inches in length vibrate in $2\frac{1}{2}$ minutes? Ans. 187·65 times.

7. What is the length of a pendulum that vibrates 20 times in a minute? Ans. $352\frac{1}{8}$ in.

8. If a clock lose a minute an hour, how much must the pendulum be shortened to make it keep true time? Ans. 1·337 in.

9. Suppose a clock was observed to lose 30 sec. in 12 hours; what must the pendulum be shortened to make it keep true time? Ans. ·055 in.

10. Required the length of a pendulum that vibrates siderial seconds, the length of the siderial day being $23^h. 56^m. 4^s$. Ans. 38·911 in.

Obs. I. The length of the seconds pendulum varies in different latitudes on account of the centrifugal force.

Thus, it has been found that, for any latitude L,

$$l = l' + d \sin.^2 L,$$

where l' is the length of the seconds pendulum at the equator (= 39·0265 in.) and d the difference between that and the length at the poles (= ·1608 inches nearly); which gives the expression $l = 39\cdot0265 + \cdot1608 \sin.^2 L$.

Ex. 1. Required the length of the seconds pendulum at London in latitude 51° 30′. Ans. 39·124986 in.

Ex. 2. Find the length of the seconds pendulum at St. Petersburgh in latitude 59° 56′. Ans. 39·1469 in.

Ex. 3. What is the length of the pendulum at the Cape of Good Hope in latitude 34° 30′? Ans. 39·078 in.

Obs. II. If l' denote the length of a seconds pendulum at any height h, above the earth's surface, r the radius of the earth, and l the length at the surface, we have $l' = \frac{rl}{r+h}$ correctly, $= l - \frac{2hl}{r}$ nearly*.

Ex. 1. Required the length of the pendulum that would vibrate seconds on the top of Chimborazo, 21,000 feet above the level of the sea, in lat. 1° 36′, the radius of the earth being considered = 3960 miles. Ans. 38·9486 in.

2. What would be the length of the seconds pendulum on the highest peak of the Himalaya mountains, supposed to be 27,000 feet high, and in latitude 20°? Ans. 38·9634 in.

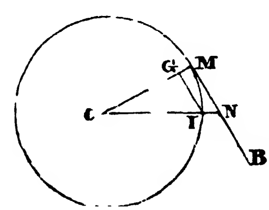
CENTRAL FORCES.

208. *Def.* 1. *Centripetal force* is a force which tends constantly to solicit or to impel a body towards a certain fixed point or centre.

2. *Centrifugal force* is that by which it would recede from such a centre, were it not prevented by the centripetal force.

3. These two forces are, jointly, called *central* forces.

209. Prop. If a body, M, drawn continually towards a fixed point, C, by a constant force, ϕ, and projected in a direction, MB, perpendicular to CM, describe the circumference of a circle about the centre C, the central force ϕ, is to the weight of the body, as the altitude due to the velocity of projection, is to half the radius CM.



Let v be the velocity of projection in the tangent MB, and r the radius CM. Independently of the action of the central force, the body would describe, along MB, during the very small time t, a space MN $= tv$, and would recede from the point C by the quantity IN, which may, without error, be regarded as equal to GM, when the arc MI is exceedingly small. If, therefore, the body instead of moving in the tangent, were kept in the circumference by the central force ϕ, its operation in the time t, would (art. 137) be equal to $\frac{1}{2}\phi t^2$, and at the same time = MG. But by the nature of the circle MG $= \frac{\text{MI}^2}{2r} = \frac{\text{MN}^2}{2r}$.

(in an extremely small arc) $= \frac{t^2v^2}{2r}$, by the above.

Making, therefore, $\frac{1}{2}\phi t^2 = \frac{t^2v^2}{2r}$, it reduces to

$$\phi = \frac{v^2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

* The variation in the length of a pendulum oscillating seconds at any point *upon* or *for three miles above* the earth's surface, never amounts to *three-tenths* of an inch.

Putting a for the altitude due to the velocity v, since (by art. 158) $v^2 = 2ag$, we have $\phi = \frac{2ag}{r}$, whence there results

$$\phi : g :: a : \tfrac{1}{2}r.$$

Thus far, we have, in reality, considered only the unit of mass; but, if we multiply the first two terms of the above proportion by the mass of the body, the whole will still remain a correct proportion, and the general result may be thus enunciated: viz.

The centripetal force of any body, if it be free, or its centrifugal force, if it be retained to the centre C, by a thread (or otherwise), is to the weight of that body, as the height due to the velocity v, is to the half of the radius CM *.

210. Hence, it appears that, so long as ϕ and r remain constant, the velocity v will be constant.

211. If both members of the equation 1 be multiplied by the mass M of the body, and we put F to represent the centrifugal force of that mass, we shall have $F = \frac{Mv^2}{r}$. In like manner, if F′ is the centrifugal force of another body which revolves with the velocity v' in a circle whose radius is r', we shall have

$$F : F' :: \frac{v^2}{r} : \frac{v'^2}{r'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

212. If T and T′ denote the times of revolution of the two bodies, because $v = \frac{2\pi r}{T}$, and $v' = \frac{2\pi r'}{T'}$, we have

$$F : F' :: \frac{r}{T^2} : \frac{r'}{T'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

213. If the times of revolution are equal, we shall have

$$F : F' :: r : r' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

214. And, if we assume $T^2 : T'^2 :: r^3 : r'^3$, as in the planetary motions, the proportion (3) will become

$$F : F' :: r'^2 : r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

as the pupil may at once shew.

SCHOLIUM.

215. The subject of central forces is too extensive and momentous to be adequately pursued here. The student may consult the treatises of mechanics by *Gregory* and *Poisson*, and those on fluxions by *Simpson*, *Dealtry*, &c. We shall simply present in this place, one example connected with practical mechanics.

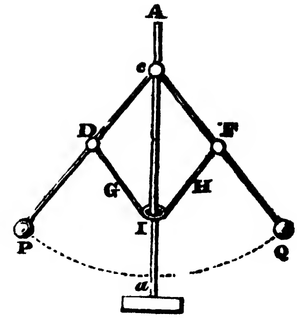
Ex. Investigate the characteristic property of a conical pendulum applied as a regulator or *governor* to steam-engines, &c.

This contrivance will be readily comprehended from the marginal figure,

* Here, since $\frac{\phi}{w} = \frac{v^2 \div 2g}{\frac{1}{2}r} = \frac{v^2}{gr} \therefore \phi = \frac{v^2 w}{gr} = \frac{v^2 w}{32r}$ nearly, from which ϕ may be found n terms of w the weight, when v and r are given.

If r and t be given, then, since $t = \frac{2\pi r}{v} \therefore v = \frac{2\pi r}{t}$, and

$$\phi = \frac{4\pi^2 r^2}{t^2} \cdot \frac{w}{gr} = \frac{4\pi^2 rw}{gt^2} = \frac{1{\cdot}2273r}{t^2} w.$$



where Aa is a vertical shaft capable of turning freely upon the sole a. cD, cF, are two bars which move freely upon the centre c, and carry at their lower extremities two equal weights, P, Q: the bars cD, cF, are united, by a proper articulation, to the bars G, H, which latter are attached to a ring, I, capable of sliding up and down the vertical shaft, Aa. When this shaft and connected apparatus are made to revolve, in virtue of the centrifugal force the balls P, Q, fly out more and more from Aa, as the rotatory velocity increases: if, on the contrary, the rotatory velocity slackens, the balls descend and approach Aa. The ring I *ascends* in the former case, *descends* in the latter: and a lever connected with I may be made to correct appropriately, the energy of the moving power. Thus, in the steam-engine, the ring may be made to act on the valve by which the steam is admitted into the cylinder; to augment its opening when the motion is slackening, and reciprocally diminish it when the motion is accelerated.

The construction is, often, so modified that the flying out of the balls causes the ring I to be depressed, and *vice versâ;* but the general principle is the same. If FQ = FI = DP = DI, then I, P, Q, are always in some one horizontal plane: but that is not essential to the construction.

Now, let t denote the time of one revolution of the shaft, x the variable horizontal distance of each ball from that shaft, π as usual $= 3{\cdot}141593$: then will the velocity of each ball be $= \frac{2\pi x}{t}$, and (art. 209.) its centrifugal force $= \left(\frac{2\pi x}{t}\right)^2 \div x = \frac{4\pi^2 x}{t^2}$. The balls being operated upon simultaneously by the centrifugal force and the force of gravity, of which one operates horizontally, the other vertically, the resultant of the two forces is, evidently, always in the actual position of the handle CD, CF. It follows therefore, that the ratio of the gravity to the centrifugal force, is that of cos. ICQ to sin. ICQ, or that of the vertical distance of Q below C to its horizontal distance from Aa. Call the former d, the latter being x:

$$\text{then } d : x :: g : \frac{4\pi^2 x}{t^2},$$

$$\text{therefore } \frac{gt^2}{4\pi^2 x} = \frac{d}{x} \text{ and } t = 2\pi\sqrt{\frac{d}{g}} = 1{\cdot}10784\,\sqrt{d}.$$

Hence, the periodic time varies as the square root of the altitude of the conic pendulum, let the radius of the base be what it may.

Hence, also, when ICQ = ICP = 45°, the centrifugal force of each ball is equal to its weight.

FARTHER EXAMPLES.

1. A ball whose weight is 10lbs. is whirled round in a circle whose radius is 10 feet, with a velocity of 30 feet per second: what is the measure of ϕ the centrifugal force?

2. If the same ball be made to move uniformly through the same circle, in 2 seconds, what will the centrifugal force then be?

3. Given the diameter of the orbit, 10 feet, and the centrifugal force equal to

the weight of the revolving body; required the time of a revolution, and velocity per second.

4. Given the diameter, 14·59 feet, and the velocity, 15·279, to find the periodic time and central force.

5. Given the diameter, 14·59 feet, the central force equal to twice the weight of the body; what is the velocity and time of a revolution?

6. Let the diameter be 29·18 feet, the time of a revolution 3 seconds; required the velocity and central force.

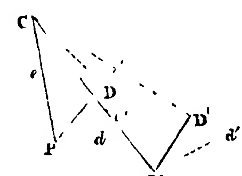
7. If a fly, 12 feet diameter, and 3 tons weight, revolves in 8 seconds, and another of the same weight revolves in 3 seconds; what must be the diameter of the last, when they have the same centrifugal force?

8 If a fly, 12 feet diameter, revolves in 8 seconds, and another of the same diameter in 3 seconds; what is the ratio of their weights, when the central forces are equal?

9. If a fly, 2 tons weight, and 16 feet diameter, is sufficient to regulate an engine, when it revolves in 4 seconds; what must be the weight of one 12 feet diameter, when it revolves in 2 seconds, so that it may have the same power upon the engine?

ON ROTATORY MOTION.

216. **Prop.** If a body revolve about an axis, the particles of which that body is composed resist, by their inertia, the communication of motion to any given point, with forces which are as the particles themselves, and the squares of their distances from the axis of motion jointly.



Let an axis of rotation pass through C perpendicular to the plane of the figure, and let a body fixed firmly to this axis be acted upon by several accelerating forces; we are to enquire into the circumstances of the motion produced. Suppose, at first, that a particle p, situated at P in the figure, is urged about the fixed axis by a force ϕ, applied in the direction PD: that force tends to impress upon the particle p a certain velocity in the direction PD, yet, in consequence of the mutual cohesion between the different moleculæ of the body, and the connection of the whole with the fixed axis, the velocity can only be produced actually in the initial direction Pd perpendicular to CP. Drawing, therefore, Pd perpendicular to CP, the force ϕ must be decomposed into two others, in the directions Pd, PC, of which the one Pe will be extinguished by the resistance of the axis, and the other Pd has place; where, of consequence, we have P$d = \phi$ cos. DPd. If, therefore, we denote the distance CP of p from the axis of rotation by r, and the perpendicular distance DC of the force from the axis by δ, we have, in the triangle CPD, sin. DPC = cos. DP$d = \frac{\delta}{r}$; and, consequently, P$d = \phi \frac{\delta}{r}$. This expression would represent the effective accelerating force of the particle p, if that particle were alone; but the connection of this particle with the others, and the operation of the forces acting on the latter, change this effect: if, then, the arc a be run over at the

end of the time t by a particle at a unit of distance from the fixed axis, ar will be the arc described in the same time by the particle p; so that the velocity, and the effective accelerating force of this latter will be $r\frac{\dot{a}}{\dot{t}}$, and $r\frac{\ddot{a}}{\dot{t}^2}$ respectively.

Now comparing this force with the former, and observing that they are both directed in the same right line, we may reason thus. The accelerating force $\phi\frac{\delta}{r}$ which is impressed in the direction of the initial motion, must be decomposed into

$r\frac{\ddot{a}}{\dot{t}^2}$ . . . effective accelerating force.

$\phi\frac{\delta}{r} - r\frac{\ddot{a}}{\dot{t}^2}$ . . . accel. force destroyed by the action of the other powers in the system.

In like manner we may proceed to investigate the effects of other forces ϕ', ϕ'', &c. acting upon the moleculæ p', p'', &c. (whose distances are r', r'', &c.) in the directions P′D′, P″D″, &c., and at the distances δ', δ'', &c. the expressions being the same with the letters accented similarly: the whole, therefore, will be in equilibrio when impressed by the moving forces

$$\frac{\delta}{r}\phi p - \frac{\ddot{a}}{\dot{t}^2} rp,\ \frac{\delta'}{r'}\phi' p' - \frac{\ddot{a}}{\dot{t}^2} r'p',\ \frac{\delta''}{r''}\phi'' p'' - \frac{\ddot{a}}{\dot{t}^2} r''p'', \text{ \&c.}$$

Now as the system is attached to a fixed axis, only one equation is necessary to express the state of equilibrium. And, if we suppose that the forces ϕ, ϕ', ϕ'', &c. are parallel to the plane of the figure, or perpendicular to the plane of rotation (and they may all be resolved to such planes by an obvious process), it will be merely requisite to make the sum of the moments of the powers with respect to the fixed axis equal to nothing. Here, that of the first force will be $\delta\phi p' - \frac{\ddot{a}}{\dot{t}^2} r^2 p$, and the moments of the other forces will be expressed in the same manner, adding the accents. Thus, then, we shall have

$$\delta\phi p + \delta'\phi' p' + \text{\&c.} = (r^2 p + r'^2 p' + \text{\&c.})\frac{\ddot{a}}{\dot{t}^2}.$$

Or taking the character $\int$ as before, and denoting the angular velocity by U, we shall have $\int\phi\delta p = \frac{\ddot{a}}{\dot{t}^2}\int r^2 p$: whence, $\frac{\ddot{a}}{\dot{t}^2} = \frac{\dot{U}}{\dot{t}} = \int\frac{\phi\delta p}{r^2 p}$.

Hence, if the quantity $\int r^2 p$, which is the sum of the products of the several moleculæ into the squares of their respective distances from the axis, be called the *momentum of inertia*, and if $\frac{\dot{U}}{\dot{t}}$ be called the *angular accelerating force*, the equation just given may be thus stated in words at length:

The angular accelerating force is the quotient of the sum of the momenta of the moving forces, or of their resultant, divided by the momentum of inertia.

Corol. 1. The force which accelerates the point A of any body revolving on an axis, to which point that force ϕ is applied, is equal to the product of the force into the square of the distance AC, divided by the sums of the products of all the moleculæ into the squares of their respective distances from C, the centre of motion.

For the mass moved is $\int \frac{p \,.\, CP^2}{AC^2}$, and the moving force is ϕ: but the accelerating force is equivalent to the quotient of the moving force by the mass, and is therefore represented by $\int \frac{\phi \,.\, AC^2}{p \,.\, PC^2}$.

Corol. 2. *The angular velocity of a system, generated in a given time, by any force ϕ at A, perpendicular to AC, is proportional to the rectangle of the force into the distance at which it acts, divided by the sums of the products of all the moleculæ into the squares of their respective distances.*

For the absolute velocity of the point A is as the accelerating force, and the angular velocity is as the absolute velocity directly and the distance reciprocally; therefore the angular velocity is as $\int \frac{\phi \,.\, AC^2}{p \,.\, PC^2} \times \frac{1}{AC}$ or as $\phi \,.\, AC \div \int p \,.\, PC^2$.

Corol. 3. *The angular motion of any system, generated by a uniform force, will be a motion uniformly accelerated.*

This is evident, because the accelerating force is in a constant ratio to the uniform force ϕ.

Corol. 4. What has been here shewn with respect to moleculæ situated on a right line passing through a centre of motion will hold equally with regard to a body or system moving upon an axis: for all the particles of such body may be conceived to be transferred to the plane in which the axis of suspension CP performs its motion, by an orthographical projection, the lines of transference being all parallel to the axis of motion; this supposition will, it is obvious, neither affect the place of the centre of gravity (with regard to the axis of motion) nor the angular motion of the body.

Corol. 5. From the above final equation we may readily obtain an expression for the angular velocity. For $\phi = \frac{\dot{v}}{\dot{t}}$ &c. give $\int \phi \delta p = \frac{\dot{v}}{\dot{t}} \delta p = \frac{\dot{v}}{\dot{t}} \delta p + \frac{\dot{v}'}{\dot{t}} \delta' p'$ + &c. whence we have $\dot{U} = \frac{\dot{v}\delta p + \dot{v}'\delta' p' + \&c.}{r^2 p + r'^2 p' + \&c.}$; and, taking the fluents,

$$U = \frac{v\delta p + v'\delta' p' + \&c.}{r^2 p + r'^2 p' + \&c.} = \int \frac{v\delta p}{r^2 p}.$$

From this it appears that cor. 2 is not confined to a single force; but may be extended to as many forces as we please, and applied to the body in any directions whatever.

Corol 6 As to the part P*e* of the force ϕ, which operates as a pressure upon the fixed axis, and is entirely destroyed by its re-action, it may be easily determined. For $Pe = \phi$ cos. DPC, and DP $= \sqrt{(r^2 - \delta^2)}$; therefore, cos. DPC $= \frac{DP}{PC} = \sqrt{\left[1 - \left(\frac{\delta}{r}\right)^2\right]}$

$$\text{consequently, } Pe = \phi \sqrt{\left[1 - \left(\frac{\delta}{r}\right)^2\right]}.$$

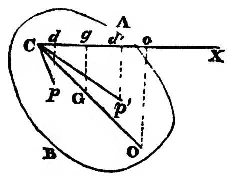
And the same may be shown with regard to the effect of any other forces, ϕ', ϕ'', &c. upon the axis of motion.

217. Def. The *centre of oscillation* is that point in the axis of suspension of a vibrating body in which, if all the matter of the system were collected, any force applied there would generate the same angular velocity in a given time as the same force at the centre of gravity, the parts of the system revolving in their respective places.

Or, since the force of gravity upon the whole body may be considered as a single force (equivalent to the weight of the body) applied at its centre of gravity, the *centre of oscillation* is that point in a vibrating body into which, if the whole were concentrated and attached to the same axis of motion, it would then vibrate in the same time the body does in its natural state.

Corol. From the first definition it follows that the centre of oscillation is situated in a right line passing through the centre of gravity, and perpendicular to the axis of motion *.

218. Prop. If a body vibrate about an axis by the force of gravity, the distance of the centre of oscillation from that axis is equal to the quotient arising from dividing the sum of the products of each particle into the square of its distance from the axis, by the product of the mass or body into the distance of its centre of gravity from the axis.



Let AB be a plane passing through the centre of gravity G of the body perpendicular to the axis of vibration on which the body is orthographically projected; C the centre of motion of that plane, or of the whole body reduced to that plane; O the centre of oscillation in the line CG produced, and p, p', &c. the constituent moleculæ of the body thus projected. Through C draw CX parallel to the horizon, and upon that line demit the verticals or perpendiculars pd, $p'd'$, Gg, Oo, &c. Now since the angular velocity of the several particles is not changed by this projection, we have $p \cdot p\mathrm{C}^2 + p' \cdot p'\mathrm{C}^2 + \&c. = \int p \cdot p\mathrm{C}^2$, the momentum of inertia of the whole body B, while $\mathrm{B} \cdot \mathrm{OC}^2$ will express the momentum of inertia of an equal body concentrated into the point O. And since the force of gravity ϕ acts in the parallel directions pd, $p'd'$, &c. we shall have $\phi \cdot p \cdot d\mathrm{C} + \phi \cdot p' \cdot d'\mathrm{C} + \&c.$ for the effect of gravity to turn the body about C, by the nature of parallel forces, and this is equal to $\phi \cdot \mathrm{B} \cdot \mathrm{C}g$, by the nature of the centre of gravity. And if B were concentrated in the point O, then would $\phi \cdot \mathrm{B} \cdot \mathrm{C}o$ be the accelerating force of gravity to turn the body about C. But, that the same angular velocity may be generated in both cases, the quotients of the accelerating forces with respect to C, by the momenta of inertia, must be equal (art. 216, cors 2—5), that is, $\frac{\phi \cdot \mathrm{B} \cdot \mathrm{C}g}{\int p \cdot p\mathrm{C}^2} = \frac{\phi \cdot \mathrm{B} \cdot \mathrm{C}o}{\mathrm{B} \cdot \mathrm{OC}^2}$, whence $\mathrm{OC}^2 = \frac{\phi \cdot \mathrm{B} \cdot \mathrm{C}o \cdot \int p \cdot p\mathrm{C}^2}{\phi \cdot \mathrm{B} \cdot \mathrm{B} \cdot \mathrm{C}g}$. Now, the triangles CG$g$, CO$o$ being similar, we have $\frac{\mathrm{CO}}{\mathrm{CG}} = \frac{\mathrm{C}o}{\mathrm{C}g}$: hence, substituting the former fraction for the latter in the preceding value of OC^2, it becomes $\mathrm{OC}^2 = \frac{\mathrm{CO}}{\mathrm{CG}} \cdot \frac{\phi \cdot \mathrm{B} \cdot \int p \cdot p\mathrm{C}^2}{\phi \cdot \mathrm{B} \cdot \mathrm{B}.} = \frac{\mathrm{CO}}{\mathrm{CG}} \cdot \frac{\int p \cdot p\mathrm{C}^2}{\mathrm{B}}$: consequently

$$\mathrm{CO} = \frac{\int p \cdot p\mathrm{C}^2}{\mathrm{B} \cdot \mathrm{CG}} = \frac{\int pr^2}{\mathrm{B} \cdot \mathrm{CG}}.$$

This expression, being independent of the line CX, will continue the same for all inclinations of the line CO; so that the centre of oscillation O, thus determined, is a *fixed point*, agreeably to the definition.

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* Since, according to the former of these definitions, the centre of oscillation is a definite point, and according to the latter it is variable; it might, perhaps, be better to say, that a body has a *centre* of oscillation when it is attached to a centre of motion, and has a *line* of oscillation when it moves upon an axis; the *line* of oscillation in the latter case being parallel to the axis of motion.

Corol. 1. Hence any body AB, suspended at C, and vibrating, may, so far as regards the time of vibration, be considered as a simple pendulum whose length is CO.

Corol. 2. *That* CO *may be the equivalent pendulum, and* O *the centre of oscillation,* O *must be in the line* CG, otherwise it would not rest in the same position with the body, when no force was keeping it out of its vertical position.

Corol. 3. *If a body be turned about its centre of gravity in a direction perpendicular to the axis of motion, the place of the centre of oscillation will remain unaltered.* For the quantity $\int pr^2$ will be not at all affected by such a motion of the body.

Corol. 4. The distance of the centre of gravity from that of oscillation (if the plane of the body's motion remain unaltered) will be reciprocally as the distance of the former from the point of suspension. *If, therefore, that distance be found when the point of suspension is in the vertex, or so situated that the operation may become the most simple, the value thereof in any other proposed position of that point will likewise be given by one single proportion.*

Corol. 5. The product of the distances of the centre of gravity, and that of oscillation, from the axis of motion, is manifestly a constant quantity for the same plane of vibration. *If, therefore, the centre of oscillation be made the point of suspension, the point of suspension will become the centre of oscillation.*

Corol. 6. *Hence also, if upon the plane of vibration passing through the centre of gravity of any body, two concentric circles be described, having the common centre* G, *and radii* GC, GO, *the body suspended from any point in the periphery of either circle will perform an oscillation in the same time.*

Some other corollaries depending upon the relation subsisting between the centres of oscillation and of gyration will be given after we have treated of that centre: when also we shall apply the propositions to the determination of these centres in various bodies. Previous to this, however, we may here show how—

219. To find the distance of the centre of oscillation from the point of suspension experimentally.

Suspend the body proposed very freely by the given point, and make it vibrate in small arcs; then count the number n of vibrations it makes in a minute, by a good stop-watch, applying when necessary the correction requisite for finite arcs (as below); so shall the distance CO of the centre of oscillation and point of suspension be denoted by $\frac{140850}{n^2}$. For the length of the simple second pendulum being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in a given time; therefore $n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{n^2} = \frac{140850}{n^2}$, the length of the simple pendulum which oscillates n times in a minute, or the distance CO in the compound pendulum.

Or, two-thirds of the length of a thin cylindrical rod, suspended at one end, and vibrating in the same time as the body, will give CO; the reason of which will soon appear.

Or, CO $= 39\frac{1}{8}t^2$, in inches, t being the time of one oscillation in a very small arc. If the arc be of finite appreciable magnitude, the time of oscillation must be reduced in the ratio 8 + versin. of semi-arc to 8, before the rule is applied.

220. Prop. In a compound pendulum, consisting of several bodies revolving about a common axis, the centre of oscillation may be determined by this process: add together the several products of each mass into the distances of the

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respective centres of oscillation and gravity from the axis of motion, and divide the sum by the product of the whole system into the distance of the common centre of gravity from the axis of motion; the distance of the centre of oscillation from the same axis will be represented by that quotient.

For it appears from art. 218, that with respect to any body B in the compound mass, $\int pr^2 = B \,.\, CO \,.\, CG$; and again, with regard to another body B′ in the same system $\int p'r'^2 = B' \,.\, CO' \,.\, CG'$, and so on; consequently $S \,.\, \int .\, pr^2 = \int B \,.\, CO \,.\, CG$; that is, the sum of all the pr^2 in the whole pendulum is equal to the sum of all the products in each part of the pendulum, which arise from multiplying each part into the product of the distances of its centres of gravity and of oscillation from the common axis of motion. If, therefore, this latter sum be divided by the product of the compound mass into the distance of the common centre of gravity from the axis of motion, the quotient (art. 218) will be the distance of the centre of oscillation from the same axis: for $\frac{\int . \int pr}{\int B \,.\, CO \,.\, CG}$ varies as $\frac{\int pr^2}{B \,.\, CO \,.\, CG}$: whence $CO = \frac{\int pr^2}{B \,.\, CG}$, as well with respect to the compound mass as to any of its constituent parts.

221. Def. The *centre of gyration* is that point in which, if all the matter contained in a revolving system were collected, the same angular velocity will be generated in the same time by a given force acting at any place as would be generated by the same force acting similarly in the body or system itself.

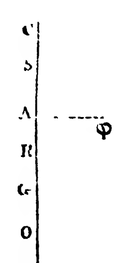
When the axis of motion passes through the centre of gravity, then is this centre called the *principal* centre of gyration.

222. Prop. If the sum of the products formed by multiplying each particle of a system into the square of its distance from the axis of motion, that is, if the momentum of inertia be divided by the whole mass, the square root of the quotient will be the distance of the centre of gyration from the axis of motion.

For, if CR be the distance from the axis of motion to the centre of gyration, the expression for the angular motion $\frac{\phi \,.\, AC}{\int p \,.\, PC^2}$ or $\frac{\phi \,.\, AC}{\int pr^2}$ (art. 217, cor. 2.) will be transformed to $\frac{\phi \, AC}{B \,.\, CR^2}$; and these by the def. must be equal. Consequently, $B \,.\, CR^2 = \int pr^2$, and $CR = \sqrt{\frac{\int pr^2}{B}}$.

Corol. 1. *The distance of the centre of gyration from the axis of motion is a mean proportional between the distances of the centres of oscillation and gravity, from the same axis.*

For $CR^2 = \frac{\int pr^2}{B}$, while $CO = \frac{\int pr^2}{B \,.\, CG}$; therefore $CO \,.\, CG = \frac{\int pr^2}{B} = CR^2$, and $CO : CR :: CR : CG$.



Corol. 2. *The distance between the centre of gravity and principal centre of gyration is a mean proportional between the distances of the centres of motion and oscillation, from the centre of gravity.*

For $PC^2 = PG^2 + GC^2 \pm 2PG \,.\, GC$; therefore, by art. 219, we have $OC = \int p \,.\, \frac{PG^2 + GC^2 \pm 2PG \,.\, GC}{B \,.\, CG}$. But from the nature of the centre of gravity, $\int p \,.\, PG$ on each side of G are equal,

and, because 2CG is constant, $\int \pm 2\text{PG} \cdot \text{GC} = 0$. Also, because $\int p \cdot \text{GC}^2 = \text{B} \cdot \text{GC}^2$, we have $\text{OC} = \text{GC} + \int \frac{p \cdot \text{PG}^2}{\text{B} \cdot \text{GC}}$, and $\text{OG} = \int \frac{p \cdot \text{PG}^2}{\text{B} \cdot \text{GC}}$; whence this corollary is manifest.

Corol. 3. *The time in which a body vibrates will be the least possible when the axis of motion passes through the principal centre of gyration.*

For if D and d represent the distances from the centre of gravity to the point of suspension, and to the centre of gyration, it will be $\text{D} : \text{D} + d :: \text{D} + d : \frac{(\text{D} + d)^2}{\text{D}}$, the distance from the point of suspension to the centre of oscillation (cor. 1.), and this latter expression will obviously be a minimum when $\text{D} = d$.

Corol. 4. If we wish to know what quantity of matter M must be placed at any other distance AC from C, so that the inertia may remain the same, we must have $\text{AC}^2 \cdot \text{M} = \text{CR}^2 \cdot \text{B}$, whence $\text{M} = \frac{\text{B} \cdot \text{CR}^2}{\text{AC}^2}$.

Corol. 5. And, for a like reason, if any number of bodies B, B′, B″, be put in motion about a common axis passing through C, by a force acting at A, the system will have the same angular velocity, if, instead of those bodies placed at the distances CB, CB′, CB″, there be substituted bodies equal to $\frac{\text{CB}^2}{\text{CA}^2}\text{B}$, $\frac{\text{CB}'^2}{\text{CA}^2}\text{B}'$, $\frac{\text{CB}''^2}{\text{CA}^2}\text{B}''$, all concentrated into the point A *.

223. From the foregoing principles are derived the following expressions for the distances of the centres of oscillation for the several figures, suspended by their vertices and vibrating flatwise, viz.

(1.) Right line or very thin cylinder, $\text{CO} = \frac{2}{3}$ of its length.
(2.) Isosceles triangle, $\text{CO} = \frac{3}{4}$ of its altitude.
(3.) Circle, $\text{CO} = \frac{5}{4}$ radius.
(4.) Common parabola, $\text{CO} = \frac{5}{7}$ of its altitude.
(5.) Any parabola, $\text{CO} = \frac{2m + 1}{3m + 1} \times$ its altitude.

Bodies vibrating laterally or sideways, or in their own plane:

(6.) In a circle, $\text{CO} = \frac{3}{4}$ of diameter.
(7.) In a rectangle suspended by one angle, $\text{CO} = \frac{2}{3}$ of diagonal.
(8.) Parabola suspended by its vertex, $\text{CO} = \frac{5}{7}$ axis $+ \frac{1}{3}$ parameter.
(9.) Parabola suspended by middle of its base, $\text{CO} = \frac{4}{7}$ axis $+ \frac{1}{2}$ parameter.
(10.) In a sector of a circle, $\text{CO} = \frac{3 \text{ arc} \times \text{radius}}{4 \text{ chord}}$.
(11.) In a cone, $\text{CO} = \frac{4}{5}$ axis $+ \frac{(\text{radius of base})^2}{5 \text{ axis}}$.

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* The distance of R, the centre of gyration, from C the centre or axis of motion, in some of the most useful cases, is as below:

In a circular wheel of uniform thickness . . .	$\text{CR} = \text{rad.} \sqrt{\frac{1}{2}}$.
In the periphery of a circle revolving about the diam. .	$\text{CR} = \text{rad.} \sqrt{\frac{1}{2}}$.
In the plane of a circle . . ditto . . .	$\text{CR} = \frac{1}{2}\text{rad.}$
In the surface of a sphere . . ditto . . .	$\text{CR} = \text{rad.} \sqrt{\frac{2}{3}}$.
In a solid sphere . . . ditto . . .	$\text{CR} = \text{rad.} \sqrt{\frac{2}{5}}$.
In a plane ring formed of circles whose radii are R, r, revolving about centre	$\text{CR} = \sqrt{\frac{\text{R}^2 + r^2}{2}}$.
In a cone revolving about its vertex	$\text{CR} = \frac{1}{2}\sqrt{\frac{12}{5}a^2 + \frac{3}{5}r^2}$.
In a cone . . its axis	$\text{CR} = r\sqrt{\frac{3}{10}}$.
In a straight lever whose arms are R and r . . .	$\text{CR} = \sqrt{\frac{\text{R}^3 + r^3}{3(\text{R} + r)}}$.

(12.) In a sphere, CO $=$ rad. $+ \; d + \dfrac{2\ \text{rad.}^2}{5\,(d + \text{rad.})}$. Where d is the length of the thread by which it is suspended.

(13.) If the weight of the thread is to be taken into the account, we have the following distance between the centre of the ball and that of oscillation, where B is the weight of the ball, d the distance between the point of suspension and its centre, r the radius of the ball, and w the weight of the thread or wire,

$$\text{GO} = \frac{(\frac{1}{3}w + \frac{2}{5}\text{B})\,4r^2 - \frac{1}{6}w\,(2\,d\,r + d^2)}{(\frac{1}{2}w + \text{B})\,d - r\,w};$$ or, if B be expressed in terms of w considered as a unit, then $\text{G O} = \dfrac{\frac{1}{6}d}{\text{B} + \frac{1}{2}}$.

(14.) If two weights W, W′, be fixed at the extremities of a rod of given length W W′, S being the centre of motion between W and W′; then if $d = \text{S W}$, $\text{D} = \text{S W}'$, and m the weight of an unit in length of the rod, we shall have $\text{CO} = \dfrac{m\,\text{D}^3 + 3\ \text{W}'\,\text{D}^2 + m\,d^3 + 3\ \text{W}\,d^2}{m\,\text{D}^2 + 2\ \text{W}'\,\text{D} - m\,d^2 - 2\ \text{W}\,d}$; the radii of the balls being supposed very small in comparison with the length of the rod.

(15.) In the bob of a clock pendulum, supposing it two equal spheric segments joined at their bases, if the radii of those bases be each $= \rho$, the height of each segment v, and d the distance from the point of suspension to G the centre of the bob, then is $\text{G O} = \frac{1}{3}\,d \,.\, \dfrac{\rho^4 + \frac{1}{2}\,\rho^2 v^2 + \frac{1}{10}\,v^4}{\rho^2 + \frac{1}{3}\,v^2}$; which shows the distance of the centre of oscillation below the centre of the bob.

If r the radius of the sphere be known, the latter expression becomes

$$\text{G O} = \frac{\frac{2}{3}\,l^2\,v\,\frac{1}{2}\,r\,v - \frac{1}{10}\,v^3}{d\,(r - \frac{1}{3}\,v)}.$$

(16.) Let the length of a rectangle be denoted by l, its breadth by $2w$, the distance (along the middle of the rectangle) from one end to the point of suspension by d, then the distance S O, from the point of suspension to the centre of oscillation, will be $\text{CO} = \dfrac{d^2 - d\,l + \frac{1}{3}l^2 + \frac{1}{3}\,w^2}{\frac{1}{2}\,l - d} = \frac{1}{2}\,l - d + \dfrac{\frac{1}{12}\,l^2 + \frac{1}{3}\,w^2}{\frac{1}{2}\,l - d}$, whether the figure be a mere geometrical rectangle, or a prismatic metallic plate of uniform density.

It follows from this theorem that a plate of 1 foot long, and $\frac{5}{6}$ of a foot broad, suspended at a fourth of a foot from either end, would vibrate as a half second pendulum.

Also, that a plate a foot long, $\frac{1}{16}$ of a foot wide, and suspended at $\frac{1}{8}$ of a foot from the middle, would vibrate 36,469 times in 5 hours.

And hence the length of a foot may be determined experimentally by vibrations.

(17.) If a thin rod, say of a foot in length, have a ball of an inch diameter at each end, A and B, and a moveable point of suspension, C; then *the time of oscillation of such a pendulum may be made as long as we please*, by bringing the point of suspension nearer and nearer to the middle of the rod.

Or, if the point of suspension be fixed, the distance CO (and consequently the time of oscillations which is as $\surd$CO) may be varied by placing A nearer or farther from C. *And this is the principle of the* METRONOME, *by which musicians sometimes regulate their time.*

(18.) If the weight of the connecting rod be evanescent with regard to the weight of the balls A and B; then if R $=$ radius of the larger ball, r that of the smaller, D and d the distances of their respective centres from S: we shall have

$$\text{CO} = \frac{\text{R}^3\,(5\ \text{D}^2 + 2\ \text{R}^2) + r^3\,(5\,d^2 + 2\,r^2)}{5\,(\text{D}\ \text{R}^3 - d\,r^3)}.$$

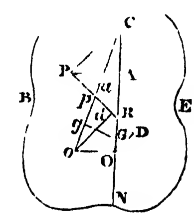
When R and r are equal, this becomes

$$CO = (D + d) + \tfrac{4}{5} \cdot \frac{r^2}{D - d}.$$

(19.) If the minor and major axes of an ellipse (or of an elliptical plate of wood or metal) be as 1 to $\sqrt{3}$, or as 1000 to 1732; then if it be suspended at one extremity of the minor axis, the centre of oscillation will be at the other extremity of that axis, or its oscillations will be performed in the same time as those of a simple pendulum whose length is equal to the minor axis.

(20.) *The same ellipse also possesses this curious and useful property:* viz. That any segment or any zone of the ellipse cut off by lines parallel to the major axis, whether it be taken near the upper part of the minor axis, near the middle, or near the bottom of the same, will vibrate in the same time as the whole ellipse, the point of suspension being at an extremity of the minor axis.

225. DEF. The *centre of percussion* is that point in a body revolving about an axis, at which, if it struck an immoveable obstacle, all its motion would be destroyed, or it would not incline either way: so that if at the instant of the impact the supports of the axis were annihilated, the body would remain in absolute rest.



During the vibration of a system of bodies round a fixed axis passing through C, if such an obstacle be opposed to any point O, as entirely to destroy the motion of that point, every other particle of the system will endeavour, by its inertia, to proceed in the direction of its motion, that is, of the tangent of the circular arc it was describing the instant O was stopped. These forces, therefore, will act on the system to turn it round O; and if the sum of the forces on each side of O should be unequal, the motion of the system will not be destroyed when O is stopped; but since the forces which act upon the pendulous body between O and C have an effect to continue the motion of the system, contrary to those which are impressed on the other side of O, if this point O be so situated that the sum of the forces to turn the system round O, on each side of that point, may be exactly equal, then the instant in which O is stopped, the whole motion of the system will be destroyed.

Corol. 1. When a pendulum, vibrating with a given angular velocity, strikes an obstacle, the effect of the impact will be the greatest if it be made at the centre of percussion.

For, in this case the obstacle receives the whole revolving motion of the pendulum; whereas, if the blow be struck in any other point, a part of the motion of the pendulum will be employed in endeavouring to continue the rotation.

Corol. 2. If a body revolving on an axis strike an immoveable obstacle at the centre of percussion, the point of suspension will not be affected by the stroke.

We can ascertain this property of the point O when we give a smart blow with a stick. If we give it motion round the joint of the wrist only, and holding it at one extremity, strike smartly with a point considerably nearer or more remote than $\frac{2}{3}$ of its length, we feel a painful wrench in the hand: but if we strike with that point which is precisely at $\frac{2}{3}$ of the length, no such disagreeable strain will be felt. If we strike the blow with one end of the stick, we must make its centre of motion at $\frac{1}{3}$ of its length from the other end; and then the wrench will be avoided.

226. Prop. The distance of the centre of percussion from the axis of motion is equal to the distance of the centre of oscillation from the same: supposing that the centre of percussion is required in a plane passing through the axis of motion and centre of gravity.

Let CBE (same fig.) be a plane passing through the centre of gravity G of the body, and perpendicular to the axis of suspension which passes through C; and conceive the whole body to be projected upon this plane in lines perpendicular to it, or parallel to the axis of motion; for then, as each particle will fall at the same distance from the axis as in the body itself, the effect from the rotatory motion will not be changed, neither will the place of the centre of gravity. Through C and G draw the line CGON, and let P be the place of one of the particles p composing the system. Now, since the angular motion of all the particles is the same, the absolute velocity will be proportional to the distance from the axis of motion; and if at the distance 1 the velocity be expressed by unity, the velocity of p will manifestly be denoted by PS, and its quantity of motion will be $p \cdot PC$, which will act in the direction PR perpendicular to PC: produce PR till it meets OD, drawn parallel to PC, in D; then the force $p \cdot PC$ acting in the direction PR will act upon O as though it had the advantage of the lever OD, and consequently $p \cdot PC \cdot OD$ will represent the force of the particle p to move the system round O. But by reason of the similar triangles CPR, ODR, we have $RC : CP :: RO : OD = \frac{CP \cdot RO}{CR} = CP \cdot \frac{CO - CR}{CR}$; and, if PA be perpendicular to CO, we shall have $RC : CP :: CP : CA = \frac{CP^2}{CR}$. Hence the entire force $p \cdot PC \cdot OD$ becomes $= p \cdot PC^2 \cdot \frac{CO - CR}{CR} = p \cdot CA \cdot CO - p \cdot PC^2$. But, when O is the centre of percussion the sum of all the $p \cdot CA \cdot CO - p \cdot PC^2$, must be equal to zero, or $\int p \cdot CA \cdot CO = \int p \cdot PC^2 = \int pr^2$. Whence it follows that $CO = \frac{\int pr^2}{\int p \cdot CA} = \frac{\int pr^2}{B \cdot CG}$, the denominators of the two fractions being equal; and this value of CO obviously corresponds with that given for the centre of oscillation in art. 218.

Corol. 1. If the body be symmetrical with regard to the plane BGE, or if it be a solid of rotation, the centre of percussion found in the axis of the body will coincide with the centre of oscillation.

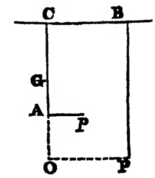
Corol. 2. If the centre of percussion be required in a plane which does not pass through G, as Co for instance, we must proceed thus: from G, the centre of gravity, let fall on Co the perpendicular Gg; and, by the same argument as above, $Co = \frac{\int pr^2}{\int p \cdot Ca} = \frac{\int pr^2}{B \cdot Cg}$. Now, $Co : CO :: CG : Cg$; hence the angles oOC and GgC are equal, and, consequently, the former is a right angle.

Corol. 3. Hence it follows, that *a body has several centres of percussion* according to the plane passing through the axis of motion in which the impact is made, *and the right line OO, at right angles to CO, is their locus.*

Prop. 227. If CB be the axis of a body's motion, CAO a plane perpendicular to CB and passing through the centre of gravity G, pA a perpendicular let fall from any particle p of the body on the plane CA, and P the centre of percussion, then will PO = CB = the sum of all the $\frac{p \cdot Ap \cdot AG}{Body \cdot CG}$.

For the sum of all the forces with which the body is liable to be turned in one

direction round PB as an axis = sum of all the p . CA . (CB — Ap) $=\int p$. CA . (CB — Ap), and the sum of all the forces which tend to turn it in the contrary direction = sum of all the p . CA (Ap — CB) $=\int p$. CA (Ap — CB). Therefore, in the case of no motion either way, we have $\int p$. AC . CB = $\int p$. Ap . AC. But, $\int p$. CB . AC = $\int p$. CB . CG + $\int p$. CB . GA; and $\int p$. CB . GA = 0, from the nature of the centre of gravity. In like manner $\int p$. pA . AC = $\int p$. pA . AG + $\int p$. pA . CG, and $\int p$. pA . CG = 0: consequently, $\int p$. CB . CG = $\int p$. pA . AG; and $\text{CB} = \text{PO} = \int \frac{p \,.\, p\text{A} \,.\, \text{AG}}{p\text{CG}} = \frac{\int p \,.\, p\text{A} \,.\, \text{AG}}{\text{B} \,.\, \text{CG}}$.



Corol. Hence the centre of percussion of a body turning round the axis CB, is determined by these conditions. 1st. It is in a line PO passing through the centre of oscillation and parallel to CB. 2dly. Its distance OP from the centre of oscillation is $\frac{\int p \,.\, p\text{A} \,.\, \text{AG}}{\text{B} \,.\, \text{CG}}$.

Practical Examples—Centre of Oscillation.

1. Five balls, whose weights are A = 4oz., B = 6oz., C = 8oz., D = 6oz., E = 4oz., are placed with their centres at distances SA = 3 inches, SB = 6, SC = 9, SD = 12, SE = 15 inches, from S, an assumed point of suspension. Required SO, the distance of the centre of oscillation from S.

2. If while the weights of the balls are the same, the respective distances are SA = 3, SB = 7, SC = 12, SD = 18, SE = 25; required SO.

3. A ball of 2oz. weight is placed at a distance of 30 inches from S, the point of suspension, in a slender wire; at what distance from S must another equal ball be fixed upon the wire, that the system should vibrate seconds?

4. Two equal balls are placed at the extremity of a thin wire 36 inches long. What will SO be, and what the times of vibration when S is 2, 4, 8, and 16 inches respectively from the middle point of the wire?

5. Each leg of an angular pendulum is 12 inches; required SO, when the two legs make an angle of 120°.

PRACTICAL SCHOLIUM.—CENTRE OF GYRATION.

228. If ρ be put for SR, the distance of the centre of gyration from the point of suspension, $W\rho^2 = A \,.\, SA^2 + B \,.\, SB^2 + C \,.\, SC^2 + \&c.$ = sum of all the pd^2.

Hence by means of the theory of the centre of gyration, and the values of ρ thence deduced in the note to art. 222. the phenomena of rotatory motion become connected with those of accelerating forces: for then, if a weight or other moving power P act at a radius r to give rotation to a body, weight W, and dist. of centre of gyration from axis of motion = ρ, we shall have for the accelerating force the expression

$$f = \frac{Pr^2}{Pr^2 + W\rho^2};$$

and consequently for the space descended by the actuating weight or power P, in a given time t, we shall have the usual formula

$$s = \tfrac{1}{2} f g t^2,$$

introducing the above value of f.

229. For applications of these formulæ and their obvious modifications, as they are exceedingly useful in rotatory motions, the student may solve the following problems.

Problems illustrative of the Principle of the Centre of Gyration.

1. Suppose a cylinder that weighs 100lbs. to turn upon a horizontal axis, and imagine motion to be communicated by a weight of 10lbs. attached to a cord which coils upon the surface of the cylinder: how far will that weight descend in 10 seconds? Ans. 268·055 ft.

2. Required the actuating weight, such that when attached in the same way to the same cylinder, it shall descend $16\frac{1}{12}$ feet in 3 seconds.

$$P = \frac{\frac{1}{2}SW}{\frac{1}{2}gT^2 - S} = 6\frac{1}{4}.$$

3. Another cylinder, which weighs 200lbs. is actuated in like manner by a weight of 30lbs. How far will the weight descend in 6 seconds?

Ans. 133·6 feet.

4. Suppose the actuating weight to be 30lbs., and that it descends through 48 feet in 2 seconds, what is the weight of the cylinder? Ans. $20\frac{5}{12}$lbs.

5. Suppose a cylinder that weighs 20lbs. to have a weight of 30lbs. actuating it, by means of a cord coiled about the surface of the cylinder; what velocity will the descending weight have acquired at the end of the first second?

Ans. $24\frac{1}{8}$.

6. Of what weight will the axis be relieved in the case of the last example, when the system is completely in motion? Ans. $22\frac{1}{2}$lbs.

7. A sphere, W, whose radius is 3 feet, and weight 500lbs. turns upon a horizontal axis, being put in motion by a weight of 20lbs. acting by means of a string that goes over a wheel whose radius is half a foot. How long will the weight, P, be in descending 50 feet? Ans. $33\frac{1}{2}''$.

8. Of what weight will the axle be relieved as soon as motion is commenced?

Ans. $\frac{20}{381}$lbs.

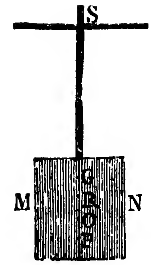
9. If in example seventh the radius of the wheel be equal to that of the sphere, what ratio will the accelerating force bear to that of gravity?

10. A paraboloid, W, whose weight is 200lbs. and radius of base 20 inches, is put in motion upon a horizontal axis by a weight P of 15lbs. acting by a cord that passes over a wheel whose radius is 6 inches. After P has descended for 10 seconds, suppose it to reach a horizontal plane and cease to act, then how many revolutions would the paraboloid make in a minute?

BALLISTIC PENDULUM.

230. Prop. To explain the construction of the Ballistic Pendulum, and show its use in determining the velocity with which a cannon or other ball strikes it.

The ballistic pendulum is a heavy block of wood MN, suspended vertically by a strong horizontal iron axis at S, to which it is connected by a firm iron stem. This problem is the application of the preceding articles, and was invented by Mr. Robins, to determine the initial velocities of military projectiles; a circum-



stance very useful in that science; and it is the best method yet known for determining them with any degree of accuracy.

Let G, R, O, be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expressed by the force f, acting at P in the foregoing propositions. Now,

Put $p =$ the whole weight of the pendulum,
$b =$ the weight of the ball,
$g =$ SG the distance of the centre of gravity,
$o =$ SO the distance of the centre of oscillation,
$r =$ SR $= \surd go$ the distance of centre of gyration,
$i =$ SP the distance of the point of impact,
$v =$ the velocity of the ball,
$u =$ that of the point of impact P,
$c =$ chord of the arc described by O.

By art. 235, if the mass p be placed all at R, the pendulum will receive the same motion from the blow in the point P: and as $SP^2 : SR^2 :: p : \frac{SR^2}{SP^2} \cdot p$ or $\frac{r^2}{i^2}p$ or $\frac{go}{ii}p$, (art. 222), the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii}p$, the former moving with the velocity v, and striking the latter at rest; to determine their common velocity u, with which they will jointly proceed forward together after the stroke. In which case, by the law of the impact of non-elastic bodies, we have $\frac{go}{ii}p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii}u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of u, we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends again to the vertical position, it will have acquired the same velocity with which it began to ascend, and by the laws of falling bodies, the velocity of the centre of oscillation is such as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the versed sine of the arc described by O. Then, by the laws of falling bodies $\surd 16\frac{1}{12} : \surd\frac{cc}{2o} :: 32\frac{1}{6} : c\surd\frac{2a}{o}$, the velocity acquired by the point O in descending through the arc whose chord is c, where $a = 16\frac{1}{12}$ feet: and therefore $o : i :: c\surd\frac{2a}{o} : \frac{ci}{o}\surd\frac{2a}{o}$, which is the velocity u, of the point P.

Then, by substituting this value for u, the velocity of the ball, before found, becomes $v = \frac{bii + gop}{bio} \times c\surd\frac{2a}{o}$. So that the velocity of the ball is directly as the chord of the arc described by the pendulum in its vibration.

SCHOLIUM.

231. In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

For an example in numbers, suppose the weights and dimensions to be as follow: namely,

$p = 570$ lbs.
$b = 18$ oz. $1\frac{1}{2}$ dr.
$\quad = 1{\cdot}131$ lb.
$g = 78\frac{1}{2}$ inches.
$o = 84\frac{7}{9}$ inches.
$\quad = 7{\cdot}065$ feet.
$i = 94\frac{3}{10}$ inches.
$c = 18{\cdot}73$ inches.

Then

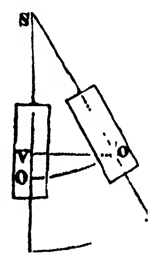
$$\frac{bii + gop}{bio} \times c = \frac{1{\cdot}131 \times 94{\cdot}3^2 + 78\frac{1}{2} \times 84\frac{7}{9} \times 570}{1{\cdot}131 \times 94\frac{3}{10} \times 84\frac{7}{9}}$$

$$\times \frac{18{\cdot}73}{12} = 656{\cdot}56,$$

And $\sqrt{\dfrac{2a}{o}} = \sqrt{\dfrac{32\frac{1}{6}}{7{\cdot}065}} = \sqrt{\dfrac{193}{42{\cdot}39}} = 2{\cdot}1337.$

Therefore $656{\cdot}56 \times 2{\cdot}1337$, or 1401 feet, is the velocity, per second, with which the ball moved when it struck the pendulum.

When the impact is made upon the centre of oscillation, the computation becomes simplified.



In that case, since the whole mass, p, of the pendulum, may be regarded as concentered at O, and the ball, b, strikes that point, we shall have $bv = (b + p)v'$; v being the velocity of the ball before the impact, and v' that of the ball and pendulum together, after the impact. Now, if the centre of oscillation O, after the blow, describes the arc OO′, before the motion is destroyed, the velocity v' will be equal to that acquired by falling through the versed sine VO, of the arc OO′ or angle S to the radius SO. But if the time t of a very minute oscillation of the pendulum be known or inferred from that in an ascertained arc, we have (art 219), $\text{SO} = 39\frac{1}{8}t^2$ inches $= 3\frac{25}{96}t^2$ feet.

$$\begin{aligned} \text{Hence VO} &= \text{SO nat versin S}, \\ &= 3{\cdot}2604\tfrac{1}{6}t^2 \text{ versin S}, \\ \text{and (art. 154)}v' &= \sqrt{(64\tfrac{1}{3} \times 3{\cdot}2604\tfrac{1}{6}t^2 \text{ versin S})}, \\ &= 14{\cdot}48286t\sqrt{\text{versin S}}. \end{aligned}$$

$$\text{Conseq. } v = \frac{(b + p)v'}{b} = \frac{b + p}{b} \,.\, 14{\cdot}48286t\sqrt{\text{versin S}}.$$

EXAMPLES.

1. Given the weight, p, of a ballistic pendulum, 7408 lbs., that of the ball, b, 6·1 lbs., SO = 11·6 feet, semi-arc of oscillation 3° 34′. Required the velocity of the ball. Ans. 1461·3 feet.

2. If a cannon ball of 1lb. weight be fired against a pendulous block of wood, and, striking the centre of oscillation, cause it to vibrate an arc whose chord is 30 inches; the radius of that arc or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the centre of oscillation of the pendulum, at the lowest point of the arc; the whole weight of the pendulum being 500 lbs.

Ans. veloc. ball 1956·6054 feet per sec. and
veloc. cent. oscil. 3.9054 feet per sec.

OF HYDROSTATICS.

232. **Hydrostatics** is the science which treats of the pressure, or weight, and equilibrium of water and other liquids or fluids, especially those that are non-elastic.

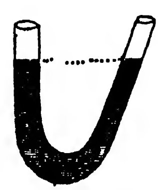
A fluid may be regarded as a collection of infinitely small spheres, capable of moving freely on each other without friction.

233. A fluid is *elastic* when it can be reduced into a less bulk by compression, and which restores itself to its former bulk again when the pressure is removed; as air. And it is *non-elastic*, when it is not compressible or expansible, as water, &c. In the latter case it is often called a liquid.

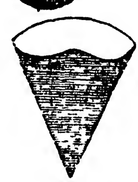
234. Prop. If a fluid be enclosed in a vessel, and no external forces but that of gravity act upon it, its surface will be horizontal.

For, if any particles be above the horizontal surface of the rest, they will, by their gravity, press upon the body of the fluid with a force equivalent to their weight; and the particles of the portion whose upper boundary is the horizontal surface being free to move in all directions, will be pressed from their places to admit of the descent of those above the surface.

As this will take place so long as any portion of the fluid surface is above the other portions of it, the fluid can only be in a state of equilibrium when all the particles of its surface are in one general horizontal position.

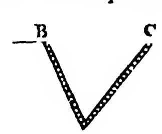


Corol. 1. Hence, water that communicates with other water, by means of a close canal or pipe, will stand at the same height in both places. It is thus with water in the two legs of a syphon.



Corol. 2. For the same reason, if a fluid gravitate towards a centre; it will dispose itself into a spherical figure, the centre of which is the centre of force: like the sea in respect of the earth. Such, indeed, the horizontal surface of a fluid upon the surface of the earth is, the centre of attraction being the centre of the mass of the earth; but the distance of that centre being so great, the surface does not sensibly differ from a plane.

235. Prop. The pressures exercised by the particles of the fluid in equilibrio, upon any one of the particles within it, are the same in all directions.



For (by 234,) the surface of the fluid is horizontal. Let A be the particle, and AC any direction which is proposed for consideration. Since the pressure in that direction is the sum of the pressures of all the particles which lie in that direction, this will be the same if we cut off that filament of particles from the surrounding ones on all sides by a slender tube interposed so as just to enclose the one in AC: since the pressures of the adjacent particles upon the filament, so as to keep them in equilibrio, are supplied by the tube, and the pressures of the particles of the filament in all directions but that of the axis of the tube are counteracted by the resistances of the sides of the tube.

Let us also conceive another inclined tube, AB, containing the particles immediately superincumbent in the vertical plane upon the particle A: then for the

same reason, the pressure on A will be the sum of the pressures of all the particles in AB. It remains now to prove that the pressure upon A by the particles in the two tubes are equal.

Here, the number of particles in AB : number in AC :: AB : AC; and by the property of the inclined plane,

Force of any particle in AB : force of any particle in AC :: $\frac{1}{AB}$: $\frac{1}{AC}$.

Whence, compounding these two proportions, we have

No. of partic. in AB $\times$ force of each in AB = no. of partic. in AC $\times$ force of each in AC.

That is, the whole force of pressure in the direction AC is equal to the whole force of pressure in AB.

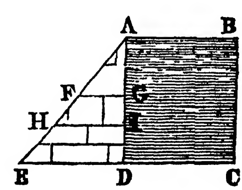
And as AC and AB are taken in *any* direction from A, it follows that in all directions the pressure is the same as that exercised by the vertical column of the fluid interposed between the particle A and the surface; and hence that the pressures in all directions are equal.

Corol. 1. If the tubular canal be curved instead of straight, the same consequence follows; since the whole mass of fluid being in equilibrio, all its parts are mutually in equilibrio, and the tube counteracts the forces of the particles within it upon the adjacent particles of the mass, exactly as stated in the proposition itself.

Corol. 2. The pressures upon particles at equal depths are equal. For in all cases these are as the altitudes of the superincumbent masses of fluid.

Corol. 3. In the same fluid, the pressures at different depths are as those depths. For in each case the pressures are produced by the superincumbent filaments of the fluid; and these (the fluids being incompressible) are as the depths.

Corol. 4. In a vessel containing a fluid, the pressure is the same against the bottom as against the sides, or even upwards, at the same depth.



Corol. 5. Hence, and from the last proposition, if ABCD be a vessel of water, and there be taken, in the base produced, DE, to represent the pressure at the bottom; joining AE, and drawing any parallels to the base, as FG, HI; then shall FG represent the pressure at the depth AG, and HI the pressure at the depth AI, and so on; because the parallels . . FG, HI, ED, by sim. triangles, are as the depths . . . . . AG, AI, AD: which are as the pressures, by the proposition.

And hence the sum of all the FG, HI, &c. or the area of the triangle ADE, is as the pressure against all the points G, I, &c. that is against the line AD. But as every point in the line CD is pressed with a force as DE, and that thence the pressure on the whole line CD is as the rectangle ED . DC, while that against the side is as the triangle ADE or $\frac{1}{2}$DA . DE; therefore the pressure on the horizontal line DC is to the pressure against the vertical line DA, as DC to $\frac{1}{2}$DA. And hence, if the vessel be an upright rectangular one, the pressure on the bottom, or whole weight of the fluid, is to the pressure against one side, as the base is to half that side. Therefore the weight of the fluid is to the pressure against all the four upright sides, as the base is to half the upright surface. And the same holds true also in any upright vessel, whatever the sides be, or in a cylindrical vessel. Or, in the cylinder, the weight of the fluid is to the

pressure against the upright surface, as the radius of the base is to the altitude. The whole pressure sustained by the cylinder is expressed by $\pi ars\ (a+r)$, s being the specific gravity of the fluid, a = alt., r = rad. of the base.

Also, when the rectangular prism becomes a cube, it appears that the weight of the fluid on the base is double the pressure against one of the upright sides, or half the pressure against the whole upright surface.

Corol. 6. The pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid whose base is equal to the surface pressed, and its altitude the same as the altitude of that surface. For the pressure on a horizontal base equal to the upright surface, is equal to that column; and the pressure on the upright surface is but half that on the base, of the same area.

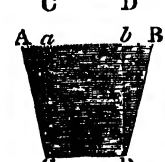
So that, if b denote the breadth, and d the depth of such a gate or upright surface: then the pressure against it is equal to the weight of the fluid whose magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}\text{AB} \cdot \text{AD}^2$. Hence, if the fluid be rain-water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}\text{AB} \cdot \text{AD}^2$, is 1440 feet; and the pressure is 1,440,000 ounces, or 90,000 pounds, or $40\frac{1}{5}$ tons weight nearly.

236. Prop. The perpendicular pressure of a fluid on a surface any way immersed in it, whether perpendicular or horizontal, or oblique, is equal to the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude equal to the depth of the centre of gravity of the surface pressed below the top or surface of the fluid.

For, conceive the surface pressed to be divided into innumerable sections parallel to the horizon; and let s denote any one of those horizontal sections, also d its distance or depth below the top surface of the fluid. Then, by art. 235, *corol.* 3, the pressure of the fluid on the section is equal to the weight of ds; consequently the total pressure on the whole surface is equal to all the weights ds. But, if b denote the whole surface pressed, and g the depth of its centre of gravity below the top of the fluid; then, by art. 91, bg is equal to the sum of all the ds. Consequently the whole pressure of the fluid on the body or surface b, is equal to the weight of the bulk bg of the fluid, that is, of the column whose base is the given surface b, and its height is g the depth of the centre of gravity in the fluid.

Corol. Hence the perpendicular pressure on all planes of equal areas, whatever be their figure, immersed in the same fluid, will be the same provided the depths of their centres of gravity remain the same. And if any plane revolve round its centre of gravity, which remains fixed, the pressure perpendicular to its surface will remain the same as when it was horizontal.

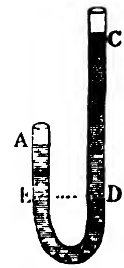
237. Prop. The pressure of a fluid, on the base of the vessel in which it is contained, is as the base and perpendicular altitude, whatever be the figure of the vessel that contains it.



For if any portion of the superior part of a fluid be replaced by a part of the vessel, the pressure against this from below will be the same which before supported the weight of the fluid removed; and every part remaining in equilibrio, the pressure on the bottom will be the same as if the horizontal section of the vessel were throughout of equal dimensions.

Corol. 1. Hence when the heights are equal, the pressures are as the bases. And when the bases are equal, the pressure is as the height. But when both the heights and bases are equal, the pressures are equal in all though their contents be ever so different.

Corol. 2. The pressure on the base of any vessel is the same as on that of a cylinder, of an equal base and height.



Corol. 3. If there be an inverted syphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or rest in equilibrio; then their heights in the two legs, AE, CD, above the point of meeting, will be reciprocally as their densities.

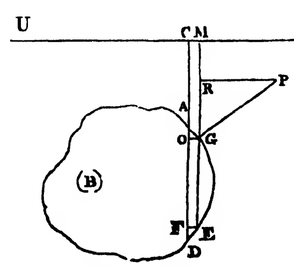
For if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the surface at D is the same, where they act against each other, therefore AE : CD :: density of CD : density of AE.

So if CD be water, and AE quicksilver, which is about 13·6 times heavier; then CD will be = 13·6AE; that is, if AE be 1 inch, CD will be 13·6 inches; if AE be 2 inches, CD will be 27·2 inches; and so on.

Corol. 4. From the reasoning in the demonstration of the proposition, it will readily appear that the smallest given quantity of a fluid may be made to produce a pressure capable of sustaining a weight of any magnitude, either by diminishing the diameter of the column and increasing its height, or by increasing the surface which supports the weight.

This property has been denominated the *hydrostatic paradox;* and it is the foundation of that powerful machine, *Bramah's Hydrostatic Press.*

238. Prop. A body immersed in a fluid loses as much weight, as an equal bulk of the fluid weighs. And the fluid gains the same weight. Thus, if the body be of equal density with the fluid, it loses all its weight, and so requires no force but the fluid to sustain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the same bulk of water; and it requires a force to sustain it just equal to that difference. But if it be lighter, it requires a force equal to the same difference of weights to keep it from rising up in the fluid.



For, let US be the upper surface of the liquid in which the body (B) is immersed. Suppose this body to be divided into indefinitely thin vertical columns, a vertical section of one of them being AGED, AD being parallel to GE. The pressure upon G, or upon the indefinitely small surface AG, will be the pressure due to the height MG: that is, drawing GP perpendicular to the surface AG and equal to GM, it will measure the pressure there. Draw also PR, GO and EF parallel to US. Then, since GP measures the perpendicular pressure upon every part of the minute upper section of the surface, GA, and GR measures the reduced vertical pressure;—if also we represent this by p, and the whole pressure of the column CG being denoted

by P;—we shall have $p : P :: AG . RG : GP . GO$. But the triangles GRP, AOG are similar; and therefore $PG : GR :: AG : GO$, or $PG . GO = AG . GR$. Hence the pressure upon AG = pressure upon OG.

In the same manner it may be shewn that the pressures upon ED and EF are equal: and consequently the *difference of the pressures* upon ED and AG is equal to the difference of the pressures upon EF and OG; or to the difference between the weights of columns of the liquid whose magnitudes are MO and MF, or to the column whose magnitude is AGED. This proof is evidently applicable to all the separate indefinitely small columns of the body, and therefore to the whole body (B).

Hence, the weight lost by the body (B) totally immersed in a fluid is equal to the weight of a portion of the liquid whose magnitude is that of the body.

As for the *horizontal* pressures, they mutually destroy one another, or counterbalance each other, all round, at every successive horizontal section.

Corol. 1. If d denote the total *downward* pressure,
u . . the *upward* pressure,
and w the *weight* of the body; then generally
when $d + w = u$, the body will *rest* in any position;
$d + w > u$, the body will *descend*;
$d + w < u$, the body will *ascend*.

Corol. 2. The weights lost, by immerging the same body in different fluids, are as the specific gravities of the fluids. And bodies of equal weight, but different bulk, lose, in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

Corol. 3. The whole weight of a body which will float in a fluid, is equal to the weight of as much of the fluid, as the immersed part of the body displaces when it floats. For the pressure under the floating body, is just the same as so much of the fluid as is equal to the immersed part; and therefore the weights are the same.

Corol. 4. Hence the magnitude of the whole body, is to the magnitude of the part immersed, as the specific gravity of the fluid, is to that of the body. For, in bodies of equal weight, the densities, or specific gravities, are reciprocally as their magnitudes.

Corol. 5. And because, when the weight of a body taken in a fluid, is subtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the specific gravity of the fluid is to that of the body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the specific gravity of the body, and
s the specific gravity of the fluid;
then $W - w : W :: s : S$, which proportion will give either of those specific gravities, the one from the other.

$$\text{Thus } S = \frac{W}{W - w}\, s, \text{ the specific gravity of the body;}$$

$$\text{and } s = \frac{W - w}{W}\, S, \text{ the specific gravity of the fluid.}$$

So that the specific gravities of bodies, are as their weights in the air directly, and their loss in the same fluid inversely.

Corol. 6. And hence, for two bodies connected together, or mixed together

into one compound, of different specific gravities, we have the following equations, denoting their weights and specific gravities, as below, viz.

H = weight of the heavier body in air,
h = weight of the same in water, } S its specific gravity;
L = weight of the lighter body in air,
l = weight of the same in water, } s its specific gravity;
C = weight of the compound in air.
c = weight of the same in water, } f its specific gravity;
w = the specific gravity of water. Then,

1st, $(H - h)\,S = Hw$,
2d, $(L - l)\,s = Lw$,
3d, $(C - c)\,f = Cw$,
4th, $H + L = C$,
5th, $h + l = c$,
6th, $\frac{H}{S} + \frac{L}{s} = \frac{C}{f}$.

From which equations may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as $s = \frac{Lw}{-l}$, by dividing the absolute weight of the body by its loss in water, and multiplying by the specific gravity of water.

But if the body L be lighter than water; then l will be negative, and we must divide by $L + l$ instead of $L - l$, and to find l we must have recourse to the compound mass C: and because, from the 4th and 5th equations, $L - l = \overline{C - c} - \overline{H - h}$, therefore $s = \frac{Lw}{(C - c) - (H - h)}$; that is, divide the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or thus, $s = \frac{SfL}{CS - Hf}$, as found from the last equation.

Also if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$$H = \frac{(f - s)S}{(S - s)f}C, \text{ and } L = \frac{(S - f)s}{(S - s)f}C,$$

the quantities of the two ingredients H and L, in the compound C. And so for any other demand.

Note.—If M, magnitude, S spec. grav. } of a body heavier than water:
m magnitude, s spec. grav. } of a body lighter than water: δ, spec. grav. of water:

Then, when $MS + ms = (M + m)\,\delta$, the compound mass will not sink, and then $m = M \cdot \frac{S - \delta}{\delta - s}$.

These expressions will often be useful.

Scholium—The resultant of all the vertical pressures upon the immersed body, or part of the body, passes through the centre of gravity of that body or that part of the body.

For every pressure may be resolved into two forces, one parallel to the horizon, and the other perpendicular to it. The forces parallel to the horizon destroy each other; and the effective forces are those perpendicular to the horizon, to the consideration of which we are therefore confined.

For if it do not, then let it pass through some other point of the body not in the vertical line through the centre of gravity. Then this resultant pressure would cause the body to rotate: but by hypothesis it is in equilibrio, which is incompatible; and hence this resultant does pass through the centre of gravity.

Prop. To find the specific gravity of a body.

239. **Case i.**—***When the body is heavier than water:*** weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, art. 238, $s = \frac{Bw}{B - b}$, where B is the weight of the body out of water, b its weight in water, s its specific gravity, and w the specific gravity of water. That is,

As the weight lost in water,
Is to the whole or absolute weight,
So is the specific gravity of water,
To the specific gravity of the body *.

Example.—If a piece of stone weigh 10lb., but in water only $6\frac{3}{4}$lb., required its specific gravity, that of water being 1000? Ans. 3077.

240. Case ii.—*When the body is lighter than water, so that it will not sink:* annex to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass, separately, both in water, and out of it: then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say, by proportion,

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

That is, the specific gravity is $s = \frac{Lw}{(C - c) - (H - h)}$, by cor. 6, art. 238.

Example. Suppose a piece of elm weighs 15lb. in air; and that a piece of copper, which weighs 18lb. in air and 16lb. in water, is affixed to it, and that the compound weighs 6lb. in water; required the specific gravity of the elm?

Ans. 600.

241. Case iii.—*For a fluid of any sort.*—Take a piece of a body of known specific gravity; weigh it both in and out of the fluid, finding the loss of weight by taking the difference of the two; then say,

As the whole or absolute weight,
Is to the loss of weight,
So is the specific gravity of the solid,
To the specific gravity of the fluid.

That is, the spec. grav. $w = \frac{B - b}{B} s$, by cor. 6, art. 238.

Example. A piece of cast iron, sp. grav. 7425, weighed 34·61 ounces in a fluid, and 40 ounces out of it; of what specific gravity is that fluid?

Ans. 1000.

242. **Prop.** To find the quantities of two ingredients in a given compound.

Take the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply each specific gravity by the difference of the other two. Then say, by proportion,

* In the Lectures on Natural Philosophy, in the Royal Military Academy, Coates's Hydrostatic *steelyard* is employed for this purpose. It is an improvement upon the one described in *Gregory's Mathematics for Practical Men.*

As the greatest product,
Is to the whole weight of the compound,
So is each of the other two products,
To the weights of the two ingredients.

That is, $H = \frac{(f - s)S}{(S - s)f}C$ = the one, and $L = \frac{(S - f)s}{(S - s)f}C$, the other, by cor. 6, art. 238.

Example. A composition of 112lb. being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and that of copper 9000?

Answer, there is 100lb. of copper and consequently 12lb. of tin } in the composition.

SCHOLIUM.

The specific gravities of several sorts of matter, as found from experiments, are expressed by the numbers annexed to their names in the following Tables.

TABLES OF SPECIFIC GRAVITIES.

Platina	20,722
Gold, pure, hammered	19,362
Guinea of George III.	17,629
Tungsten	17,600
Mercury, at 32° Fahr.	13,598
Lead	11,352
Palladium	11,300
Rhodium	11,000
Virgin Silver	10,744
Shilling of George III.	10,534
Bismuth, molten	9,822
Copper, wiredrawn	8,878
Red Copper, molten	8,788
Molybdena	8,611
Arsenic	8,308
Nickel, molten	8,279
Uranium	8,100
Steel	from 7,767 to 7,816
Cobalt, molten	7,812
Bar Iron	7,788
Pure Cornish Tin	7,291
Do. hardened	7,299
Cast Iron	7,207
Zinc	6,862
Antimony	6,712
Tellurium	6,115
Chromium	5,900
Spar, heavy	4,430
Jargon of Ceylon	4,416
Oriental Ruby	4,283
Sapphire, Oriental	3,994
Do. Brazilian	3,131
Oriental Topaz	4,019
Oriental Beryl	3,549
Diamond	from 3,501 to 3,531
English Flint-Glass	3,329
Tourmalin	3,155
Asbestus	2,996
Marble, green, Campan.	2,742
——— Parian	2,837
——— Norwegian	2,728
——— green, Egyptian	2,668
Emerald	2,775
Pearl	2,752
Chalk, British	2,784
Jasper	2,710
Coral	2,680
Rock Crystal	2,653
English Pebble	2,619
Limpid Feldspar	2,564
Glass, green	2,642
—— white	2,892
—— bottle	2,733
Paving stone	2,416
Porcelaine, China	2,385
———— Limoges	2,341
Native Sulphur	2,033
Ivory	1,917
Alabaster	1,874
Alum	1,720
Brick	from 1,400 to 1,550
Copal, opaque	1,140

Sodium	973	Orange-Wood . .	705
Oak, heart of . .	950	Pear-Tree . . .	661
Gunpowder, about . .	937	Linden-Tree . . .	604
Ice	930	Cypress . . .	598
Potassium . . .	866	Cedar	561
Beech	852	Fir	550
Ash	845	Poplar	383
Apple-Tree . . .	793	Cork	240

LIQUIDS.

Sulphuric Acid . .	1,841	Burgundy Wine . .	991
Nitrous Acid . .	1,550	Olive Oil . .	915
Water from the Dead Sea	1,240	Muriatic Ether . .	874
Nitric Acid . . .	1,218	Oil of Turpentine . .	870
Sea-Water . . .	1,026	Liquid Bitumen . .	848
Milk	1,030	Alcohol, absolute . .	792
Distilled Water . .	1,000	Sulphuric Ether . .	716
Wine of Bordeaux .	994	Air at the Earth's Surface, about	$1\frac{2}{9}$

*** Since a cubic foot of water at the temperature 40° Fahrenheit, weighs 1000 ounces avoirdupois, or $62\frac{1}{2}$ pounds, $= \frac{5}{9}$ cwt. nearly, the numbers in the preceding Tables exhibit very nearly the respective weights of a cubic foot of the several substances tabulated.

243. Prop. To find the magnitude of any body, from its weight.

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or, inches, respectively.

Exam. 1. Required the content of an irregular block of green marble, spec. grav. 2742, which weighs 1 cwt. or 112lb? Ans. 1129·3 cubic inches.

Exam. 2. How many cubic inches of gunpowder, spec. grav. 937, are there in 1lb. weight? Ans. $29\frac{1}{2}$ cubic inches nearly.

Exam. 3. How many cubic feet are there in a ton weight of dry oak? Spec. grav. 925. Ans. $38\frac{138}{185}$ cubic feet.

244. Prop. To find the weight of a body from its magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is the tabular specific gravity,
To the weight of the body.

Ex. 1. Required the weight of a block of marble, spec. grav. 2700, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans. $683\frac{4}{10}$ ton, which is nearly equal to the burden of an East-India ship.

Ex. 2. What is the weight of 1 pint, ale measure, of gunpowder?

Ans. 19 oz. nearly.

Ex. 3. What is the weight of a block of dry oak, spec. grav. 925, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep or thick?

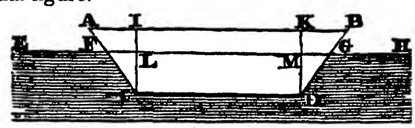
Ans. $4335\frac{15}{16}$lb.

BUOYANCY OF PONTOONS.

GENERAL SCHOLIUM.

245. The principles established in art. 238 have an interesting application to military men, in the use of pontoons, and the buoyancy by which they become serviceable in the construction of temporary bridges. When the dimensions, magnitude, and weight of a pontoon are known, that weight can readily be deducted from the weight of an equal bulk of water, and the remainder is evidently the weight which the pontoon will carry before it will sink.

Pontoons as usually constructed, are prisms whose vertical sections are equal trapezoids, as exhibited in the marginal figure.



Suppose $\text{AB} = \text{L}$

$\text{CD} = l$

$\text{AI} = \text{KB} = \frac{1}{2}(\text{L} - l) = \delta$

$\text{CI} = \text{D}.$

Uniform width of the pontoon $= b$: all in feet and parts. Suppose also $\text{CL} = d$, depth of the part immersed; $w =$ weight in avoirdupois pounds of the water displaced; and $c = 62\frac{1}{2}$lbs. weight of a cubic foot of rain water. Then, by the following expressions, which are left for the student to investigate, d may be found when w and the rest are given, and w may be found when d and the rest are given; also the maximum value of w.

1. $w = bcd\left(l + \frac{d\delta}{\text{D}}\right)$

2. w when a max. $= bc\text{D}\,(l + \delta) = \frac{1}{2}bc\text{D}\,(\text{L} + l)$

3. $d = \sqrt{\left[\frac{\text{D}}{\delta}\left(\frac{w}{bc} + \frac{l^2\text{D}}{4\delta}\right)\right]} - \frac{l\text{D}}{2\delta}.$

Ex. 1. Given $\text{AB} = 21\frac{1}{2}$ feet, $\text{CD} = 17\frac{1}{8}$ feet, $\text{CI} = 2\frac{1}{4}$ feet, $b = 4\frac{3}{4}$ feet. Required the weight of the pontoon and its load, when it is immersed to the depth CL of $1\frac{1}{2}$ feet. Ans. $8287\frac{5}{8}$lbs. nearly.

Ex. 2. Suppose the weight of such a pontoon to be 900lbs. what is the greatest weight it will carry? Ans. $12014\frac{1}{16}$lbs.

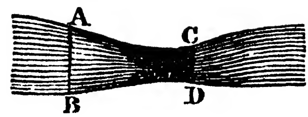
Ex. 3. Suppose the weight of the above pontoon and its load to be 6000lbs. how deep will it sink in water? Ans. 1·1084 *f.* = 13·3 inches.

HYDRAULICS OR HYDRODYNAMICS.

246. Hydraulics or Hydrodynamics is that part of mechanical science which relates to the motion of fluids, and the forces with which they act upon bodies against which they strike, or which move in them.

This is a very extensive subject; but we shall here give only a few elementary propositions.

247. Prop. If a fluid run through a canal or river, or pipe of various widths, always filling it; the velocity of the fluid in different parts of it, AB, CD, will be reciprocally as the transverse sections in those parts.



That is, veloc. at A : veloc. at C :: CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or sections there.

For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the same time; that is, the column through AB is equal to the column through CD, in the same time; or AB $\times$ length of its column $=$ CD $\times$ length of its column; therefore AB : CD :: length of column through CD : length of column through AB. But the uniform velocity of the water, is as the space run over, or length of the columns; therefore AB : CD :: velocity through CD : velocity through AB.

248. *Corol.* Hence, by observing the velocity at any place AB, the quantity of water discharged in a second, or any other time, will be found, namely, by multiplying the section AB by the velocity there.

But if the channel be not a close pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the section will not be the same, because the velocity towards the bottom and sides will be diminished by the friction against the bed or channel; and therefore a medium among the three ought to be taken. So if the

velocity at the top be . .	82 feet per minute,
that at the bottom . . .	68
and that at the sides . .	60
	3)210 sum;

dividing their sum by 3, gives 70 for the mean velocity, which is to be multiplied by the section, to give the quantity discharged in a minute: and in many cases still greater accuracy will be necessary in determining the mean.

249. Prop. The velocity with which a fluid runs out by a hole in the bottom or side of a vessel, is equal to that which is generated by gravity through the height of the water above the hole; that is, the velocity of a heavy body acquired by falling freely through the height AB.

The momenta, or quantities of motion, generated in two given bodies, by the same force, acting during the same or an equal time, are equal. And the force in this case, is the weight of the superincumbent column of the fluid over the hole. Let then the one body to be moved, be that column itself, expressed by ah, where a denotes the altitude AB, and h the area of the hole; and the other body is the column of the fluid that runs out uniformly in one second suppose, with the middle or medium velocity of that interval of time, which is $\frac{1}{2}hv$, if v be the whole velocity required. Then the mass $\frac{1}{2}hv$, with the velocity v, gives the quantity of motion $\frac{1}{2}hv \times v$, or $\frac{1}{2}hv^2$, generated in one second, in the spouting water: also g, or $32\frac{1}{6}$ feet, is the velocity generated in the mass ah, during the same interval of one second; consequently $ah \times g$, or ahg, is the motion generated in the column ah in the same time of one second. But as these two momenta must be equal, this gives $\frac{1}{2}hv^2 = ahg$: hence then $v^2 = 2ag$, and $v = \sqrt{2ag}$, for the value of the velocity sought: which therefore is exactly the same as the velocity generated by the gravity falling through the space a, or the whole height of the fluid.

250. For example, if the fluid were air, of the whole height of the atmosphere, supposed uniform, which is about $5\frac{1}{4}$ miles, or 27720 feet $= a$. Then $\sqrt{2ag} = 2\sqrt{(27720 \times 16\frac{1}{12})} = 1335$ feet $= v$ the velocity, that is, the velocity with which common air would rush into a vacuum.

251. *Corol.* 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{AB}$.

252. *Corol.* 2. The fluid spouts out with the same velocity, whether it be downward or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if an adjutage be turned upward, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the air and from the oblique motion of the fluid in the hole.

253. *Corol.* 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if a denote the altitude of the fluid,
and h the area of the orifice,
also $\frac{1}{2}g = 16\frac{1}{12}$ feet, or 193 inches;
then $2h\sqrt{\frac{1}{2}ag}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{48}h\sqrt{a}$ cubic feet, when a and h are taken in feet.

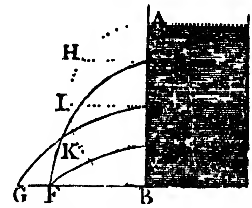
So, for example, if the height a be 25 inches, and the orifice $h = 1$ square inch; then $2h\sqrt{\frac{1}{2}ag} = 2\sqrt{25 \times 193} = 139$ cubic inches, which is the quantity that would be discharged per second.

SCHOLIUM.

254. When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

255. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion: but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from which it happens, that it is the particles in the centre of the hole only that issue out with the whole velocity due to the entire height of the fluid, while the other particles towards the sides of the orifices pass out with decreased velocities; and hence the medium velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments on the quantity of water discharged through apertures, show that the quantity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity nearly equal to that in a body falling through $\frac{1}{2}$ the height of the fluid above the orifice. If the velocity be taken as that due to the whole altitude above the orifice, then instead of the area of the orifice, the area of the contracted vein at a small distance from it must be taken. See *Gregory's Mechanics* and *Bossut's Hydrodynamique.*

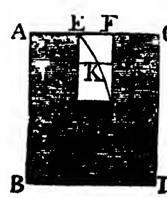
256. Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet will spout on the horizontal plane BG, will be as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the spaces BF, BG, are as the times and horizontal velocities; but the velocity is as $\sqrt{AC}$; and the time of the fall, which is the same as the time of moving, is as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{AC \cdot CB}$; and the distance BG as $\sqrt{AD \cdot DB}$. And hence, if two holes are made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB : which makes BF the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . CB, &c.; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows, that the projection from the middle point D will be farthest, for DI is the greatest ordinate.



These are the *proportions* of the distances; but for the absolute distances, it will be thus. The velocity through any hole C, is such as will carry the water horizontally through a space equal to 2AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC \cdot CB} = 2CH =$ BF, the space ranged on the horizontal plane. And the greatest range BG = 2DI, or 2AD, or equal to AB.

And as these ranges answer very nearly to the experiments, this confirms the theory, as to the velocity assigned.

257. Prop. If a notch or slit EH in form of a parallelogram, be cut in the side of a vessel, full of water, AD; the quantity of water flowing through it will be ⅔ of the quantity flowing through an equal orifice, placed at the whole depth EG, or at the base GH, in the same time; it being supposed that the vessel is always kept full.



For the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK, to the sum of all the IL's; namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE, to EGHF, or as 2 to 3; the area of a parabola being ⅔ of its circumscribing parallelogram.

Corol. 1. The mean velocity of the water in the notch, is equal to ⅔ of that

Corol. 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. This appears from the demonstration.

ON PNEUMATICS.

258. PNEUMATICS is the science which treats of the properties of air, or elastic fluids.

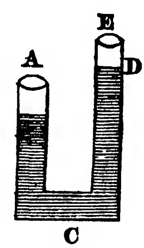
259. PROP. Air is a fluid body; which surrounds the earth, and gravitates on all parts of its surface.

These properties of air are proved by experience.—That it is a fluid, is evident from its easily yielding to any the least force impressed on it, without making a sensible resistance.

But when it is moved briskly, by any means, as by a fan or a pair of bellows; or when any body is moved very briskly through it; in these cases we become sensible of it as a body, by the resistance it makes in such motions, and also by its impelling or blowing away any light substances. So that, being capable of resisting, or moving other bodies, by its impulse, it must itself be a body, and be heavy, like all other bodies, in proportion to the matter it contains; and therefore it will press on all bodies that are placed under it.

Also, as it is a fluid, it spreads itself all over on the earth; and, like other fluids, it gravitates and presses every where on the earth's surface.

260. The gravity and pressure of the air are also evident from many experiments. Thus, for instance, if water, or quicksilver, be poured into the tube ACE, and the air be suffered to press on it, in both ends of the tube, the fluid will rest at the same height in both legs. but if the air be drawn out of one end as E, by any means; then the air pressing on the other end A, will press down the fluid in this leg at B, and raise it up in the other to D, as much higher than at B, as the pressure of the air is equal to. From which it appears, not only that the air does really press, but also how much the intensity of that pressure is equal to. And this is the principle of the barometer.

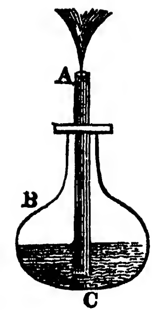


261. PROP. The air is also an elastic fluid, being condensible and expansible: and the law it observes is this, that its density and elasticity are proportional to the force or weight which compresses it.

This property of the air is proved by many experiments. Thus, if the handle of a syringe be pushed inward, it will condense the inclosed air into less space, thereby showing its condensibility. But the included air, thus condensed, is felt to act strongly against the hand, resisting the force compressing it more and more; and, on withdrawing the hand, the handle is pushed back again to where it was at first. Which shows that the air is elastic.

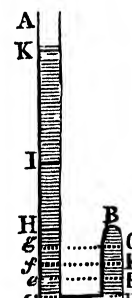
262. Again, fill a strong bottle half full of water; then insert a small glass tube into it, putting its lower end down near to the bottom, and cementing it very close round the mouth of the bottle. Then, if air be strongly injected

through the pipe, as by blowing with the mouth or otherwise, it will pass through the water from the lower end, ascending into the parts before occupied with air at B, and the whole mass of air become there condensed, because the water is not compressible into a less space. But, on removing the force which injected the air at A, the water will begin to rise from thence in a jet, being pushed up the pipe by the increased elasticity of the air at B, by which it presses on the surface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the same density as at first, and, the balance being restored, the jet will cease.



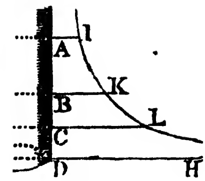
263. Likewise, if into a jar of water AB, be inverted an empty glass tumbler CD, or such-like, the mouth downward; the water will enter it, and partly fill it, but not near so high as the water in the jar, compressing and condensing the air into a less space in the upper parts C, and causing the glass to make a sensible resistance to the hand in pushing it down. Then, on removing the hand, the elasticity of the internal condensed air throws the glass up again. All these showing that the air is condensible and elastic.

264. Again, to show the relation of the elasticity to the condensation: take a long crooked glass tube, equally wide throughout, or at least in the part BD, and open at A, but close at the other end B. Pour in a little quicksilver at A, just to cover the bottom to the bend at CD, and to stop the communication between the external air and the air in BD. Then pour in more quicksilver, and mark the corresponding heights at which it stands in the two legs: so, when it rises to H in the open leg AC, let it rise to E in the close one, reducing its included air from the natural bulk BD to the contracted space BE, by the pressure of the column He; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weights of the columns If, Kg. Then it is always found, within moderate limits, that the condensations and elasticities are as the compressing weights and columns of the quicksilver, and the atmosphere together. So, if the natural bulk of the air BD be compressed into the spaces BE, BF, BG, which are $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$ of BD, or as the numbers 3, 2, 1; then the atmosphere, together with the corresponding columns He, If, Kg, are also found to be in the same proportion reciprocally, viz. as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, or as the numbers 2, 3, 6. And then He = $\frac{1}{3}$A, If = A, and Kg = 3A; where A is the weight of the atmosphere. Which show that the condensations are directly as the compressing forces. And the elasticities are in the same ratio, since the columns in AC are sustained by the elasticities in BD.



From the foregoing principles may be deduced many useful remarks, as in the following corollaries, viz.

265. *Corol.* 1. The space in which any quantity of air is confined, is reciprocally as the force that compresses it. So, the forces which confine a quantity



of air in the cylindrical spaces AG, BG, CG, are reciprocally as the same, or reciprocally as the heights AD, BD, CD. And therefore if to the two perpendicular lines DA, DH, as asymptotes, the hyperbola IKL be described, and the ordinates AI, BK, CL be drawn; then

the forces which confine the air in the spaces AG, BG, CG,
will be directly as the corresponding ordinates AI, BK, CL,
since these are reciprocally as the abscisses AD, BD, CD,
by the nature of the hyperbola.

Corol. 2. All the air near the earth is in a state of compression, by the weight of the incumbent atmosphere.

Corol. 3. The air is denser near the earth, than in high places; or denser at the foot of a mountain, than at the top of it. And the higher above the earth the less dense it is.

Corol. 4. The spring or elasticity of the air, is equal to the weight of the atmosphere above it; and they will produce the same effects: since they always sustain and balance each other.

Corol. 5. If the density of the air be increased, preserving the same heat or temperature, its spring or elasticity is also increased, and in the same proportion.

Corol. 6. By the pressure and gravity of the atmosphere, on the surface of fluids, the fluids are made to rise in any pipes or vessels, when the spring or pressure within is decreased or taken off.

266. Prop. Heat increases the elasticity of the air, and cold diminishes it. Or, heat expands, and cold condenses the air.

This property is also proved by experience

Thus, tie a bladder very close with some air in it; and lay it before the fire: then as it warms it will more and more distend the bladder, and at last burst it, if the heat be continued, and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was on this principle that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire beneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of the balloon.

Also, if a cup or glass, with a little air in it, be inverted into a vessel of water; and the whole be heated over the fire, or otherwise; the air in the top will expand till it fill the glass, and expel the water out of it; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments, to the same effect, might be adduced, all proving the properties mentioned in the proposition.

SCHOLIUM.

267. So that, when the force of the elasticity of air is considered, regard must be had to its heat or temperature; the same quantity of air being more or less elastic, as its heat is more or less. And it has been found, by experiment, that the elasticity is increased by the 435th part, for each degree of heat, of which there are 180, between the freezing and boiling heat of water, in Fahrenheit's thermometer.

N.B.—Water expands about the $\frac{2}{20000}$ part, with each degree of heat. (Sir Geo. Shuckburgh, *Philos. Trans.* 1777, p. 560, &c.)

Also, the

Spec. grav. of air	1·201 or $1\frac{1}{5}$	when the barom. is 29·5
water	1000	and the therm. is 55°
mercury	13592	which are their mean heights in this country.
Or thus, air	1·222 or $1\frac{2}{9}$	when the barom. is 30,
water	1000	and thermometer 55°.
mercury	13600	

268. Prop. The weight or pressure of the atmosphere, on any base at the earth's surface, is equal to the weight of a column of quicksilver, of the same base, and the height of which is between 28 and 31 inches.

This is proved by the barometer, an instrument which measures the pressure of the air, and which is described below (art. 289). For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches : but at other times it balances a column of 29, or 30, or near 31 inches high; seldom in the extremes 28 or 31, but commonly about the means 29 or 30. This variation depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost continually happening in any one place. But the medium state is commonly about $29\frac{1}{2}$ or 30 inches.

269. *Corol.* 1. Hence the pressure of the atmosphere on every square inch at the earth's surface, at a medium, is very near 15 pounds avoirdupois, or rather $14\frac{3}{4}$ pounds. For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almost 8 ounces, or nearly half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

270. *Corol.* 2. Hence also the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high. For, water and quicksilver are in weight nearly as 1 to 13·6; so that the atmosphere will balance a column of water 13·6 times as high as one of quicksilver; consequently

13·6 times 28 inches = 381 inches, or $31\frac{3}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

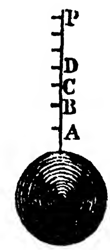
And hence a common sucking pump (art. 279) will not raise water higher than about 33 or 34 feet. And a siphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet (art. 278).

271. *Corol.* 3. If the air were of the same uniform density at every height up to the top of the atmosphere, as at the surface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium. For, the weights of the same bulk of air and water, are nearly as 1·222 to 1000; therefore as 1·222 : 1000 :: $33\frac{3}{4}$ feet : 27600 feet, or $5\frac{1}{4}$ miles nearly. And so high the atmosphere would be, if it were *homogeneous*, or all of uniform density, like water. But, instead of that, from its expansive and elastic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion, which will be treated of below, as also the method of measuring heights by the barometer, which depends on it.

272. *Corol* 4. From this proposition and the last it follows, that the height s always the same, of a *homogeneous atmosphere* above any place, which shall be

all of the uniform density with the air there, and of equal weight or pressure with the real height of the atmosphere above that place, whether it be at the same place, at different times, or at any different places or heights above the earth; and that height is always about $5\frac{1}{4}$ miles, or 27600 feet, as above found. For, as the density varies in exact proportion to the weight of the column, therefore it requires a column of the same height in all cases, to make the respective weights or pressures. Thus, if W and w be the weights of atmosphere above any places, D and d their densities, and H and h the heights of the uniform columns, of the same densities and weights; then $H \times D = W$, and $h \times d = w$; therefore $\frac{W}{D}$ or H is equal to $\frac{w}{d}$ or h: the temperature being the same.

273. Prop. With regard to the atmosphere, at different heights above the earth, this law obtains that when the heights increase in arithmetical progression, the densities decrease in geometrical progression.



Let the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts, A, B, C, D, &c. forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata A, B, C, D, &c. will be in geometrical progression decreasing.

For, as the strata A, B, C, &c. are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C; that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is proportional to itself, leaves the next density. But when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals: consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its nth part; there will remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the series of densities will be D, $\frac{m}{n}$ D, $\frac{m^2}{n^2}$ D, $\frac{m^3}{n^3}$ D, $\frac{m^4}{n^4}$ D, &c. the common ratio of the series being that of n to m.

SCHOLIUM.

274. Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series: therefore different altitudes above the earth's surface, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A,

and d . the density at the altitude a;

then A being as the log. of D, and a as the log. of d,

the dif. of alt. $A - a$ will be as the log. D $-$ log. d, or log. $\frac{D}{d}$.

And if $A = 0$, or D the density at the surface of the earth; then any altitude above the surface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer, which (art. 289) is an instrument that measures the pressure or density of the air at any place. For, by taking, with this instrument, the pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

275. But as this formula expresses only the relations between different altitudes with respect to their densities, recourse must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; assume h so that $a = h \times \log. \frac{D}{d}$, where h will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude a corresponding to a known density d; as for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log. \frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \log. \frac{n}{n-1}$. But $\log. \frac{n}{n-1} = M \left(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \&c\right)$ $= \frac{M}{n}$ nearly, when n is a large number. Therefore, since $M = {\cdot}43429448$, &c. we have 1 nearly $= h \times \frac{.43429448}{27600} = \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log. \frac{D}{d}$, or $= 63551 \times \log. \frac{M}{m}$ feet, or $10592 \times \log. \frac{M}{m}$ fathoms: where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

276. Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which must be added, when that medium exceeds 55°, otherwise subtracted.

Note, also, that a column of 30 inches of mercury varies its length by about the $\frac{1}{320}$ part of an inch for every degree of heat, or rather $\frac{1}{9600}$ of the whole volume.

277. But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°; thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the corresponding change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

278. Taking, instead of the logarithms, the first term of the logarithmic series we have $55000 . \frac{B - b}{B + b}$, for the altitude in feet: B and b, being the heights of the barometrical columns observed at the bottom and top of the hill. This formula is for the mean temperature 55°, and is easily remembered because the effective figures of the co-efficient are also 55. The reductions for any other temperature are the same as in the logarithmic rule.

Ex. 1. To find the height of a hill when the pressure of the atmosphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top; the mean temperature being 50°? Ans. 4352·4 feet, or 725·4 fathoms.

Ex. 2. To find the height of a hill when the atmosphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°? Ans. 406·28 fathoms.

Ex. 3. At what altitude is the density of the atmosphere only the 4th part of what it is at the earth's surface? As. 6020 fathoms.

By the weight and pressure of the atmosphere, the effect and operations of pneumatic engines may be accounted for, and explained; such as siphons, pumps, barometers, &c.; of which it will be proper here to give a brief description.

OF THE SIPHON.

279. A Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.

If it be filled with water, and then inverted, with the two open ends downward, and held level in that position; the water will remain suspended in it, if the two legs be equal. For the atmosphere will press equally on the surface of the water in each end, and support them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact counterpoise by their equal weights; so that the one has no power to move more than the other; and they are both supported by the atmosphere.

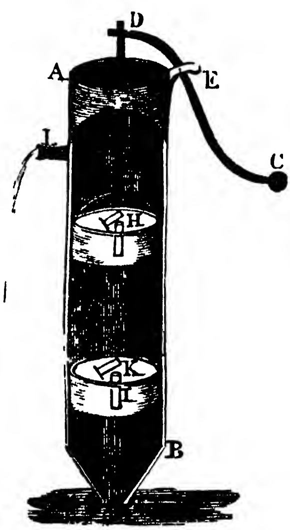
But if now the siphon be a little inclined to one side, so that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the same thing; then the equilibrium is destroyed, and the water will all descend out by the lower end, and rise up in the higher. For, the air pressing equally, but the two ends weighing unequally, a motion must commence where

the power is greatest, and so continue till all the water has run out by the lower end. And if the shorter leg be immersed into a vessel of water, and the siphon be set running as above, it will continue to run till all the water be exhausted from the vessel, or at least as low as that end of the siphon. Or, it may be set running without filling the siphon as above, by only inverting it, with its shorter leg into the vessel of water; then, with the mouth applied to the lower orifice A, suck out the air; and the water will presently follow, being forced up into the siphon by the pressure of the air on the water in the vessel.

If a siphon be fixed in a vessel of water capable of rotation upon a vertical axis, and the orifice be lateral instead of at the bottom of the pipe, the reaction may be advantageously employed as a motive force. This is the principle of Mr Busby's *Hydraulic Orrery*.

OF THE PUMP.

280. There are three sorts of pumps: the Sucking, the Lifting, and the Forcing Pump. By the first, water can be raised only to about 33 feet, viz. by the pressure of the atmosphere; but by the others, to any height; but then they require more apparatus and power.



The annexed figure represents a common sucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and smooth within, or of wood for very common purposes. CD is the handle, moveable about the pin E, by moving the end C up and down. DT an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piston, bucket, or sucker, TG, by which this is moved up and down within the barrel, which it must fit very tight and close, that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, also perforated, and covered by a valve K opening upwards.

281. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, then the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air above it, the external atmosphere by its pressure keeping the valve H shut: the air in the barrel being thus exhausted, or rarefied, is no longer a counterpoise to that which presses on the surface of the water in the well; this

is forced up the pipe, and through the valve K, into the barrel of the pump. Then pushing the piston down again into this water, now in the barrel, its weight shuts the lower valve K, and its resistance forces up the valve of the piston, and enters the upper part of the barrel, above the piston. Then, the bucket being raised, lifts up with it the water which had passed above its valve, and it runs out by the cock L; and taking off the weight below it, the pressure of the external atmosphere on the water in the well again forces it up through the pipe and lower valve close to the piston, all the way as it ascends, thus keeping the barrel always full of water. And thus, by repeating the strokes of the piston, a continued discharge is made at the cock L.

282. There is a farther limitation of the operation, than that which relates to the 33 feet. If the elastic force of the air within the tube joined to the weight of water in the tube equal the pressure of the atmosphere, the water cannot rise in the pump. To prevent this, the product of the stroke of the piston into 33 must always exceed the square of half the greatest altitude of the piston above the surface of the water in the well. Otherwise diminish the diameter of the sucking-pipe proportionally.

OF THE AIR-PUMP.

283. Nearly on the same principles as the water-pump, is the invention of the air pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by the sides of it, and furnished with a proper valve opening upward. Then, by lifting up the piston, the air in the close vessel below it follows the piston, and fills the barrel; and being thus diffused through a larger space than before, when it occupied the vessel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together exceeds the receiver alone. Another stroke of the piston exhausts another barrel of this now rarer air, which again rarefies it in the same proportion as before. And so on, for any number of strokes of the piston, still exhausting in the same geometrical progression, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhausted to any proposed degree, or as far as the nature of the machine is capable of performing; which happens when the elasticity of the included air is so far diminished, by rarefying, that it is too feeble to push up the valve of the piston, and escape.

284. From the nature of this exhausting, in geometrical progression, we may easily find how much the air in the receiver is rarefied by any number of strokes of the piston; or what number of such strokes is necessary, to exhaust the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone, as c to r, and 1 denote the natural density of the air at first; then

$$c : r :: 1 : \frac{r}{c}, \text{ the density after 1 stroke of the piston,}$$

$$c : r :: \frac{r}{c} : \frac{r^2}{c^2}, \text{ the density after 2 strokes,}$$

$$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}, \text{ the density after 3 strokes,}$$

$$\&\text{c., and } \frac{r^n}{c^n}, \text{ the density after } n \text{ strokes.}$$

So if the barrel be equal to $\frac{1}{4}$ of the receiver; then $c : r :: 5 : 4$; and $\frac{4^n}{5^n} = 0{\cdot}8^n$ is $= d$ the density after n turns. And if n be 20, then $0{\cdot}8^{20} = {\cdot}0115$ is the density of the included air after 20 strokes of the piston; which being the $86\frac{7}{10}$ part of 1, or the first density, it follows that the air is $86\frac{7}{10}$ times rarified by the 20 strokes.

285. Or, if it were required to find the number of strokes necessary to rarefy the air any number of times; because $\frac{r^n}{c^n}$ is $=$ the proposed density d; therefore, taking the logarithms, $n \times \log. \frac{r}{c} = \log. d$, and $n = \frac{\log. d}{\text{l. } r - \text{l. } c}$, the number of strokes required. So if r be $\frac{4}{5}$ of c, and it be required to rarefy the air 100 times; then $d = \frac{1}{100}$ or $\cdot 01$; and hence $n = \frac{\log. 100}{\text{l. } 5 - \text{l. } 4} = 20\frac{2}{3}$ nearly. So that in $20\frac{2}{3}$ strokes the air will be rarefied 100 times.

OF THE DIVING BELL AND CONDENSING MACHINE.

286. On the same principles, too, depend the operations and effect of the Condensing Engine, by which air may be condensed to any degree, instead of rarefied as in the air-pump. And, like as the air-pump rarefies the air, by extracting always one barrel of air after another; so, by this other machine, the air is condensed by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner; so that, as they both open upward or outward in the air-pump, or rarefier, they will both open downward or inward in the condenser.

287. And on the same principles, namely, of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downward; only in this case the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell, by increasing always the force which compresses it.

288. If a vessel of any sort be inverted into water, and pushed or let down to any depth; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less

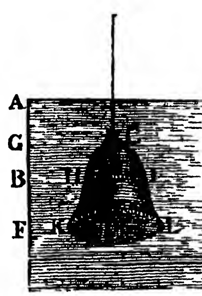
space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion, as its space in the vessel is diminished.

So, if the tube CE be inverted, and pushed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the space CD; then that air is pressed both by a column of water of the height AB, and by the whole atmosphere which presses on the upper surface of the water; consequently the space CD is to the whole space CE, as the weight of the atmosphere is to the weights both of the atmosphere and the column of water AB. So that, if AB be about 34 feet, which is equal to the force of the atmosphere, then CD will be equal to $\frac{1}{2}$CE; but if AB be double of that, or 68 feet, then CD will be $\frac{1}{3}$CE; and so on. And hence, by knowing the depth AF, to which the vessel is sunk, we can easily find the point D, to which the water will rise within it at any time. For let the weight of the atmosphere at that time be equal to that of 34 feet of water; also, let the depth AF be 20 feet, and the length of the tube CE 4 feet: then, putting the height of the internal water DE $= x$,

it is $34 + AB : 34 :: CE : CD$,
that is $34 + AF - DE : 34 :: CE : CE - DE$,
or $54 - x : 34 :: 4 : 4 - x$;

hence, multiplying extremes and means, $216 - 58x + x^2 = 136$, and the root is $x = \sqrt{2}$ very nearly $= 1{\cdot}414$ of a foot, or 17 inches nearly; being the height DE to which the water will rise within the tube.

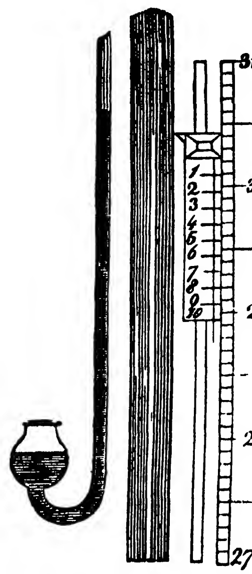
289. But if the vessel be not equally wide throughout, but of any other shape, as of a bell-like form, such as is used in diving; then the altitudes will not observe the proportion above, but the spaces or bulks only will accord with that proportion, namely $34 +$ AB : 34 :: capacity CKL : capacity CHI, if it be common or fresh-water; and $33 +$ AB : 33 :: capacity CKL : capacity CHI, if it be sea-water. From which proportion the height DE may be found, when the nature and shape of the vessel or bell CKL are known.



OF THE BAROMETER.

290. The Barometer is an instrument for measuring the pressure of the atmosphere and elasticity of the air, at any time. It is commonly made of a glass tube, of near 3 feet long, close at one end, and filled with mercury. When the tube is full, by stopping the open end with the finger, then inverting the tube, and immersing that end with the finger into a bason of quicksilver, on removing the finger from the orifice, the fluid in the tube will descend into the bason, till what remains in the tube be of the same weight with a column of the

atmosphere, which is commonly between 28 and 31 inches of quicksilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no pressure downwards but from the column of quicksilver, and therefore that will be an exact balance to the counter-pressure of the whole column of atmosphere, acting on the orifice of the tube by the quicksilver in the bason. The upper 3 inches of the tube, namely, from 28 to 31 inches, have a scale attached to them, divided into inches, tenths, and hundredths, for measuring the length of the column at all times, by observing which division of the scale the top of the quicksilver is opposite to; as it ascends and descends within these limits, according to the state of the atmosphere.



The weight of the quicksilver in the tube, above that in the bason, is at all times equal to the weight or pressure of the column of atmosphere above it, and of the same base with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound avoirdupois for every inch of quicksilver in the tube, on every square inch of base; or more exactly it is $\frac{59}{120}$ of a pound on the square inch, for every inch in the altitude of the quicksilver weighs just $\frac{59}{120}$lb., or nearly $\frac{1}{2}$ a pound, in the mean temperature of 55° of heat. And consequently, when the barometer stands at 30 inches, or $2\frac{1}{2}$ feet high, which is nearly the medium or standard height, the whole pressure of the atmosphere is equal to $14\frac{3}{4}$ pounds, on every square inch of the base: and so in proportion for other heights.

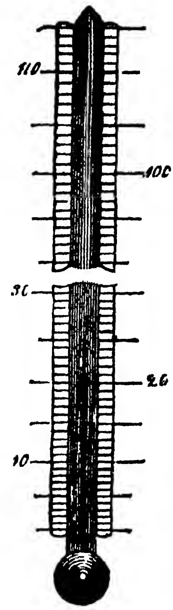
Barometers are now constructed so as to be susceptible of convenient motion from place to place without derangement; thus facilitating the pneumatic method of determining the heights of hills, &c.

OF THE THERMOMETER.

291. The Thermometer is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold: and since the expansion is, to a certain extent, uniform, the degrees of expansion become the measures of the degrees of heat. Fluids are more convenient for this purpose than solids: and quicksilver or mercury is now most commonly used for it. A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain

point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands, and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of the quicksilver stands. The method of preparing the scale, as used in England, is thus:—Bring the thermometer into the temperature of freezing, by immersing the ball in water just freezing, or in ice just thawing, the latter is best, and mark the scale where the mercury then stands, for the point of freezing. Next immerge it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale, for the boiling point, or the heat of boiling water. Then the distance between these two points is divided into 180 equal divisions, or degrees; and the like equal degrees are also continued to any extent below the freezing point, and above the boiling point. The divisions are then numbered as follows, namely, at the freezing point is set the number 32, and consequently 212 at the boiling point; and all the other numbers in their order.



This division of the scale is commonly called *Fahrenheit's*. According to this division, 55° is at the mean temperature of the air in this country; and it is in this temperature, and in an atmosphere which sustains a column of 30 inches of quicksilver in the barometer, that all measures and specific gravities are taken, unless when otherwise mentioned; and in this temperature and pressure, the relative weights, or specific gravities of air, water, and quicksilver, are as

$1\frac{2}{9}$ for air,
1000 for water,
13600 for mercury; } these also are the weights of a cubic foot of each, in avoirdupois ounces, in that state of the barometer and thermometer. For other states of the thermometer, each of these bodies expands or contracts according to the following rate, with each degree of heat, viz.

Air about - $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{6666}$ part of its bulk,
Mercury about $\frac{1}{9600}$ part of its bulk.

Another division is that of 100 equal degrees between the freezing and the boiling points, the 0 or *zero* being at the former. This is called the *centigrade* thermometer. It is now very common to put Fahrenheit's division on the left of the tube, and the centigrade division on the right.

ON THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

292. From the principles laid down in arts. 273 to 277, concerning the measuring of altitudes by the barometer, and the foregoing descriptions of the barometer and thermometer, we may now collect together the precepts for the practice of such measurements, which are as follow:

First. Observe the height of the barometer at the bottom of any height or depth intended to be measured; with the temperature of the quicksilver, by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detached thermometer.

Secondly. Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other, that is, augment the height of the mercury in the colder temperature, or diminish that in the warmer, by its $\frac{1}{9600}$ part for every degree of difference of the two.

Thirdly. Take the difference of the common logarithms of the two heights of the barometer, corrected as above if necessary, cutting off 3 figures next the right hand for decimals, when the log. tables go to 7 figures, or cut off only 2 figures when the tables go to 6 places, and so on; or in general remove the decimal point 4 places more towards the right hand, those on the left hand being fathoms in whole numbers.

Fourthly. Correct the number last found for the difference of temperature of the air, as follows: take half the sum of the two temperatures, for the mean one; and for every degree which this differs from the temperature 31°, take so many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but subtract them if the mean temperature be below 31°; and the sum or difference will be the true altitude in fathoms; or, being multiplied by 6, will be the altitude in feet.

Ex. 1. Let the state of the barometers and thermometers be as follows; to find the altitude, viz.

Barom.	Thermom. attach.	Thermom. detach.	
Lower 29·68	57	57	Ans. the alt is 720 fathoms.
Upper 25·28	43	42	

Ex. 2. To find the altitude, when the state of the barometers and thermometers is as follows, viz.

Barom.	Thermom. attach.	Thermom. detach.	
Lower 29·45	38	31	Ans. the alt. is $409\frac{8}{13}$ fathoms, or 2458 feet.
Upper 26·82	41	35	

This is a highly useful method within certain limits; but is by no means susceptible of that degree of accuracy which many have imputed to it.

A very useful table, computed by Mr. Simms from Mr. F. Baily's formula, is inserted in his useful book on Instruments used in Surveying and Levelling, also in White's Ephemeris for 1837.

EXERCISES ON HYDROSTATIC PRESSURE, &c.

1. Required the pressure of rain or river water, on a square inch, at the respective depths of 30 feet, 30 yards, 300 feet, and 300 yards.

2. If the density of mercury be 13·6 times that of rain water, required its pressure on a square inch at the depth of a foot.

3. A sluice-gate of 10 feet square is placed vertically in water, so that its top just coincides with the upper surface of the liquid; required the hydrostatic pressure on the upper and lower halves of that gate.

4. Required the respective pressures on the two triangular sections of the same gate, found by a line drawn diagonally from top to bottom.

5. Required the respective pressures on each foot in depth of the same gate from top to bottom.

6. If the same gate, instead of standing vertically, be placed at an angle of 30° with the horizon, its top being 10 feet below the upper surface of the water, what will be the entire pressure it will then sustain?

7. Compare the pressures on the three faces of an equilateral triangular prism, just immersed in a fluid in such a manner that one face of the prism may be perpendicular to the surface of the fluid.

8. How deep will a cube of oak sink in common water; each side of the cube being 1 foot, spec. grav. = ·925? Ans. $11\frac{1}{10}$ inches.

9. How deep will a globe of oak sink in water, the diameter being 1 foot? Ans. 9·9867 inches.

10. If a cube of wood, floating in common water, have 3 inches of it dry above the water, and $4\frac{8}{103}$ inches dry when in sea-water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of? Ans. the wood is oak, and each side 40 inches.

11. An irregular piece of lead ore weighs in air 12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative densities, or specific gravities. Ans. as 145 to 132.

12. An irregular fragment of glass, in the scale, weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79: what then will their specific gravities turn out to be? Ans. glass to magnet as 3933 to 5202, or nearly as 10 to 13.

13. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that substituting part silver was only a proper perquisite; which being suspected, Archimedes was appointed to examine it; who, on putting it into a vessel of water, found it raised the fluid 8·2245 cubic inches: and having discovered that the inch of gold more critically weighed 10·36 ounces, and that of silver but 5·85 ounces, he found by calculation what part of the king's gold had been changed. And you are desired to repeat the process. Ans. 28 8 ounces.

14. Supposing the cubic inch of common glass weigh 1·4921 ounces troy, the same of sea-water ·59542, and of brandy ·5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3·84lb. out of water, and to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up? Ans. 14·1496 ounces.

15. Another person has half an anker of brandy, of the same specific gravity as in the last question; the wood of the cask suppose measures $\frac{1}{8}$ of a cubic

foot: it is proposed to assign what quantity of lead is just requisite to keep the cask and liquor under water? Ans. 89·743 ounces.

16. Suppose, by measurement, it be found that a man-of-war, with its ordnance, rigging, and appointments, sinks so deep as to displace 50000 cubic feet of fresh water, what is the whole weight of the vessel? Ans. $1395\frac{1}{10}$ tons.

17. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 30 inches: and also, what would be the height of a water barometer at the same time?

Ans. height of the air $28636\frac{4}{11}$ feet, or 5·4235 miles,
height of water 35 feet.

18. With what velocity would each of those three fluids, viz. quicksilver, water, and air, issue through a small orifice in the bottom of vessels, of the respective heights of 30 inches, 35 feet, and 5·5240 miles, estimating the pressure by the whole altitudes, and the air rushing into a vacuum?

Ans. the veloc. of quicksilver 12·681 feet.
the veloc. of water . 47·447
the veloc. of air . 1369·8

19. A very large vessel of 10 feet high (no matter what shape) being kept constantly full of water, by a large supplying cock at the top; if 9 small circular holes, each $\frac{1}{2}$ of an inch diameter, be opened in its perpendicular side at every foot of the depth: it is required to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by all of them in 10 minutes?

Ans. the distances are
$\sqrt{36}$ or 6·00000
$\sqrt{64}$ - 8·00000
$\sqrt{84}$ - 9·16515
$\sqrt{96}$ - 9·79796
$\sqrt{100}$ - 10·00000
$\sqrt{96}$ - 9·79796
$\sqrt{84}$ - 9·16515
$\sqrt{64}$ - 8·00000
$\sqrt{36}$ - 6·00000

and the quantity discharged in 10 min. 123·8849 ale gallons.

Note.—In this solution, the velocity of the water is supposed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice, and that it is the same in every part of the holes.

20. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet, what thickness must it be of, that it may just float in the air?

Ans. ·02688 of an inch.

21. If a spherical balloon of copper, of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{10}$ of the gravity of common air, what weight will just balance it, and prevent it from rising up into the atmosphere? Ans. 21273lb.

22. If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at $29\frac{1}{2}$ inches? Ans. 2·26545 inches.

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

The weight and dimensions of Balls and Shells might be found from the problems given under the head of specific gravity. But they may be found still easier by means of the experimental weight of a ball of a given size, from the known proportion of similar figures, namely, as the cubes of their diameters, or like linear dimensions.

PROBLEM I.

To find the weight of an iron ball from its diameter.

An iron ball of 4 inches diameter weighs 9lb. and the weights being as the cubes of the diameters, it will be, as 64 (which is the cube of 4) is to 9 its weight, so is the cube of the diameter of any other ball, to its weight. Or, take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or, take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight. Or, $d = \sqrt[3]{3w} + \frac{1}{3}\sqrt[3]{3w}$.

Ex. 1. The diameter of an iron shot being 6·7 inches, required its weight?
Ans. 42·294lb.

Ex. 2. What is the weight of an iron ball, whose diameter is 5·54 inches?
Ans. 24lb. nearly.

PROBLEM II.

To find the weight of a leaden ball.

A leaden ball of 1 inch diameter weighs $\frac{3}{14}$ of a pound; therefore as the cube of 1 is to $\frac{3}{14}$, or as 14 is to 3, so is the cube of the diameter of a leaden ball, to its weight. Or, take $\frac{3}{14}$ of the cube of the diameter, for the weight, nearly.

Ex. 1. Required the weight of a leaden ball of 6·6 inches diameter?
Ans. 61·606lb.

Ex. 2. What is the weight of a leaden ball of 5·30 inches diameter?
Ans. 32lb. nearly.

Ex. 3. How many shot, each $\frac{1}{40}$ of an inch diameter, may be made out of 10lb. of lead?
Ans. 2986667.

PROBLEM III.

To find the diameter of an iron ball.

Multiply the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

Ex. 1. Required the diameter of a 42lb. iron ball? Ans. 6·685 inches.

Ex. 2. What is the diameter of a 24lb. iron ball? Ans. 5·54 inches.

PROBLEM IV.

To find the diameter of a leaden ball.

Multiply the weight by 14, and divide the product by 3; then the cube root of the quotient will be the diameter.

Ex. 1. Required the diameter of a 64lb. leaden ball? Ans. 6·684 inches.

Ex 2. What is the diameter of an 8lb. leaden ball? Ans. 3·343 inches.

PROBLEM V.

To find the weight of an iron shell.

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the shell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

Ex. 1. The outside diameter of an iron shell being 12·8, and the inside diameter 9·1 inches; required its weight? Ans. 188·941lb.

Ex. 2. What is the weight of an iron shell, whose external and internal diameters are 9·8 and 7 inches? Ans. $84\frac{1}{8}$lb.

PROBLEM VI.

To find how much powder will fill a shell.

Divide the cube of the internal diameter, in inches, by 57·3, for the lbs. of powder *.

Ex. 1. How much powder will fill a shell whose internal diameter is 9·1 inches? Ans. $13\frac{2}{13}$lb. nearly.

Ex. 2. How much powder will fill a shell whose internal diameter is 7 inches? Ans. 6lb.

PROBLEM VII.

To find how much powder will fill a rectangular box.

Find the content of the box in inches, by multiplying the length, breadth, and depth altogether. Then divide by 30 for the pounds of powder.

Ex. 1. Required what quantity of powder will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches? Ans. 60lb.

Ex. 2. How much powder will fill a cubical box whose side is 12 inches? Ans. $57\frac{3}{5}$lb.

PROBLEM VIII.

To find how much powder will fill a cylinder.

Multiply the square of the diameter by the length, then divide by 38·2 for the pounds of powder.

Ex. 1. How much powder will the cylinder hold, whose diameter is 10 inches, and length 20 inches? Ans. $52\frac{1}{3}$lb. nearly.

Ex. 2. How much powder can be contained in the cylinder whose diameter is 4 inches, and length 12 inches? Ans. $5\frac{5}{191}$lb.

* This and the following are only approximative rules, founded upon the supposition that, at a medium, 30 cubic inches of gunpowder weigh a pound. Of 18 different kinds of gunpowder used in the Royal Laboratory, Woolwich, the weights vary from 58lb. 1oz. to 49lb. 13oz. per cubic foot, and the specific gravities, consequently, from 929 to 727. The specific gravity of French gunpowder usually lies between narrower limits; viz. those of 944 and 897.

PROBLEM IX.

To find the size of a shell to contain a given weight of powder.

Multiply the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

Ex. 1. What is the diameter of a shell that will hold $13\frac{1}{8}$lb. of powder?
Ans. 9·1 inches.

Ex. 2. What is the diameter of a shell to contain 6lb. of powder?
Ans. 7 inches.

PROBLEM X.

To find the size of a cubical box, to contain a given weight of powder.

Multiply the weight in pounds by 30, and the cube root of the product will be the size of the box in inches.

Ex. 1. Required the side of a cubical box, to hold 50lb of gunpowder?
Ans. 11·44 inches.

Ex. 2. Required the side of a cubical box, to hold 400lb. of gunpowder?
Ans. 22·89 inches.

PROBLEM XI.

To find what length of a cylinder will be filled by a given weight of gunpowder.

Multiply the weight in pounds by 38·2, and divide the product by the square of the diameter in inches for the length.

Ex. 1. What length of a 36-pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12lb. of gunpowder? Ans. 10·314 inches.

Ex. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20lb. of powder? Ans. $11\frac{15}{16}$ inches.

OF DISTANCES BY THE VELOCITY OF SOUND.

From various experiments recently made, with great care, by the editor of this volume, it has been found that sound flies through the air uniformly at the rate of about 1110 feet per second, when the air is quiescent, and at a medium temperature. At the temperature of freezing, or a little below, the velocity is 1100 feet; at the temperature of 75°, on Fahrenheit's thermometer, the velocity is about 1120. The approximate velocity under different temperatures may be found, by adding to 1100, *half a foot*, for every degree, on Fahrenheit's thermometer, above the freezing point. The mean velocity may be taken at 370 yards per second; or a mile in $4\frac{3}{4}$ seconds.

Hence, multiplying any time employed by sound in moving, by 370, will give the corresponding space in yards. Or, dividing any space in yards by 370, will give the time which sound will occupy in passing uniformly over that space.

If the wind blow briskly, as at the rate of from 20 to 60 feet per second, in the

direction in which the sound moves, the velocity of the sound will be proportionably augmented: if the direction of the wind is opposed to that of the sound, the difference of their velocities must be employed.

Note.—The time for the passage of sound in the interval between seeing the flash of a gun, or lightning, and hearing the report, may be observed by a watch, or a small pendulum. Or, it may be observed by the beats of the pulse in the wrist, counting, on an average, about 70 to a minute for persons in moderate health, or $5\frac{5}{9}$ pulsations to a mile; and more or less according to circumstances.

Ex. 1. After observing a flash of lightning, it was 12 seconds before the thunder was heard; required the distance of the cloud from whence it came?

Ans. 2·52 miles.

Ex. 2. How long, after firing the Tower guns, may the report be heard at Shooter's-Hill, supposing the distance to be 8 miles in a straight line?

Ans. $38\frac{2}{9}$ seconds.

Ex. 3. After observing the firing of a large cannon at a distance, it was 7 seconds before the report was heard; what was its distance? Ans. 1·47 mile.

Ex, 4. Perceiving a man at a distance hewing down a tree with an axe, I remarked that four of my pulsations passed between seeing him strike and hearing the report of the blow; what was the distance between us, allowing 70 pulses to a minute?

Ex. 5. How far off was the cloud from which thunder issued, whose report was 5 pulsations after the flash of lightning, counting 75 to a minute?

Ex. 6. If I see the flash of a cannon, fired by a ship in distress at sea, and hear the report 33 seconds after, how far is she off?

A FEW EXERCISES IN MECHANICS, STATICS, AND OTHER BRANCHES OF NATURAL PHILOSOPHY.

Question 1. Required the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 0·258lb. avoirdupois.

Ans. 3·64739lb.

Quest. 2. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch. Ans. 13·78lb.

Quest. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted, by my watch, 17 seconds between the time of seeing the flash and hearing the report; what then was the distance? Ans. $3\frac{5}{9}$ miles.

Quest. 4. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

Quest. 5. What difference is there, in point of weight, between a block of marble, containing 1 cubic foot and a half, and another of brass of the same dimensions? Ans. 496lb. 14oz.

Quest. 6. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that measure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet

broad: now if this block be marble, what power would balance it, so as to prepare it for moving? Ans. $683\frac{7}{8}$ tons, the burden of an East India ship.

Quest. 7. The battering-ram of Vespasian weighed, suppose 10,000 pounds; and was moved, let us admit, with such a velocity, by strength of hand, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is, with what velocity a 32lb. ball must move, to do the same execution? Ans. 6250 feet.

Quest. 8. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier: but the less moves with 1000 times the velocity of the greater; in what proportion then are the momenta, or forces, with which they move? Ans. the less moves with a force 40 times greater.

Quest. 9. A body, weighing 20lb. is impelled by such a force, as to send it through 100 feet in a second; with what velocity then would a body of 8lb. weight move, if it were impelled by the same force?

Ans. 250 feet per second.

Quest. 10. There are two bodies, the one of which weighs 100lb. the other 60; but the less body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which these bodies move, is required?

Ans. the velocity of the greater to that of the less, as 3 to 40.

Quest. 11. There are two bodies, the greater contains 8 times the quantity of matter in the less, and is moved with a force 48 times greater: the ratio of the velocities of these two bodies is required?

Ans. the greater is to the less, as 6 to 1.

Quest. 12. There are two bodies, one of which moves 40 times swifter than the other; but the swifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the spaces described by these two bodies is required? Ans. the swifter is to the slower, as 1 to 3.

Quest. 13. Supposing one body to move 30 times swifter than another, as also the swifter to move 12 minutes, the other only 1: what difference will there be between the spaces described by them, supposing the last has moved 5 feet?

Ans. 1795 feet.

Quest. 14. There are two bodies, the one of which has passed over 50 miles, the other only 5; and the first had moved with 5 times the celerity of the second: what is the ratio of the times they have been in describing those spaces? Ans. as 2 to 1.

Quest. 15. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c. lb. on the one side, will just balance a constant weight of 95lb. at the distance of 2 inches on the other side of the fulcrum; the weight of the beam being 10lb. and its whole length 36 inches? Ans. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

Quest. 16. Two men carrying a burden of 200lb. weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet? Ans. 125lb. and 75lb.

Quest. 17. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For instance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itself when the prop is 13 feet from the less end; but removing the

prop a foot nearer to the said end, it takes a man's weight of 210lb. standing on the less end, to hold it in equilibrium. Required the weight of the tree?

Ans. 2520lb.

Quest. 18. If AB be a cane or walking-stick, 40 inches long, suspended by a string SD fastened to the middle point D: now a body being hung on at E, 6 inches distance from D, is balanced by a weight of 2lb. hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2lb. weight on the other side is moved to G, within 8 inches of D, before the cane will rest in equilibrio. Required the weight of the body? Ans. 24lb.

Quest. 19. A certain body on the surface of the earth weighs a cwt., or 112lb.; the question is, whither this body must be carried, that it may weigh only 10lb.?

Ans. either at 3·3466 semi-diameters, or $\frac{5}{56}$ of a semi-diameter, from the centre.

Quest. 20. If a body weigh 1 pound, or 16 ounces, on the surface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles? Ans. 15 oz. $9\frac{6}{8}$ dr. nearly.

Quest. 21. Whereabouts, in the line between the earth and moon, is their common centre of gravity; supposing the earth's diameter to be 7930 miles, and the moon's 2160; also the density of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters?

Ans. at $\frac{105}{231}$ parts of a diameter from the earth's centre,
or $\frac{41}{302}$ parts of a diameter, or 648 miles below the surface.

Quest. 22. Whereabouts, between the earth and moon, are their attractions equal to each other? Or where must another body be placed, so as to remain suspended in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimensions being as in the last question.

Ans. From the earth's centre $26\frac{9}{11}$ } of the earth's
From the moon's centre $3\frac{2}{11}$ } diameters.

Quest. 23. If a ball vibrate in the arch of a circle, 10 degrees on each side of the perpendicular; or a ball roll down the lowest 10 degrees of the arch; required the velocity at the lowest point? the radius of the circle, or length of the pendulum, being 20 feet. Ans. 4·4213 feet per second.

Quest. 24. If a ball descend down a smooth inclined plane, whose length is 100 feet, and altitude 10 feet; how long will it be in descending, and what will be the last velocity?

Ans. the veloc. 25·364 feet per sec. and time 7·8852 sec.

ON MODELS.

FROM an experiment made to ascertain the firmness of the model of a machine, or of an edifice, certain precautions are necessary before we can infer the firmness of the structure itself.

The classes of forces must be distinguished; as, whether they tend to *draw* asunder the parts, to *break* them transversely, or to *crush* them by compression.

To the first class belongs the stretching suffered by key-stones, or bonds of vaults, &c.; to the second, the load which tends to bend or break horizontal or inclined beams; to the third, the weight which presses vertically upon walls and columns.

Prop. 1. If the side of a model be to the corresponding side of the structure as 1 to n, the stress which tends to *draw* asunder, or to *break transversely*, the parts, increases from the smaller to the greater scale as 1 to n^3; while the resistance to those ruptures increases only as 1 to n^2.

The structure, therefore, will have so much less firmness than the model, as n is greater.

If W be the greatest weight which one of the beams of the model can bear, and w the weight or stress which it actually sustains, then the limit of n will be $n = \frac{W}{w}$.

Prop. 2. The side of a model being to the corresponding side of the structure as 1 to n, the stress which tends to crush the parts by compression, increases from the smaller to the greater scale, as 1 to n^3, while the resistance increases only in the ratio of 1 to n.

Hence, if W were the greatest load which a modular wall, or column, could carry, and w the weight with which it is actually loaded; then the greatest limit of increased dimensions would be found from the expression $n = \sqrt{\frac{W}{w}}$.

If, retaining the length or height nh, and the breadth nb, we wished to give to the solid such a thickness xt, as that it should not break in consequence of its increased dimensions, we should have $x = n^3 \sqrt{\frac{w}{W}}$.

In the case of a pilaster with a square base, or of a cylindrical column, if the dimension of the model were d, and of the largest pillar, which should not crush with its own weight when n times as high, xd, we should have

$$x = n^3 \sqrt{\frac{n^2 w}{W}}.$$

These theorems will often find their application in the profession of an engineer, whether civil or military.

PRACTICAL EXERCISES CONCERNING FORCES, &c.

Before we enter on the following problems, it will be convenient to lay down a modified synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces described: they are all comprehended in the following 12 theorems, collected from the principles already advanced.

Let f, F, be any two constant accelerative forces, acting on any body, during the respective times t, T, at the end of which are generated the velocities v, V, and described the spaces s, S. Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

I. *In Constant Forces.*

$$1.\quad \frac{s}{S} = \frac{tv}{TV} = \frac{t^2f}{T^2F} = \frac{v^2F}{V^2f}.$$

$$2.\quad \frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}}.$$

$$3.\quad \frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Fs}{fS}}.$$

$$4.\quad \frac{f}{F} = \frac{Tv}{tV} = \frac{T^2s}{t^2S} = \frac{v^2S}{V^2s}.$$

And if one of the forces, as F, be the force of gravity at the surface of the earth, and be called one, and its time T be $= 1''$; then it is known by experiment that, in the latitude of London, the corresponding space S is $= 16\frac{1}{12}$ feet, and consequently its velocity $V = 2S = 32\frac{1}{6}$, which we call g. Then the above four theorems, in this case, become as here below:

$$5.\quad s = \tfrac{1}{2}tv = \tfrac{1}{2}gft^2 = \frac{v^2}{2gf}.$$

$$6.\quad v = \frac{2s}{t} = gft = \sqrt{2gfs}.$$

$$7.\quad t = \frac{2s}{v} = \frac{v}{gf} = \sqrt{\frac{s}{\frac{1}{2}gf}}.$$

$$8.\quad f = \frac{v}{gt} = \frac{2s}{gt^2} = \frac{v^2}{2gs}.$$

And from these are deduced the following four fluxional theorems for variable forces, viz.

II. *In Variable Forces.*

$$9\quad \dot{s} = v\dot{t} = \frac{v\dot{v}}{gf}.$$

$$10\quad \dot{v} = gf\dot{t} = \frac{gf\dot{s}}{v}.$$

$$11.\quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{gf}.$$

$$12.\quad f = \frac{v\dot{v}}{g\dot{s}} = \frac{\dot{v}}{g\dot{t}}.$$

In these last four theorems, the force f, though variable, is supposed to be constant for the indefinitely small time $\dot{t}$, and they are to be used in all cases of variable forces, as the former ones in constant forces; that is to say, from the circumstances of the problem under consideration, an expression is deduced for the value of the force f, which being substituted in one of these theorems, that may be proper to the case in hand; the equation thence resulting will determine the corresponding values of the other quantities, required in the problem.

When a motive force is concerned in the question, it may be proper to observe, that the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter q in it; and the relation between these three

quantities being universally expressed by this equation $m = qf$, it follows that, by means of it, any one of the three may be expelled from the calculation, or brought into it, as the investigation requires.

Also, the momentum, or quantity of motion in a moving body, is qv, the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same, by means of accelerating forces.

To the following problems, which are all resolved by the application of these theorems, we subjoin their solutions.

PROBLEM I.

To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 *feet, and its height* 1 *foot.*

Here, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as $20 : 1 :: 1$ (the force of gravity) $: \frac{1}{20} = f$, the force on the plane.

Therefore, by theor. 6, v or $\sqrt{2gfs}$ is $\sqrt{(4 \times 16\frac{1}{12} \times \frac{1}{20} \times 20)} = \sqrt{(4 \times 16\frac{1}{12})} = 2 \times 4\frac{1}{96}$ or $8\frac{1}{48}$ feet nearly, the last velocity per second. And,

By theor. 7, t or $\sqrt{\frac{s}{\frac{1}{2}gf}}$ is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}} = \frac{20}{4\frac{1}{96}} = 4\frac{76}{77}$ seconds, the time of descending.

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 *feet per second, up a smooth inclined plane, which rises* 1 *foot in* 20: *it is proposed to assign the length which it will ascend up the plane, before it stops and begins to return down again, and the time of its ascent.*

Here $f = \frac{1}{20}$ as before.

Then by theor. 5, $s = \frac{v^2}{2gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{60000000}{193} = 310880\frac{160}{193}$ feet, or nearly 59 miles, the distance moved.

And, by theor. 7, $t = \frac{v}{gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193} = 621''\frac{147}{193} = 10'21''\frac{147}{193}$, the time of ascent.

PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 *foot in* 10, *and ascend* 100 *feet before it stop: required the time of ascent, and the velocity of projection.*

First, by theor. 6, $v = \sqrt{2gfs} = \sqrt{(4 \times 16\frac{1}{12} \times \frac{1}{10} \times 100)} = 8\frac{1}{48}\sqrt{10} = 25\cdot36408$ feet per second, the velocity.

And, by theor. 7, $t = \sqrt{\frac{s}{\frac{1}{2}gf}} = \sqrt{\frac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}}\sqrt{10} = \frac{192}{77}\sqrt{10} = 7\cdot88516$ seconds, the time in motion.

PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 *feet in* 10 *seconds, before it stop, to return back again: required the velocity of projection, and the angle of the plane's inclination.*

First, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per second, the velocity.

And, by theor. 8, $f = \frac{2s}{gt^2} = \frac{2\cdot100}{2\cdot16\frac{1}{12}\times100} = \frac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore 193 : 12 :: 100 : 6·2176 the height of the plane, or the sine of elevation to radius 100, which answers to $3^\circ\ 34'$, the angle of elevation of the plane.

PROBLEM V.

By a mean of several experiments, I have found, that a cast-iron ball, of 2 *inches diameter, fired perpendicularly into the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of* 1500 *feet per second, penetrated* 13 *inches deep into its substance. It is proposed thence to determine the time of the penetration, and the resisting force of the wood, as compared to the force of gravity, supposing that force to be a constant quantity.*

First, by theor. 7, $t = \frac{2s}{v} = \frac{2\times13}{1500\times12} = \frac{1}{692}$ part of a second, the time in penetrating.

And, by theor. 8, $f = \frac{v^2}{2gs} = \frac{1500^2}{4\times16\frac{1}{12}\times\frac{13}{12}} = \frac{81000000}{13\times193} = 32284$. That is, the resisting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its density or specific gravity. For, since f is as $\frac{v^2}{s}$ by theor. 4, the density and size of the ball remaining the same; if the density, or specific gravity, n, vary, and all the rest be constant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the size of the ball only is constant. But when only the diameter d varies, all the rest being constant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the resisting surface, or force of resistance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only, when all the rest are constant. Consequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And so $\frac{f}{F} = \frac{dnv^2S}{DNV^2s}$, and $\frac{s}{S} = \frac{dnv^2F}{DNV^2f}$; where f denotes the strength or firmness of the substance penetrated, and is here supposed to be the same, for all balls and velocities, in the same substance, which is either accurately or nearly so. See page 214, vol. iii. of my Tracts.

Hence taking the numbers in the problem, it is . . . .

$f = \frac{dnv^2}{s} = \frac{\frac{2}{12}\times7\frac{1}{3}\times1500^2}{\frac{13}{12}} = \frac{44\times1500^2}{39} = 2538462$ the value of f for elm wood. Where the specific gravity of the ball is taken $7\frac{1}{3}$, which is a little less

than that of solid cast iron, as it ought, on account of the air-bubble which is found in all cast balls.

PROBLEM VI.

To find how far a 24*lb. ball of cast iron will penetrate into a block of sound elm, when fired with a velocity of* 1600 *feet per second.*

Here, because the substance is the same as in the last problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $s = \frac{DV^2 s}{dv^2} = \frac{5{\cdot}55 \times 1600^2 \times 13}{2 \times 1500^2} = 41\frac{2}{15}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. i. p. 273, *of his works*), *that an* 18*-pounder ball, fired with a velocity of* 1200 *feet per second, penetrated* 34 *inches into sound dry oak. It is required thence to ascertain the comparative strength or firmness of oak and elm.*

The diameter of an 18lb. ball is 5·04 inches = D. Then, by the numbers given in this problem for oak, and in prob. 5, for elm, we have . . . . . .

$$\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 34}{5{\cdot}04 \times 1200^2 \times 13} = \frac{100 \times 17}{5{\cdot}04 \times 16 \times 13} = \frac{1700}{1048} \text{ or } = \tfrac{8}{5} \text{ nearly.}$$

From which it would seem, that elm timber resists more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be suspected that the great penetration in Mr. R.'s experiment was owing to the splitting of the timber in some degree.

PROBLEM VIII.

A 24*-pounder ball being fired into a bank of firm earth, with a velocity of* 1300 *feet per second, penetrated* 15 *feet. It is required thence to ascertain the comparative resistances of elm and earth.*

Comparing the numbers here with those in prob. 5, it is

$$\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 15 \times 12}{5{\cdot}55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0{\cdot}37} = \tfrac{1300}{271} = \tfrac{20}{3} \text{ nearly } = 6\tfrac{2}{3}$$

nearly. That is, elm timber resists about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ *of an inch diameter, will penetrate dry elm; supposing it fired with a velocity of* 1700 *feet per second, and that the lead does not change its figure by the stroke against the wood.*

Here $D = \frac{3}{4}$, $N = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then, by the numbers and theorem in prob. 5, it is $S = \frac{DNV^2 s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} = 9\frac{2}{3}$ inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted about twice as much; or else, which is more probable, that the defect in the penetration arose from the change of figure in the leaden ball he used, from the blow against the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 feet per second, having been found to penetrate 13 inches deep into dry elm: it is required to ascertain the time of passing through every single inch of the 13, and the velocity lost at each of them; supposing the resistance of the wood constant or uniform.

The velocity v being 1500 feet, or $1500 \times 12 = 18000$ inches, and velocities and times being as the roots of the spaces, in constant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a second, the whole time of passing through the 13 inches; therefore as

$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

veloc. lost   Time in the

$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}} v = 58{\cdot}9 :: t : \frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}} t = {\cdot}00005$ 1st inch.

$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}} v = 61{\cdot}4 :: t : \frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}} t = {\cdot}00006$ 2d

$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}} v = 64{\cdot}2$, &c. $\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}} t = {\cdot}00006$ 3d

$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}} v = 67{\cdot}5$ $\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}} t = {\cdot}00007$ 4th

$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}} v = 71{\cdot}4$ $\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}} t = {\cdot}00007$ 5th

$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}} v = 76{\cdot}0$ $\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}} t = {\cdot}00007$ 6th

$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}} v = 81{\cdot}7$ $\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}} t = {\cdot}00008$ 7th

$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}} v = 88{\cdot}8$ $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}} t = {\cdot}00008$ 8th

$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}} v = 98{\cdot}2$ $\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}} t = {\cdot}00009$ 9th

$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}} v = 111{\cdot}4$ $\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}} t = {\cdot}00011$ 10th

$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}} v = 132{\cdot}2$ $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}} t = {\cdot}00013$ 11th

$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}} v = 172{\cdot}3$ $\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}} t = {\cdot}00017$ 12th

$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}} v = 416{\cdot}0$ $\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}} t = {\cdot}00040$ 13th

Sum $1500{\cdot}0$ Sum $\frac{1}{692}$ or ${\cdot}00144$

Hence, as the motion lost at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it,

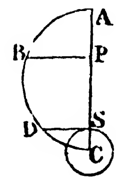
---

* In all such calculations as these, it is best, in carrying out the work, to rationalise the denominators: thus, instead of computing from the expression $\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}$, to use its equivalent one, $\frac{\sqrt{156}-\sqrt{143}}{13}$.

is very small also, we can conceive how such a plank may be shot through, when standing upright, without oversetting it.

PROBLEM XI.

The force of attraction above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{12}$ feet, or 193 inches, in the first second of time.



Put
r = CS the radius of the earth
a = CA the dist. fallen from,
x = CP any variable distance,
v = the velocity at P,
t = time of falling there, and
$\frac{1}{2}g$ = $16\frac{1}{12}$, half the veloc. or force at S,
f = the force at the point P.

Then we have the three following equations, viz.

$x^2 : r :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity at the surface is considered as 1;

$\dot{t}v = -\dot{x}$, because x decreases; and

$v\dot{v} = -gf\dot{x} = -\frac{gr^2\dot{x}}{x^2}$.

The fluents of the last equation give $v^2 = \frac{2gr^2}{x}$. But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{2gr^2}{x} - \frac{2gr^2}{a} = 2gr^2 \times \frac{a-x}{ax}$; or, $v = \sqrt{(\frac{2gr^2}{a} \times \frac{a-x}{x})}$, a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{(2gr \times \frac{a-r}{a})}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r, the last velocity becomes $(1 - \frac{r}{2a} \times \sqrt{2gr}$ very nearly, or nearly $\sqrt{2gr}$ only, which is accurately the greatest velocity by falling from an infinite height. And this, when r = 3965 miles, is 6·9506 miles per second. Also, the velocity acquired in falling from the distance of the sun, 12000 diameters of the earth, is 6·9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6·8927 miles per second.

Again, to find the time; since $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{2gr^2}} \times \frac{-x\dot{x}}{\sqrt{(ax-xx)}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{2gr^2}} \times (\sqrt{ax-xx}$ + arc to diameter a and vers. $a-x$); or the time of falling to any point P $= \frac{1}{2r}\sqrt{\frac{a}{\frac{1}{2}g}} \times$ (AB + BP). And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{\frac{1}{2}g}} \times \frac{AD + DS}{SC}$ for the whole time of falling to the surface at S; which is evidently infinite when a or AC is infinite, though the velocity is then only the finite quantity $\sqrt{2gr}$.

When the height above the earth's surface is given $= g$; because r is then nearly $= a$, and AD nearly $=$ DS, the time t for the distance g will be nearly $\sqrt{\frac{1}{2gr^2}} \times 2\text{DS} = \sqrt{\frac{1}{2gr}} \times \sqrt{2gr} = 1''$, as it ought to be.

If a body, at the distance of the moon at A, fall to the earth's surface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19 h. 46 m. 46 s. which is the time of falling from the moon to the earth.

In like manner the time of falling from the distance of the sun would be 64 d. 13 h. 46 m. 46 s.

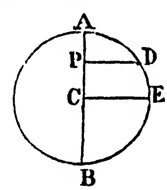
When the attracting body is considered as a point C; the whole time of descending to C will be . . . .

$$\frac{1}{2r}\sqrt{\frac{a}{\frac{1}{2}g}} \times \text{ABCD} = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{\frac{1}{2}g}} = \frac{10a}{51r}\sqrt{a} = \frac{\cdot 7854}{r}\sqrt{\frac{a^3}{\frac{1}{2}g}}.$$

Hence, the times employed by bodies, in falling from quiescence to the centre of attraction, are as the square roots of the cubes of the heights from which they respectively fall.

PROBLEM XII.

The force of attraction below the earth's surface being directly as the distance from the centre: it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth: abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere 3965 *miles radius.*



Put $r =$ AC the radius of the earth,
$x =$ CP the dist. from the centre,
$v =$ the velocity at P,
$t =$ the time there,
$\frac{1}{2}g = 16\frac{1}{12}$, half the force at A,
$f =$ the force at P.

Then CA : CP :: 1 : f; and the three equations are $rf = x$, and $v\dot{v} = -gf\dot{x}$, and $\dot{t}v = -\dot{x}$.

Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-gx\dot{x}}{r}$; the correct fluent of which gives $=$ $\sqrt{g \times \frac{r^2 - x^2}{r}} = \text{PD}\sqrt{\frac{g}{r}} = \text{PD}\sqrt{\frac{g}{\text{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular or sine PD at that point.

When the body arrives at C, then $v = \sqrt{gr} = \sqrt{g \cdot \text{AC}} = 25950$ feet or 4·9148 miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time; $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; and the fluents give $t = \sqrt{\frac{r}{g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

When P arrives at C, the above becomes $t = \sqrt{\frac{1}{gr}} \times$ quadrant AE $= \frac{\text{AE}}{\text{AC}}\sqrt{\frac{r}{g}} = 1\cdot 5708\sqrt{\frac{r}{g}} = 1267\frac{1}{4}$ seconds $= 21$ m. $7\frac{1}{4}$ s. for the time of falling to the centre C.

The time of falling to the centre is the same quantity $1{\cdot}5708 \sqrt{\frac{r}{g}}$, from whatever point in the radius AC the body begins to move. For, let n be any given distance from C at which the motion commences: then by correction, $v = \sqrt{(\frac{g}{r} \cdot \overline{n^2 - x^2})}$, and hence $\dot{t} = \sqrt{\frac{r}{g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$, the fluents of which give $t = \sqrt{\frac{r}{g}} \times$ arc to cosine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{g}} \times$ quadrant $= 1{\cdot}5708 \sqrt{\frac{r}{g}}$, for the time of descent to the centre C, the same as before.

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A. Then from B it will return again in the same manner, through C to A; and so oscillate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3{\cdot}1416 \sqrt{\frac{r}{g}} =$ 1h. 24m. 29s.

PROBLEM XIII.

To find the time of a pendulum vibrating in the arc of a cycloid.

Let S be the point of suspension;
SA the length of pendulum;
CAB, the whole cycloidal arc;
AIKD, the generating circle, to which FKE, HIG, are perpendiculars.
SC, SB two other equal semicycloids, on which the thread wrapping, the end A is made to describe the cycloid BAC.

By the nature of the cycloid, AD = DS; and SA = 2AD = SC = SB = CA = AB. Also, if at any point G be drawn the tangent GP; GQ parallel and PQ perpendicular to AD: then PG is parallel to the chord AI, by the nature of the curve. And, by the nature of forces, the force of gravity: force in direction GP :: GP : GQ :: AI : AH :: AD : AI; in like manner, the force of gravity: force in the curve at E :: AD : AK; that is, the accelerative force in the curve, is every where as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI, by the nature of the curve. So that the process and conclusions, for the velocity and time of describing any arc in this case, will be the very same as in the last problem, the nature of the forces being the same, viz. as the distance to be passed over to the lowest point A.

From which it follows, that the time of a semi-vibration, in all arcs, AG, AE, &c. is the same constant quantity $1{\cdot}5708 \sqrt{\frac{r}{g}} = 1{\cdot}5708 \sqrt{\frac{AS}{g}} = 1{\cdot}5708 \sqrt{\frac{l}{g}}$; and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416 \sqrt{\frac{l}{g}}$; where $l =$ AS = AB is the length of the pendulum, and 3·1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, by the nature of descents, is $\sqrt{\frac{l}{g}}$, which being in proportion to $3{\cdot}1416 \sqrt{\frac{l}{g}}$, as 1 is to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

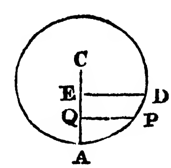
If the time of the whole vibration be 1 second, this equation arises, viz. $1'' = 3{\cdot}1416 \sqrt{\frac{l}{g}}$; hence $l = \frac{g}{3{\cdot}1416^2} = \frac{\frac{1}{2}g}{4{\cdot}9348}$, and $\frac{1}{2}g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348\ l$. So that if one of these, g or l, be given by experiment, these equations will give the other. See page 337.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating n times in a minute, or 60″ is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{n^2}$; as at page 346.

When a pendulum vibrates in a circular arc: as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that when it is very small, the times of vibration will be nearly equal. And hence it happens that $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds, in the very small arc of a circle.

PROBLEM XIV.

To find the velocity and time of a heavy body descending down the arc of a circle, or vibrating in the arc by a line fixed in the centre.



Let D be the beginning of the descent, C the centre, and A the lowest point of the circle; draw DE and PQ perpendicular to AC. Then the velocity in P being the same as in Q by falling through EQ, it will be $v = 2\sqrt{(\frac{1}{2}g \times EQ)} = 8\sqrt{(a - x)}$ nearly, when $a = AE$, $x = AQ$; since $AQ \cdot 2r = AP^2$, $v \cdot \text{in } A \propto$ chord AP.

But the fluxion of the time $\dot{t}$ is $= \frac{-\dot{AP}}{v}$, and

$\dot{AP} = \frac{r\dot{x}}{\sqrt{(2rx - x^2)}}$ where $r =$ the radius AC. Therefore

$\dot{t} = \frac{r}{8} \times \frac{-\dot{x}}{\sqrt{(2rx - x^2} \times \sqrt{(d - x)}} = \frac{d}{16} \times \frac{-\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(d - x)}}$, because $(2rx - x^2)\ (a - x) = (dx - x^2)\ (a - x) = (ax - x^2)\ (d - x) = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(1 - \frac{x}{d})}}$, where $d = 2r$ the diameter.

Or $\dot{t} = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)}} (1 + \frac{x}{2d} + \frac{1 \cdot 3x^2}{2 \cdot 4d^2} + \frac{1 \cdot 3 \cdot 5x^3}{2 \cdot 4 \cdot 6d^3}$ &c.), by developing $1 \div \sqrt{(1 - \frac{x}{d})}$, or $(1 - \frac{x}{d})^{-\frac{1}{2}}$, in a series.

But the fluent of $\frac{\dot{x}}{\sqrt{(ax - x^2)}}$ is $\frac{2}{a} \times$ arc to radius $\frac{1}{2}a$ and vers. x, or it is the arc whose rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the succeeding terms, without the co-efficients, be B, C, D, E, &c. Then will the flux. of any one as $\dot{Q}$, at n distance from A, be $\dot{Q} = x^n\dot{A} = x\dot{P}$, which suppose also = the flux. of $bP - dx^{n-1}\sqrt{(ax - x^2)} = b\dot{P} - d(n-1)\dot{x}x^{n-2}\sqrt{(ax - x^2)} - d\dot{x}x^{n-2} \times \frac{\frac{1}{2}ax - x^2}{\sqrt{(ax - x^2)}} = b\dot{P} - d\dot{x} \times \frac{(n - \frac{1}{2})\,ax^{n-1} - nx^n}{\sqrt{(ax - x^2)}} = b\dot{P} - d(n - \frac{1}{2})\,a\dot{P} + dnx\dot{P}$.

Hence, by equating the co-efficients of the like terms,

$$d = \frac{1}{n}\,;\; b = \frac{2n-1}{2n}a\,;\;\text{ and } Q = \frac{(2n-1)\,aP - 2x^{n-1}\sqrt{(ax - x^2)}}{2n}.$$

Which being substituted, the fluential terms become

$$\frac{\sqrt{d}}{16} \times \left(-A - \frac{1}{2d} \cdot \frac{aA - 2\sqrt{(ax - x^2)}}{2} - \frac{1\,.\,3}{2\,.\,4d^2} \cdot \frac{3aB - 2x\sqrt{(ax - x^2)}}{4} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6a^3} \cdot \frac{5aC - 2x^2\sqrt{(ax - x^2)}}{6} - \&c.\right)$$

But when $x = a$, those terms become barely $\frac{3{\cdot}1416\sqrt{d}}{16} \times \left(-1 - \frac{1^2a}{2^2d} - \frac{1^2}{2^2} \cdot \frac{3^2a^2}{4^2d^2} - \frac{1^2\,.\,3^2\,.\,5^2a^3}{2^2\,.\,4^2\,.\,6^2d^3} - \&c.\right)$; which being subtracted, and x taken $= 0$, there arises for the whole time of descending down DA, or the corrected value of

$$t = \frac{3{\cdot}1416\sqrt{d}}{16} \times \left(1 + \frac{1^2a}{2^2d} + \frac{1^2\,.\,3^2a^2}{2^2\,.\,4^2d^2} + \frac{1^2\,.\,3^2\,.\,5^2a^3}{2^2\,.\,4^2\,.\,6^2d^3} + \&c.\right).$$

When the arc is small, as in the vibration of the pendulum of a clock, all the terms of the series may be omitted after the second, and then the time of a semi-vibration t is nearly $= \frac{1{\cdot}5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{a}{8r})$ And theref. the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the versed sine of the arc.

If D be the degrees of the pendulum's vibration, on each side of the lowest point of the small arc, the radius being r, the diameter d, and $3{\cdot}1416 = p$; then is the length of that arc $A = \frac{prD}{180} = \frac{pdD}{360}$. But the versed sine in terms of the arc is $a = \frac{A^2}{2r} - \frac{A^4}{24r} + \&c. = \frac{A^2}{d} - \frac{A^4}{3d^3} + \&c.$ Therefore $\frac{a}{d} = \frac{A^2}{d^2} - \frac{A^4}{3d^4} + \&c. = \frac{p^2D^2}{360^2} - \frac{p^4D^4}{3{\cdot}360^4} + \&c$, or only $= \frac{p^2D^2}{360^2}$ the first term, by rejecting all the rest of the terms on account of their smallness, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{D^2}{13131}$. This value then being substituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the last near value of the time, it becomes $t = \frac{1{\cdot}5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{D^2}{52524})$ nearly. And therefore the times of vibration in different small arcs, are as $52524 + D^2$, or as 52524 added to the square of the number of degrees in the arc.

Hence it follows that the time lost in each second, by vibrating in a circle

instead of the cycloid, is $\frac{D^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or $24 \times .60 \times 60$ seconds, is $\frac{5}{3}D^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$. Therefore, if the pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3}(D^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees, and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees; and so on.

And in like manner we might proceed for any other curve, as the ellipse, hyperbola, parabola, &c.

Scholium.—By comparing this with the results of the problem 13, and the doctrine of the inclined plane in this vol., it will appear that the times in the cycloid, and in the arc of a circle, and in any chord of the circle, are respectively as the three quantities.

$$1,\ 1 + \frac{a}{8r},\ \&c., \text{ and } \frac{1}{\cdot 7854},$$

or nearly as the three quantities $1,\ 1 + \frac{a}{8r},\ 1{\cdot}27324$; the first and last being constant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Also the time in the cycloid is the least, but in the chord the greatest; for the greatest value of the series, in this prob. when $a = r$, or the arc AD is a quadrant, is $1{\cdot}18014$; and in that case the proportion of the three times is as the numbers 1, 1·18014, 1·27324. Moreover the time in the circle approaches to that in the cycloid, as the arc decreases, and they are very nearly equal when that arc is very small.

PROBLEM XV.

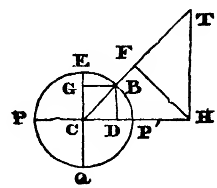
To find the weight of a column infinitely high, whose base is B.

Let r = radius of the earth, x = any height above it, and x' any indefinitely small height; also, let s the specific gravity of the matter of which the pillar is constituted. Then Bsx' = weight of an indefinitely small portion of the column at the earth's surface. And by the laws of gravity $\frac{1}{r^2} : Bsx' :: \frac{1}{(r+x)^2} : \frac{r^2Bsx'}{(r+x)^2}$, the weight of an equal part Bx' at the height x. Consequently the fluxion of the weight is $\frac{r^2Bs\dot{x}}{(r+x)^2} = r^2Bs\dot{x}\,(r+x)^{-2}$; the fluent of this is $-\,r^2Bs \times (r+x)^{-1}$, or $\frac{-\,r^2Bs}{r+x}$; or corrected, $\frac{r^2Bs}{r} - \frac{r^2Bs}{r+x}$, the weight of the pillar whose height is x. This reduces to $r^2Bs\,(\frac{1}{r} - \frac{1}{r+x}) = r^2Bs\,(\frac{x}{r(r+x)}) = rBs\,(\frac{x}{r+x})$. When x is infinite, the weight of the infinite pillar becomes $rBs\,(\frac{x}{x})$ or simply rBs. That is, the weight would be equal to that of a bent pillar of the same matter, and the same base which should be in horizontal contact with an arc of 57·2957795 degrees of a great circle of the earth.

Note.—This is independent of the consideration of centrifugal force, which would take off the stones from the top of the column, when its height was little more than $5{\cdot}6r$, at the equator, &c. See the next problem.

PROBLEM XVI.

Required the altitude of the highest edifice that could possibly be raised on any part of the earth's surface.



Let C be the centre of the earth, PP' its poles, EQ the equator, BT the altitude of the edifice erected perpendicularly at B, whose latitude ECB is l. Draw BD, TH, each perpendicular to PP', or to PP' produced, and from H, HF perpendicular to CT. Now, it is evident that the edifice cannot be raised any higher than the point T, where the centrifugal force reduced to the direction CT is equal to the force of gravity; for at a greater altitude the centrifugal force would exceed the gravitating force, and the materials would fly off.

Let $CE = CB = CP = 1$, $CD = BG = \sin. l$, $BD = \cos. l$, $C = \frac{1}{289} =$ centrifugal force at E, that of gravity being expressed by unity, and $CT = x$. Then, as $CB : BD :: CT : TH$, or $1 : \cos. l :: x : x \cos. l = TH$: hence, as $CE : TH$, or $1 : x \cos. l :: C : Cx \cos. l =$ centrifugal force at T in direction HT, T revolving about H. And, as $HT : FT :: CB : BD ::$ rad. : $\cos. l ::$ $Cx \cos. l : Cx \cos.^2 l$, the said centrifugal force reduced to the direction TC, opposite that of gravity. Again, as $\frac{1}{CB^2} : \frac{1}{T^2} ::$ grav. at B : grav. at T $= \frac{1}{x^2}$. Hence, by the above, $Cx \cos.^2 l = \frac{1}{x^2}$, and $x^3 = \frac{1}{C \cos.^2 l} = \frac{\sec.^2 l}{C}$, and $x = \sqrt[3]{(289 \sec.^2 l)}$. Consequently, $BT = CT - CB = \sqrt[3]{(289 \sec.^2 l)} - 1$.

1. Suppose the place to be the equator, then $\sec. l = 1$, and $BT = \sqrt[3]{289} - 1 = 6{\cdot}611489 - 1 = 5{\cdot}611489$ radii of the earth.

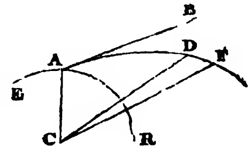
2. Suppose the latitude to be 45°, then $BT = \sqrt[3]{(289 \sec.^2 45^\circ)} - 1 = \sqrt[3]{289 \times 2} - 1 = 8{\cdot}320335 - 1 = 7{\cdot}320335$ radii.

3. If the latitude be 60°, then $BT = \sqrt[3]{(289 \sec.^2 60^\circ)} - 1 = \sqrt[3]{(289 \times 4)} - 1 = 10{\cdot}49508 - 1 = 9{\cdot}49508$ radii.

4. At either pole $\sec. l$ is infinite, and the height of the edifice would have no limit.

PROBLEM XVII.

Required the least velocity with which a cannon ball must be projected from the surface of the earth (suppose at an angle of 40° elevation) so that it shall never return.



Let C be the earth's centre, EAR a portion of the surface, AB the projectile's direction, and ADF its trajectory. Suppose CD, CF indefinitely near each other, and call CA (the earth's radius = 21000000 feet), r, CD, x; 32·2 feet, the velocity generated in a second at the earth's surface g; v the velocity in D; V the required velocity. Then the centripetal

force in D will be $\frac{r^2g}{x^2}$ (being reciprocally as the square of the distance from the earth's centre), and the force to retard the motion in the direction DF $\frac{r^2gx}{x^2 \times \mathrm{DF}}$; this retarding force drawn into the fluxion of the time, being equal to the fluxion of the velocity, $\frac{r^2g\dot{x}}{vx^2}$ will be $= -\dot{v}$; therefore $v\dot{v} = -\frac{r^2g\dot{x}}{xx}$, and the fluent $\frac{vv}{2} = \frac{r^2g}{x}$. But in A ($v$ being $=$ V, and $x = r$) the correct fluent gives $v = \sqrt{(\mathrm{VV} - 2rg + 2\,r^2gx^{-1})}$. After an infinite time, x will be infinitely great, and $\sqrt{(\mathrm{VV} - 2rg + 2r^2gx^{-1})}$ infinitely small, and therefore may be put $= 0$, in which equation r is nothing in respect of the value of x; and therefore $\mathrm{V} = \sqrt{2rg} = 36775$ feet $= 6{\cdot}9655$ miles. Hence, there is no limit with regard to the angle of direction; but if a body be projected from the earth's surface, in *any* direction whatever above the horizon, with such a velocity as will carry it about or above 7 miles per second, it will never return.

PROBLEM XVIII.

To determine the time of filling the ditches of a work with water, at the top, by a sluice of 2 *feet square; the head of water above the sluice being* 10 *feet, and the dimensions of the ditch being* 20 *feet wide at bottom,* 22 *at top,* 9 *deep, and* 1000 *feet long.*

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{\frac{1}{2}g} : \sqrt{10} :: g : 2g\sqrt{5}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore $8g\sqrt{5}$ is the quantity per second running through it: and consequently $8g\sqrt{5} : 189000 :: 1'' : \frac{23625}{g\sqrt{5}} = 1863''$ or 31 m. 3 s. nearly, which is the time of filling the ditch.

PROBLEM XIX.

To determine the time of emptying a vessel of water by a sluice in the bottom of it, or in the side near the bottom: the height of the aperture being very small in respect of the altitude of the fluid.

Put $a =$ the area of the aperture or sluice;
$g = 32\frac{1}{6}$ feet, the force of gravity;
$d =$ the whole depth of water;
$x =$ the variable altitude of the surface above the aperture;
$\mathrm{A} =$ the area of the surface of the water.

Then $\sqrt{\frac{1}{2}g} : \sqrt{x} :: 2g : 2\sqrt{\frac{1}{2}gx}$ the velocity with which the fluid will issue at the sluice; and hence $\mathrm{A} : a :: 2\sqrt{\frac{1}{2}gx} : \frac{2a\sqrt{\frac{1}{2}gx}}{\mathrm{A}}$, the velocity with which the surface of the water will descend at the altitude x, or the space it would descend in 1 second with the velocity there. Now, in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times; therefore $\frac{2a\sqrt{\frac{1}{2}gx}}{\mathrm{A}} : -\dot{x} :: 1'' . \frac{-\mathrm{A}\dot{x}}{2a\sqrt{\frac{1}{2}gx}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is, $\dot{t} = \frac{-\mathrm{A}\dot{x}}{2a\sqrt{\frac{1}{2}gx}}$; which is made negative, because x is a decreasing quantity, or its fluxion negative.

Now, when the nature or figure of the vessel is given, the area A will be

given in terms of x; which value of A being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

So if, for example, the vessel be any prism, or everywhere of the same breadth; then A is a constant quantity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{\frac{1}{2}g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{\frac{1}{2}g}}$, and should be 0; therefore the correct fluent is $\dot{t} = \frac{A}{a} \times \frac{\sqrt{d}-\sqrt{x}}{\sqrt{\frac{1}{2}g}}$ for the time of the surface descending till the depth of the water be x. And when $x = 0$, the whole time of exhausting is barely $\frac{A}{a}\sqrt{\frac{d}{\frac{1}{2}g}}$.

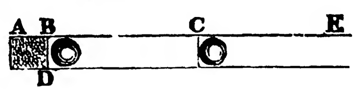
Hence, if A be $= 10000$ square feet, $a = 1$ square foot, and $d = 10$ feet; the time is $7885\frac{1}{5}$ seconds, or 2h. 11m. $25\frac{1}{5}$s.

Again, if the vessel be a ditch, or canal of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90 + x :: 20 : \frac{90+x}{9} \times 2$ the breadth of the surface of the water when its depth in the canal is x; and therefore $A = \frac{90+x}{9} \times 2000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{\frac{1}{2}gx}} = 1000 \times \frac{90+x}{9} \times \frac{-\dot{x}}{a\sqrt{\frac{1}{2}gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $1000 \times \frac{180+\frac{2}{3}d}{9a} \times \sqrt{\frac{d}{\frac{1}{2}g}} = \frac{1000 \times 186 \times 3}{9 \times 4\frac{1}{98}}$ $= 15459''\frac{2}{3}$ nearly, or 4h. 17m. $39\frac{2}{3}$s., being the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XX.

To determine the velocity with which a ball is discharged from a given piece of ordnance, with a given charge of gunpowder.

Let the annexed figure represent the bore of the gun; AD being the part filled with gunpowder.



And put

$a =$ AB, the part at first filled with powder and the bag;
$b =$ AE, the whole length of the gun-bore;
$c = \cdot 7854$, the area of a circle whose diameter is 1;
$d =$ BD, the diameter of the ball;
$e =$ the specific gravity of the ball, or weight of 1 cubic foot;
$\frac{1}{2}g = 16\frac{1}{12}$ feet, descended by a body in 1 second;
$m =$ 240oz. 15lb., the pressure of the atmosphere on a square inch;
n to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
$w =$ the weight of the ball. Also, let
$x =$ AC, be any variable distance of the ball from A, in moving along the gun-barrel.

First, cd^2 is $=$ the area of the circle BD of the ball;
theref. mcd^2 is the pressure of the atmosphere on BD;
conseq. $mncd^2$ is the first force of the powder on BD.

But the force of the inflamed powder is proportional to its density, and the density is inversely as the space it fills; therefore the force of the powder on the ball at B, is to the force on the same at C, as AC is to AB: that is, ...

$x : a :: mncd^2 : \frac{mnacd^2}{x} = \text{F}$, the motive force at C

conseq. $\frac{\text{F}}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, theor. 10, p. 392, gives $v\dot{v} = gf\dot{x} = \frac{gmnacd^2}{w} \times \frac{\dot{x}}{x}$; the fluent of which is $v^2 = \frac{2gmnacd^2}{w} \times$ hyp. log. of x.

But when $v = 0$, then $x = a$; therefore by correction, $v^2 = \frac{2gmnacd^2}{w} \times$ hyp. log. $\frac{x}{a}$ is the correct fluent; conseq. $v = \sqrt{(\frac{2gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the veloc. of the ball at C, and $v = \sqrt{(\frac{2gmnhcd^2}{w} \times \text{hyp. log. } \frac{b}{a})}$ the velocity with which the ball issues from the muzzle at E; where h denotes the length of the cylinder filled with powder; and a the length to the hinder part of the ball, which will be more than h when the ball does not touch the powder.

Or, by substituting the numbers for g, m, c, and changing the hyperbolic logarithms for the common ones, then $v = \sqrt{(\frac{2306nhd^2}{w} \times \text{com. log. } \frac{b}{a})}$, the velocity at E, in feet.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is $w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ced^3}{2592} = \frac{ed^3}{3300}$; which being substituted for w, in the value of v, it becomes

$v = 2758 \sqrt{(\frac{nh}{de} \times \text{com. log. } \frac{b}{a})}$, the velocity at E.

When the ball is of cast iron; taking $e = 7368 = 10(27{\cdot}14)^2$, the rule becomes $v = 32 \sqrt{(\frac{nh}{d} \times \log. \frac{b}{a})}$ for the veloc. of the cast iron ball.

Or, when the ball is of lead; then $e = 11325 = 10(33{\cdot}652)^2$, and $v = 26 \sqrt{(\frac{nh}{d} \times \log. \frac{b}{a})}$ for the veloc. of the leaden ball *.

* Some practical artillerists having expressed a wish to the Editor to see a solution to this problem upon the supposition *that the gunpowder explodes gradually, so as to be all ignited when the ball quits the mouth of the gun*, he avails himself of the opportunity of giving it in this place.

Here we must first investigate the relation between the time and the space described. By the hypothesis we shall have the force as $\frac{t}{x}$; and we have x varying as t^n to find n. We have, from the primitive formula, page 390, $v\dot{v} = gf\dot{x} \propto f\dot{x}$, and $f \propto \frac{t}{x} \propto \frac{x^{\frac{1}{n}}}{x} \propto \frac{1}{x^{\frac{n-1}{n}}}$. Therefore $v\dot{v} \propto \frac{\dot{x}}{x^{\frac{n-1}{n}}}$, and the fluent $v^2 \propto nx^{\frac{1}{n}}$. Farther,

Corol. From the general expression for the velocity v, above given, may be derived what must be the length of the charge of powder a, in the gun-barrel, so as to produce the greatest possible velocity in the ball; namely, by making the value of v a maximum, or, by squaring and omitting the constant quantities, the expression $a \times$ hyp. log. of $\frac{b}{a}$ a maximum, or its fluxion equal to nothing; that is, $\dot{a} \times$ hyp. log. $\frac{b}{a} - \dot{a} = 0$, or hyp. log. of $\frac{b}{a} = 1$; hence $\frac{b}{a} =$ 2·71828, the number whose hyp. log. is 1. So that $a : b :: 1 : 2{\cdot}71828$, or as 4 to 11 nearly, or nearer, as 7 to 19; that is, the length of the charge, to produce the greatest velocity, is the $\frac{4}{11}$th part of the length of the bore, or nearer $\frac{7}{19}$ of it.

$\dot{x} \propto v\dot{t} \propto v \cdot \frac{1}{n} x^{\frac{1}{n}-1}\dot{x} \propto x^{\frac{1}{2n}} \times x^{\frac{1}{n}-1}\dot{x} \propto x^{\frac{3-2n}{2n}}\dot{x}$; so that the fluent $x \propto x^{\frac{3}{2n}}$. Conseq. $1 = \frac{3}{2n}$, and $n = \frac{3}{2}$. That is, in this case, $x \propto t^{\frac{3}{2}}$.

Now, taking the notation of the problem in the text, we have $f' = \frac{mncd^2}{w}$, the accelerative force at the first instant of explosion, and, by hyp. at E, where the whole is exploded, the force will be $\frac{af'}{b}$. To find the force at C, if T be put for the whole time of explosion, we shall have

$$\frac{T}{b}\left(\frac{\text{time}}{\text{space}}\right) : \frac{af'}{b} \text{ (force at E)} :: \frac{t}{x}\left(\frac{\text{time}}{\text{space}}\right) : \frac{af't}{Tx},$$

force at C, or the value of f there, in the general formulæ.

Hence we have $v\dot{v} = gf\dot{x} = \frac{gaf't\dot{x}}{Tx}$.

But $b : T^{\frac{3}{2}} :: x : t^{\frac{3}{2}} = \frac{T^{\frac{3}{2}}x}{b} \therefore t = \frac{Tx^{\frac{2}{3}}}{b^{\frac{2}{3}}}$; and, by substituting this value of t for it, we have

$$v\dot{v} = \frac{gaf'}{Tx} \cdot \frac{Tx^{\frac{2}{3}}\dot{x}}{b^{\frac{2}{3}}} = \frac{gaf'x^{-\frac{1}{3}}\dot{x}}{b^{\frac{2}{3}}}.$$

Taking the fluents, we have

$$\tfrac{1}{2}v^2 = \frac{gaf'x^{\frac{2}{3}}}{\frac{2}{3}b^{\frac{2}{3}}} \text{ whence } v = \sqrt{\frac{3gaf'x^{\frac{2}{3}}}{b^{\frac{2}{3}}}}.$$

When x becomes $= b$, this becomes

$$v = \sqrt{(3gaf')} = \sqrt{(3gmnc \cdot \frac{ad^2}{w})}.$$

Cor. 1. Hence, so long as a and d remain the same, the velocity at the muzzle E will be the same whatever be the length of the gun; for b does not appear in the ultimate value of v.

This is contrary to all experience, and proves that the hypothesis is untenable.

Cor. 2. The powder being the same, the velocity at the muzzle (d remaining the same) will be as the square root of the charge.

Cor. 3. In guns of different bores, the velocity at the muzzle will be as

$$\sqrt{\frac{a}{d}}. \text{ For } v \propto \sqrt{\frac{ad^2}{w}} \propto \sqrt{\frac{ad^2}{d^3}} \propto \sqrt{\frac{a}{d}}.$$

Cor. 4. If the charge be given, a will be inversely as d^2, and $v \propto \sqrt{\frac{1}{d^3}}$.

Cor. 5. If b be the length of a gun in which the charge of powder will be all fired when the ball reaches the muzzle, then in a shorter gun AC, the same powder, and an equal charge, will give an ultimate velocity varying as $\sqrt{\frac{x^{\frac{2}{3}}}{b^{\frac{2}{3}}}}$ or as $\sqrt[3]{\frac{x}{b}}$

By actual experiment it is found, that the charge for the greatest velocity, is but little less than that which is here computed from theory; as may be seen by turning to page 213, vol. iii. of the Tracts, where the corresponding parts are found to be, for four different lengths of gun, thus, $\frac{3}{10}$, $\frac{1}{12}$, $\frac{3}{16}$, $\frac{3}{20}$; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the constant quantity n remains to be determined. It denotes the first strength or force of the fired gunpowder, just before the ball is moved out of its place. This value is assumed, by Mr. Robins, equal to 1000, that is, 1000 times the pressure of the atmosphere, or any equal spaces.

But the value of the quantity n may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of v, or the velocity, as computed by theory, viz. $v = 100 \sqrt{(\frac{na}{10d} \times \text{log. of } \frac{b}{a})}$, or $= 100 \sqrt{(\frac{nh}{10d} \times \text{log. of } \frac{b}{a})}$. Now, supposing that v is a given quantity, as well as all the other quantities, excepting only the number n, then by reducing this equation, the value of the letter n is found to be as follows, viz. . . . .

$n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}$, or $= \frac{dvv}{1000h} \div \text{log. of } \frac{b}{a}$, when h is different from a.

Now, to apply this to the experiments. By pa. 69, vol. iii. of the Tracts, the velocity of the ball, of 1·96 inches diameter, with 4 ounces of powder, in the gun, No. 1, was 1100 feet per second; and, by pa. 316, vol. ii., the length of the gun, when corrected for the spheroidal hollow in the bottom of the bore, was 28·53; also, by pa. 48, vol. iii., the length of the charge, when corrected in like manner, was 3·45 inches of powder and bag together, but 2·54 of powder only: so that the values of the quantities in the rule, are thus: $a = 3·45$; $b = 28·53$; $d = 1·96$; $h = 2·54$; and $v = 1100$: then, by substituting these values instead of the letters, in the theorem $n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}$, it comes out $n = 750$, when h is considered as the same as a. And so on, for the other experiments there treated of.

It is here to be noted, however, that there is a circumstance in the experiments delivered in the Tracts, just mentioned, which will alter the value of the letter a in this theorem, which is this, viz. that a denotes the distance of the shot from the bottom of the bore; and the length of the charge of powder alone ought to be the same thing; but, in the experiments, that length is included, besides the length of the real powder, the substance of the thin flannel bag in which it was always contained, of which the neck at least extended a considerable length, being the part where the open end was wrapped and tied close round with a thread. This circumstance causes the value of n, as found by the theorem above, to come out less than it ought to be, for it shows the strength of the inflamed powder when just fired, and when the flame fills the whole space a before occupied both by the real powder and the bag, whereas it ought to show the first strength of the flame when it is supposed to be contained in the space only occupied by the powder alone, without the bag. The formula will therefore bring out the value of n too little, in proportion as the real space filled

by the powder is less than the space filled both by the powder and its bag. In the same proportion therefore must we increase the formula, that is, in the proportion of h, the length of real powder, to a the length of powder and bag together. When the theorem is so corrected, it becomes $\frac{dvv}{1000h} \div$ com. log. of $\frac{b}{a}$.

Now, by pa. 48 and 49, vol. iii., Tracts, there are given both the lengths of all the charges, or values of a, including the bag, and also the length of the neck and bottom of the bag, which is 0·91 of an inch, which therefore must be subtracted from all the values of a, to give the corresponding values of h. This in the example above reduces 3·45 to 2·54.

Hence, by increasing the above result 750, in proportion of 2·54 to 3·45, it becomes 1018. And so on for the other experiments.

But it will be best to arrange the results in a table, with the several dimensions, when corrected, from which they are computed, as follows.

Tables of Velocities of Balls and First Force of Powder, &c.

Gun.		Charge of powder.			Velocity or value of v.	First force, or value of n
No	Length or value of b.	Weight in ounces.	Length or value of a	of h.		
	inches.	4	3·45	2·54	1100	1018
1	28·53	8	5·99	5·08	1340	1091
		16	11·07	10·16	1430	967
		4	3·45	2·54	1180	1077
2	38·43	8	5·99	5·08	1580	1193
		16	11·07	10·16	1660	984
		4	3·45	2·54	1300	1067
3	57·70	8	5·99	5·08	1790	1256
		16	11·07	10·16	2000	1076
		4	3 45	2·54	1370	1060
4	80·23	8	5·99	5·08	1940	1289
		16	11·07	10·16	2200	1085

Where it may be observed, that the numbers in the column of velocities, 1430 and 2200, are a little increased, as, from a view of the table of experiments, they evidently required to be. Also the value of the letter d is constantly 1·96 inch.

Hence it appears, that the value of the letter n, used in the theorem, though not yet greatly different from the number 1000, assumed by Mr. Robins, is rather various, both for the different lengths of the gun, and for the different charges with the same gun.

But this diversity in the value of the quantity n, or the first force of the inflamed gunpowder, is probably owing in some measure to the omission of a material datum in the calculation of the problem, namely, the weight of the charge of powder, which has not at all been brought into the computation. For it is manifest, that the elastic fluid has not only the ball to move and impel before it, but its own weight of matter also. The computation may therefore be renewed, in the ensuing problem, to take that datum into the account.

PROBLEM XXI.

To determine the same as in the last problem; taking both the weight of powder and the ball into the calculation.

Besides the notation used in the last problem, let $2p$ denote the weight of the powder in the charge, with the flannel bag in which it was inclosed.

Now, because the inflamed powder occupies at all times the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the same, will move with only half the velocity that the ball moves with; and this will require the same force as half the weight of the power, &c. moved with the whole velocity of the ball. Therefore, in the conclusion derived in the last problem, we are now, instead of w, to substitute the quantity $p + w$; and when that is done, the last velocity will come out, $v = \sqrt{(\frac{2230nhd^2}{p+w} \times \text{com. log. } \frac{b}{a})}$.

And from this equation is found the value of n, which is $n = \frac{p+w}{\quad} v^2 \div$ log. of $\frac{b}{a}$, $= \frac{p+w}{8567h} v^2 \div$ log. of $\frac{b}{a}$, by substituting for d its value 1·96, the diameter of the ball.

Now as to the ball, its medium weight was 16oz. 13dr. = 16·81oz. And the weights of the bags containing the several charges of powder, viz. 4oz., 8oz., 16oz., were 8dr., 12dr., and 1oz., 5dr.; then adding these to the respective contained weights of powder, the sums, 4·5oz., 8·75 oz., 17·31oz., are the values of $2p$, or the weights of the powder and bags; the halves of which, or 2·25, and 4·38, and 8·66, are the values of the quantity p for those three charges; and these being added to 16·81, the constant weight of the ball, there are obtained the three values of $p + w$ for the three charges of powder, which values therefore are 19·06oz., and 21·19oz., and 25·47oz. Then, by calculating the values of the first force n, by the last rule above, with these new data, the whole will be found as in the following table.

The Gun.		Charge of Powder.			Weight of ball and charge, or values of $p+w$.	Velocity, or the values of v.	First force, or the value of n.
No.	Length or value of b.	Weight in ounces	Length or value of a.	of h.			
	Inches.	4	3·45	2·54	19·06	1100	1155
1	28·53	8	5·99	5·08	21·19	1340	1377
		16	11·07	10·16	25·47	1430	1456
		4	3·45	2·54	19·06	1180	1167
2	38·43	8	5·99	5·08	21·19	1580	1506
		16	11·07	10·16	25·47	1660	1492
		4	3·45	2·54	19·06	1300	1210
3	57·70	8	5·99	5·08	21·19	1790	1586
		16	11·07	10·16	25·47	2000	1646
		4	3·45	2·54	19·06	1370	1203
4	80·23	8	5·99	5·08	21·19	1940	1627
		16	11·07	10·16	25·47	2200	1648

And here it appears that the values of n, the first force of the charge, are much more uniform and regular than by the former calculations in the preceding problem, at least in all excepting the smallest charge, 4 oz. in each gun; which it would seem must be owing to some general cause or causes. Nor have we long to search to find out what those causes may be. For when it is considered that these numbers for the value of n, in the last column of the table, ought to exhibit the first force of the fired powder, when it is supposed to occupy the space only in which the bare powder itself lies; and that whereas it is manifest that the condensed fluid of the charge in these experiments occupies the whole space between the ball and the bottom of the gun bore, or the whole space taken up by the powder and the bag or cartridge together, which exceeds the former space, or that of the powder alone, at least in the proportion of the circle of the gun bore, to the same as diminished by the thickness of the surrounding flannel of the bag that contained the powder; it is manifest that the force was diminished on that account. Now by gently compressing a number of folds of the flannel together, it has been found that the thickness of the single flannel was equal to the 40th part of an inch; the double of which, $\frac{1}{20}$ or ·05 of an inch, is therefore the quantity by which the diameter of the circle of the powder within the bag was less than that of the gun bore. But the diameter of the gun bore was 2·02 inches; therefore, deducting the ·05, the remainder 1·97 is the diameter of the powder cylinder within the bag; and because the areas of circles are to each other as the squares of their diameters, and the squares of these numbers, 1·97 and 2·02, being to each other as 388 to 408, or as 97 to 102; therefore, on this account alone, the numbers before found for the value of n, must be increased in the ratio of 97 to 102.

But there is yet another circumstance, which occasions the space at first occupied by the inflamed powder to be larger than that at which it has been taken in the foregoing calculations, and that is the difference between the content of a sphere and cylinder. For, the space supposed to be occupied at first by the elastic fluid was considered as the length of a cylinder measured to the hinder part of the curve surface of the ball, which is manifestly too little by the difference between the content of half the ball and a cylinder of the same length and diameter, that is, by a cylinder whose length is $\frac{1}{3}$ the semidiameter of the ball. Now that diameter was 1·96 inches; the half of which is 0·98, and $\frac{1}{3}$ of this is 0·33 nearly. Hence then it appears that the lengths of the cylinders, at first filled by the dense fluid, viz. 3·45, and 5·99, and 11·07, have been all taken too little by 0·33; and hence it follows that, on this account also, all the numbers before found for the value of the first force n, must be further increased in the ratios of 3·45 and 5·99 and 11·07, to the same numbers increased by 0·33, that is, to the numbers 3·78 and 6·32 and 11·40.

Compounding now these last ratios with the foregoing one, viz. 97 to 102, it produces these three, viz. the ratios of 334 and 581 and 1074, respectively to 385 and 647 and 1163. Therefore increasing the last column of numbers, for the value of n, viz. those of the 4 oz. charge in the ratio of 334 to 385, and those of the 8 oz. charge in the ratio of 581 to 647, and those of the 16 oz. charge in the ratio of 1074 to 1163, with every gun, they will be reduced to the numbers in the an-

Powder.	The Guns.			
	1	2	3	4
oz.				
4	1372	1387	1438	1430
8	1637	1677	1766	1812
16	1577	1616	1782	1784

nexed table; where the numbers are still larger and more regular than before *.

Thus then at length it appears that the first force of the inflamed gunpowder, when occupying only the space at first filled with the powder, is about 1800, that is, 1800 times the elasticity of the natural air, or pressure of the atmosphere, in the charges with 8 oz. and 16 oz. of powder, in the two longer guns; but somewhat less in the two shorter, probably owing to the gradual firing of gunpowder in some degree; and also less in the lowest charge 4 oz. in all the guns, which may probably be owing to the less degree of heat in the small charge. But besides the foregoing circumstances that have been noticed, or used in the calculations, there are yet several others that might and ought to be taken into the account, in order to a strict and perfect solution of the problem; such as, the counterpressure of the atmosphere, and the resistance of the air on the fore part of the ball while moving along the bore of the gun; the loss of the elastic fluid by the vent and windage of the gun; the gradual firing of the powder; the unequal density of the elastic fluid in the different parts of the space it occupies between the ball and the bottom of the bore; the difference between pressure and percussion when the ball is not laid close to the powder; and perhaps some others: on all which accounts it is probable that, instead of 1800, the first force of the elastic fluid is not less than 2000 times the strength of natural air.

Corol. From the theorem last used for the velocity of the ball and elastic fluid, viz. $v = \sqrt{(\frac{2230hd^2}{p+w}n \times \log. \frac{b}{a})} = \sqrt{\frac{8567hn}{p+w}} \times \log. \frac{b}{a}$, we may find the velocity of the elastic fluid alone, viz by taking w, or the weight of the ball $= 0$ in the theorem, by which it becomes barely $v = \sqrt{(\frac{8567hn}{p} \times \log. \frac{b}{a})}$, for that velocity. And by computing the several preceding examples by this theorem, supposing the value of n to be 2000, the conclusions come out a little various, being between 4000 and 5000, but most of them nearer to the latter number. So that it may be concluded that the velocity of the flame, or of the fired gunpowder, expands itself at the muzzle of the gun, at the rate of about 5000 feet per second nearly

ON THE MOTION OF BODIES IN FLUIDS.

PROBLEM XXII.

To determine the force of fluids in motion; and the circumstances attending bodies moving in fluids.

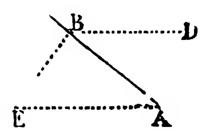
1. It is evident that the resistance to a plane, moving perpendicularly through an infinite fluid at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the

* From the experiments of 1815, 1816, and some subsequent experiments, it appears that n exceeds 2000, in the best gunpowder.

fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{2g}$, the whole resistance, or motive force m, will be $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$; g being, as we have all along assumed it, $= 32\frac{1}{6}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion must be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number of them striking the plane in a given time, is also as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any angle, the sine of that angle being s, to the radius 1; then the resistance to the plane, or the force of the fluid against the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to s^3. For, AB being the direction of the plane, and BD that of the motion, making the angle ABD, whose sine is s; the number of particles, or quantity of the fluid striking the plane, will be diminished in the ratio of 1 to s, or of radius to the sine of the angle B of inclination; and the force of each particle will also be diminished in the same ratio of 1 to s: so that on both these accounts, the whole resistance will be diminished in the ratio of 1 to s^2, or in the duplicate ratio of radius to the sine of the said angle. But again, it is to be considered that this whole resistance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is, as 1 to s. So that finally, on all these accounts, the resistance in the direction of motion, is diminished in the ratio of 1 to s^3, or in the triplicate ratio of radius to the sine of inclination. Hence, comparing this with article 1, the whole resistance, or the motive force on the plane, will be $m = \frac{anv^2s^3}{2g}$.



4. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force m; then the retarding force f, or $\frac{m}{w}$, will be $\frac{anv^2s^3}{2gw}$.

5. And if the body be a cylinder, whose face or end is a, and diameter d, or radius r, moving in the direction of its axis; because then $s = 1$, and $a = \pi r^2 = \frac{1}{4}\pi d^2$, where $\pi = 3{\cdot}1416$; the resisting force m will be . . . . $\frac{\pi nd^2v^2}{8g} = \frac{\pi nr^2v^2}{2g}$, and the retarding force $f = \frac{\pi nd^2v^2}{8gw} = \frac{\pi nr^2v^2}{2gw}$.

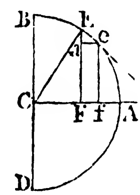
6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s: then the number of particles of the fluid striking the face being still the same, but the force of each

opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m would be $\frac{\pi nd^2v^2s^2}{8g} = \frac{\pi nr^2v^2s^2}{2g}$.

But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a further investigation becomes necessary, such as in the following proposition.

PROBLEM XXIII.

To determine the resistance of a fluid to any body, moving in it, of a curved end; as a sphere, or a cylinder, with a hemispherical end, &c.



1. Let BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also, draw the perpendicular ordinates EF, ef, indefinitely near each other; and draw ae parallel to CG.

Putting CF $= x$ EF $= y$, BE $= z$, $s =$ sine $\angle$ G to radius 1, and $\pi = 3{\cdot}1416$: then $2\pi y$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving upon the axis CA; and $2\pi y \times$ Ee or $2\pi y\dot{z}$ is the fluxion of the surface, or it is the surface described by Ee in the said revolution about CA, and which is the quantity represented by a in art. 3 of the last problem: hence $\frac{nv^2s^3}{2g} \times 2\pi y\dot{z}$ or $\frac{\pi nv^2s^3}{g} \times y\dot{z}$ is the resistance on that ring or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.

2. In the case of a spherical form: putting the radius CA or CB $= r$, we have $y = \sqrt{(r^2 - x^2)}$, $s = \frac{\text{EF}}{\text{EG}} = \frac{\text{CF}}{\text{CE}} = \frac{x}{r}$, and $y\dot{z}$ or EF $\times$ Ee = CE $\times$ $ae = r\dot{x}$; therefore the general fluxion $\frac{\pi nv^2}{g} \times s^3y\dot{z}$ becomes $\frac{\pi nv^2}{g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{\pi nv^2}{gr^2} \times x^3\dot{x}$; the fluent of which, or $\frac{\pi nv^2}{4gr^2}x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{\pi nv^2r^2}{4g}$ for the resistance on the whole hemisphere; which is also equal to $\frac{\pi nv^2d^2}{16g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 5 of the preceding problem, is $\frac{\pi nv^2d^2}{8g}$; which, being double the former, shows that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter *.

* It may be here specified as a useful practical example, that a 9lb. iron ball, whose diameter is 4 inches, when moving through the air with a velocity of 1600 feet per second, would meet a

4. Since $\frac{1}{6}\pi d^3$ is the magnitude of the globe; if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}\pi d^3 \mathrm{N}$, and therefore the retardive force f or $\frac{m}{w} = \frac{\pi n v^2 d^2}{16g} \times \frac{6}{\pi \mathrm{N} d^3} = \frac{3nv^2}{8g\mathrm{N}d}$; which is also $= \frac{v^2}{2gs}$ by* article 8 of the general theorems; hence then $\frac{3n}{4\mathrm{N}d} = \frac{1}{s}$, and $s = \frac{\mathrm{N}}{n} \times \frac{4}{3}d$; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by the accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it can increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, $\mathrm{N} - n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}\pi d^3 (\mathrm{N} - n)$ is the weight by which it is urged; also $m = \ . \ . \ . \ . \ :$ $\frac{\pi n v^2 d^2}{16g}$ is the resistance; consequently $\frac{\pi n v^2 d^2}{16g} = \frac{1}{6}\pi d^3 (\mathrm{N} - n)$ when the velocity becomes uniform; from which equation is found $v = \sqrt{(2g \cdot \frac{4}{3}d \cdot \frac{\mathrm{N} - n}{n})}$*, for the said uniform or greatest velocity.

And, by comparing this form with that in art. 6 of the general theorems, it will appear that its greatest velocity is equal to the velocity generated by the accelerating force $\frac{\mathrm{N} - n}{n}$, in describing the space $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely describing the space $\frac{\mathrm{N} - n}{n} \times \frac{4}{3}d$. If $\mathrm{N} = 2n$, or the specific gravity of the globe be double that of the fluid, then $\frac{\mathrm{N} - n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greatest velocity in describing $\frac{4}{3}d$ or $\frac{4}{3}$ of its diameter.—It is further evident that if the body be very small, it will very soon acquire its greatest velocity, whatever its density may be.

Ex. If a leaden ball, of 1 inch diameter, descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as $11\frac{1}{3}$,

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resistance which is equal to a weight of $132\frac{2}{3}$lb. over and above the pressure of the atmosphere, for want of the counterpoise behind the ball.

Resistance to ball or shell, in lbs. $= \cdot 0000032389\ d^2$ (in inches) $\times\ v^2$ (in feet) $= \cdot 00000324\ d^2 v^2$ nearly.

Log. resistance $= 2 \log. d + 2 \log. v. + \bar{6}\cdot 5103978$. These theorems will be very correct up to the velocity of 1300 or 1400 feet.

Thus we find that with velocities of 200, 400, 800, 1200, 1600 feet respectively, a 9lb. iron ball experiences a resistance of 2·070, 8·2915, 33·166, 74·624, and 132·661bs. respectively.

* About $\frac{6}{7}$ of the results furnished by this theorem give the experimental results. The reason is easily assigned.

and 1, and $\frac{3}{2500}$. Then $v = \sqrt{(4 \cdot 16\frac{1}{12} \cdot \frac{4}{36} \cdot 10\frac{1}{3})} = \frac{1}{6}\sqrt{(31 \cdot 193)} = 8{\cdot}5944$ feet, is the greatest velocity per second the ball can acquire by descending in water. And $v = \sqrt{4 \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3})}$ nearly $= \frac{50}{9}\sqrt{\frac{34}{3}193} = 259{\cdot}82$ is the greatest velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greatest velocities it could acquire, would be only $\frac{1}{10}$ of these, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were still further diminished, the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

PROBLEM XXIV.

To determine the relations of velocity, space, and time, of a ball moving in a fluid, in which it is projected with a given velocity.

1. Let a = the first velocity of projection, x the space described in any time t, and v the velocity then. Now, by art. 4 of the last problem, the accelerative force $f = \frac{3nv^2}{8gNd}$, where n is the density of the fluid, N that of the ball, and d its diameter. Therefore the general equation $v\dot{v} = gf\dot{s}$ becomes $v\dot{v} = \frac{-3nv^2}{8Nd}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8ND}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8Nd}$. The correct fluent of this, is log. a — log. v or log. $\frac{a}{v} = bx$. Or, putting $c = 2{\cdot}718281828$, the number whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time, will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx} - 1}{c^{bx}}$ part of the first velocity.

EXAMPLES.

1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8Nd} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and $\frac{c^{bx} - 1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$ is the velocity lost, or nearly $\frac{2}{3}$ of the projectile velocity.

2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $N = 7\frac{1}{3}$, $n = {\cdot}0012$; and therefore $bx = \frac{3nx}{8Nd} = \frac{3 \cdot 12 \cdot 500 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per second: having lost 202 feet, or nearly $\frac{1}{6}$ of its first velocity.

3. If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a year. Then, if the earth's mean density be about $4\frac{1}{2}$, and its distance from the sun 12000 of its diameters, we have $24000 \times 3{\cdot}1416 = 75398$ diameters $= x$, and $bx =$ . . . .

$\frac{3 \,.\, 75398 \,.\, 12 \,.\, 2}{8 \,.\, 10000 \,.\, 9} = 7{\cdot}5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{1880}{1881}$ parts are lost of the first motion in the space of a year, and only the $\frac{1}{1881}$ part remains. If the earth's mean density be taken $= 5$, the result will become $\frac{885}{886}$ for the motion lost.

4. If it be required to determine the distance moved, x, when the globe has lost any part of its motion, as suppose $\frac{1}{2}$, and the density of the globe and fluid equal; the general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log.$ of $2 = 1{\cdot}8483925d$. So that the globe loses half its motion before it has described twice its diameter.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx = \log. z$, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; consequently $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanish together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}\ (\frac{1}{v} - \frac{1}{a})$ the time sought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

Exam. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per second; and it were required to determine in what time it would pass over 500 yards or 1500 feet, and what would be its velocity at the end of that time: we should have, as in exam. 2. above,

$b = \frac{3 \,.\, 12 \,.\, 3 \,.\, 6}{8 \,.\, 22 \,.\, 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1{\cdot}7372}{1200} = \frac{1}{690}$ nearly. Consequently $v = 690$ is the velocity; and $t = \frac{1}{b}\ (\frac{1}{v} - \frac{1}{a}) = 2716 \times (\frac{1}{690} - \frac{1}{1200}) = 1\frac{31}{46}$ seconds is the time required, or 1″ and $\frac{2}{3}$ nearly.

PROBLEM XXV.

To determine the relations of space, time, and velocity, when a globe descends by its own weight in a fluid.

The foregoing notation remaining, viz. $d =$ diameter, N and n the density of the ball and fluid, and v, s, t, the velocity, space, and time, in motion; we have $\frac{1}{6}\pi d^3 =$ the magnitude of the ball, and $\frac{1}{6}\pi d^3(N - n) =$ its weight in the fluid, also $m = \frac{\pi n d^2 v^2}{16g} =$ its resistance from the fluid; conseq. $\frac{1}{6}\pi d^3(N - n) - \frac{\pi n d^2 v^2}{16g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}\pi N d^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{N} - \frac{3nv^2}{8gNd}$ for the accelerative force.

2. Hence $v\dot{v}, = gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{gf} = \frac{\mathrm{N}v\dot{v}}{g\,(\mathrm{N} - n) - \frac{3n}{8d}v^2}$

$= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8\mathrm{N}d}$, and $\frac{1}{a} = \frac{3\mathrm{N}}{g\,.\,8d\,(\mathrm{N} - n)}$, or $ab = g$ nearly; the fluent of which is $s = \frac{1}{2b} \times$ log. of $\frac{a}{a - v^2}$, an expression for the space s, in terms of the velocity v. That is, when s and v begin, or are equal to nothing, both together.

But if the body commence motion in the fluid with a certain given velocity e, or enter the fluid with that velocity, like as when the body, after falling in empty space from a certain height, falls into a fluid like water; then the correct fluent will be $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a - e^2}{a - v^2}$.

3. But now to determine v in terms of s, put $c = 2{\cdot}718281828$: then, since the log. of $\frac{a}{a - v^2} = 2bs$, therefore $\frac{a}{a - v^2} = c^{2bs}$, or $\frac{a - v^2}{a} = c^{-2bs}$; hence $v = \sqrt{a - ac^{-2bs}}$ is the velocity sought.

4. The greatest velocity is to be found, as in art. 5 of prob. 18, by making f or $1 - \frac{n}{\mathrm{N}} - \frac{3nv^2}{8g\mathrm{N}d} = 0$, which gives $v = \sqrt{(g\,.\,8d\,.\,\frac{\mathrm{N} - n}{3n}} = \sqrt{a}$. The same value of v is obtained by making the fluxion of v^2, or of $a - ac^{-2bs}, = 0$. And the same value of v is also obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$ continually approaches. It is evident, however, that it will approximate towards it the faster, the greater b is, or the less d is; and that, the diameters being very small, the bodies descend by nearly uniform velocities, which are directly in the subduplicate ratio of the diameters. See also art. 5, prob. 18, for other observations on this head.

5. To find the time t. Now $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{(1 - c^{-2bs})}$. Then, to find the fluent of this fluxion, put $z = \sqrt{(1 - c^{-2bs})} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1 - z^2}$, consequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1 - z^2}$, and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times$ log. $\frac{1 + z}{1 - z} = \frac{1}{2b\sqrt{a}} \times$ log. $\frac{1 + \sqrt{(1 - c^{-2bs})}}{1 - \sqrt{(1 - c^{-2bs})}} = \frac{1}{2b\sqrt{a}} \times$ log. $\frac{\sqrt{a} + v}{\sqrt{a} - v}$, which is the general expression for the time.

Or thus: because $\dot{s} = \frac{1}{b} \cdot \frac{v\dot{v}}{a - v^2}$, theref. $\dot{t} = \frac{1}{b} \cdot \frac{\dot{v}}{a - v^2}$; and the fluent, by form 10, is $\frac{1}{2b\sqrt{a}} \times$ log. $\frac{\sqrt{a} + v}{\sqrt{a} - v}$.

Ex. If it were required to determine the time and velocity, by descending in air 1000 feet, the ball being of lead, and 1 inch diameter.

Here $\mathrm{N} = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \cdot 16 \cdot \frac{1}{12} \cdot \frac{3}{16} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} = \frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}$,

and $b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{12}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68 \cdot 50^2}$; consequently $v = \sqrt{a} \times \sqrt{(1 - c^{-2bs})} = \sqrt{\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}} \times \sqrt{(1 - c^{-\frac{81}{85}})} = 203\frac{1}{4}$ the velocity. And $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{2 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \sqrt{\frac{34 \cdot 2500}{27 \cdot 193}} \times \log. \frac{1{\cdot}78383}{0{\cdot}21617} = 8{\cdot}5236''$, the time.

Note.—If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{8gNd}$; and then proceed in all respects as before.

SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, the few following numbers are here extracted from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made in the course of the year 1786.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude of the cone nearly equal to the diameter also, when urged by the several weights expressed in avoirdupois ounces, and standing on the same line with the velocities, each in their proper column. So, in the first line, the numbers show, that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by ·028oz. when the vertex of the cone went foremost; by ·064oz. when the base of the cone went foremost; by ·027oz. for a whole sphere; by ·050oz. for a cylinder; by ·051oz. for the flat side of the hemisphere; and by ·020oz. for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures in the nearest whole numbers. Note, the common diameter of all the figures was 6·375, or $6\frac{3}{8}$ inches; so that the area of the circle of that diameter is just 32 square inches, or $\frac{2}{9}$ of a square foot; and the altitude of the cone was $6\frac{5}{8}$ inches. Also, the diameter of the small hemisphere was $4\frac{3}{4}$ inches, and consequently the area of its base $17\frac{3}{4}$ square inches, or $\frac{1}{8}$ of a square foot nearly.

From the given dimensions of the cone, it appears, that the angle made by its side and axis, or direction of the path, is 25° 42′, very nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; consequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$oz. nearly, in those circumstances.

Veloc. per sec	Cone.		Whole globe.	Cylinder.	Hemisphere.		Small Hemis. flat.
	vertex.	base.			flat.	round.	
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	·028	·064	·027	·050	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·297	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor Numb.	126	291	124	285	288	119	140

From this table of resistances, several practical inferences may be drawn. As,

1. That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater surfaces. Thus, by comparing together the numbers in the 6th and last columns, for the bases of the two hemispheres, the areas of which are in the proportion of 17¼ to 32, or as 5 to 9 very nearly; it appears that the numbers in those two columns, expressing the resistances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the resistances on the greater surface increase gradually more and more above that proportion. And the mean resistances are as 140 to 288, or as 5 to 10⅔. This circumstance therefore agrees nearly with the theory.

2. The resistance to the same surface, is nearly as the square of the velocity; but gradually increasing more and more above that proportion, as the velocity increases. This is manifest from all the columns. And therefore this circumstance also differs but little from the theory, in small velocities.

3. When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts be alike; owing to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder, is less than that on the flat base of the hemisphere, or of the cone; because the hinder part of the cylinder is more pressed or pushed, by the following air, than those of the other two figures.

4. The resistance on the base of the hemisphere, is to that on the convex side, nearly as 2⅜ to 1, instead of 2 to 1, as the theory assigns the proportion. And the experimented resistance, in each of these, is nearly ⅛ part more than that which is assigned by the theory.

5. The resistance on the base of the cone is to that on the vertex, nearly as $2\frac{3}{10}$ to 1. And in the same ratio is radius to the sine of the angle of the inclination of the side of the cone, to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same, instead of the square of the sine.

6. Hence we can find the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity. Thus,

Let $a =$ the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion;

$r =$ the resistance to the velocity, in the table; and

$x =$ the altitude sought, of a column of air, whose base is a, and its pressure r.

Then $ax =$ the content of the column in feet,

and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces; . . . .

therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude sought in feet, namely, $\frac{5}{6}$ of the quotient of the resistance of any body divided by its transverse section; which is a constant quantity for all similar bodies, however different in magnitude, since the resistance r is as the section a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the small hemisphere: then $x = \frac{5}{6} \times \frac{r}{a}$ becomes $x = \frac{15}{4}r$, where r is the resistance in the table, to the similar body.

If, for example, we take the convex side of the large hemisphere, whose resistance is ·634oz. to a velocity of 16 feet per second, then $r = {\cdot}634$, and $x = \frac{15}{4}r = 2{\cdot}3775$ feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the pressure And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whose resistance $r = 1{\cdot}526$: then $x = \frac{15}{4}r = 5{\cdot}72$; which exceeds the height, 4, due to the velocity, in the ratio of 23 to 16 nearly. And the difference would be still greater, if the body were larger; and also if the velocity were more.

7. Also, if it be required to find with what velocity any flat surface must be moved, so as to suffer a resistance just equal to the whole pressure of the atmosphere:

The resistance on the whole circle whose area is $\frac{2}{9}$ of a foot, is ·051oz. with a velocity of 3 feet per second; it is $\frac{1}{9}$ of ·051, or ·0056oz. only, with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$oz. is the whole pressure of the atmosphere. Therefore, as $\sqrt{{\cdot}0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum *.

8. Hence may be inferred the great resistance suffered by military projectiles. For, in the table, it appears, that a globe of $6\frac{3}{8}$ inches diameter, which is equal

* Wind moving at the rate of 80 miles an hour will exert a force of 21·6lbs. on a square foot.

to the size of an iron ball weighing 26lb., moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce weight: and therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb., and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 487lb. more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb. to such a velocity.

9. Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air: let us therefore inquire what this velocity is. Now the velocity with which any fluid issues, depends on its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of a uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 14 \times 833\frac{1}{3}$ or 29167 feet; therefore $\sqrt{16} : \sqrt{29167} :: 32 : 8\sqrt{29167} = 1366$ feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1366 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its fore part, as well as the resistance arising from the *vis inertiæ* of the particles of air struck by the ball.

10. On the whole, we find that the resistance of the air, as determined by the experiments, differs very widely, both in respect to its quantity on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory; which accords with that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude therefore, that all the theories of the resistance of the air hitherto given, are erroneous. And the preceding one is only laid down, till further experiments, on this important subject, shall enable philosophers to deduce from them another, that shall be more consonant to the true phænomena of nature.

APPENDIX.

ELEMENTS OF THE DIFFERENTIAL CALCULUS.

ON FUNCTIONS IN GENERAL, AND ON DIFFERENTIALS.

1. Quantities which are supposed capable of admitting an indefinite number of different values are called *variable quantities*; while those which always retain the same value are called *constant quantities*. Constant quantities are usually denoted by the first letters of the alphabet, and variable quantities by the last letters.

An expression which comprises both variable and constant quantities is called *a function*: often, too, if two variable quantities are so related that the value of each depends on that of the other, such relation is expressed briefly by saying that each of them is *a function* of the other: thus, the sine of an arc is a function of the arc; and the arc is a function of a sine. Thus, again, for another example, the expression ax^2, where we regard a as constant and x as variable, is a function. When we give to x successively all its determinate values, the expression ax^2 will obtain a different value for each of them: for example, making $x = a, = 2a, = 3a$, &c. the value of the expression, or of the function ax^2, will become a^3, $4a^3$, $9a^3$, &c.

2. Functions are first divided, according to the number of variable quantities which they contain, into functions of 1, 2, 3, or more variable quantities. Thus, for example, the expression ax^2, specified above is a function of a single variable quantity; but the expression $ay + bx + c$ is a function of two variables, in which both x and y are regarded as variable. Moreover, the expression $az + by + cx + d$, in which z, y, and x are considered variable, is a function of three variables; and so on for many variables.

In the second place, functions are divided according to the prescribed operations which ought to be performed on the variable. When the terms containing the variable are joined by addition, subtraction, multiplication, or division, or when the variable is found in a power whose exponent is an integral or fractional number, positive or negative, but constant, the function is called ***algebraic*** (rational or irrational, according as the exponent is an integral or fractional

number). But if the function contains the variable in the exponent, or under the sign of a logarithm, or as an arc, or finally, as a trigonometrical line, it is called *transcendental.*

Functions are also divided into integral and fractional: the former containing the variable only in the numerator; the latter containing it in the denominator only, or in both numerator and denominator.

Examples. $ax + b^{\frac{1}{2}} x^2$ is an algebraic rational integral function;

$(a^2 - x^2)^{\frac{1}{2}}$, an algebraic irrational integral function;

a . arc (sin. $= x$), or a sin.^{-1}x, a transcendental integral function;

$\frac{\log a}{x}$, an algebraic fractional rational function;

$\frac{a}{b^x}$, a transcendental fractional function;

These are all functions of a single variable quantity.

$\frac{a^2x}{y}$ is an algebraic function of two variables, integral in respect of x, and fractional in respect of y.

$\frac{ax}{\log y} \frac{b}{z}$ is a function of three variables, algebraic and integral in respect of x, transcendental and fractional in respect of y, and algebraic and fractional in respect to z.

3. Equations, like functions, are divided into equations of 1, 2, 3, or more variables, according as they contain 1, 2, 3, or more variables. Thus, for example,

$ax^2 = c$ is an equation of one variable;

$\frac{ax}{y} + y = \frac{x^2}{c}$ is an equation of two variables.

Equations are also divided into *explicit* and *implicit* equations. In the former the variable is found only in one of the two members of the equation; in the latter the variables are found blended.

Any implicit equation may be changed into an explicit equation, by separating one of the variables which it contains. Thus the implicit equation $y^2 + x^2 = a^2$, may, by the separation of y, be changed into the explicit equation $y = (a^2 - x^2)^{\frac{1}{2}}$. It will be at once perceived, that this separation of one of the variables, or the proposed transformation of an implicit equation into an explicit equation, is liable to the same difficulties as those which are experienced in the solution of equations of one unknown quantity.

It is moreover evident that the function of one variable may be transformed into an explicit equation of two variables, by making the function equal to one of the variables. Thus, if y be the function of a single variable ax^2, we have the explicit equation of two variables $y = ax^2$.

In the same manner a function of two variables may be changed into an explicit equation of three variables; and so on for many variables. Thus, for example, from the expression $x^2 + y^2 - a^2$, may be produced the explicit equation $z = x^2 + y^2 - a^2$.

4. In determining x arbitrarily, in an explicit or implicit equation of two variables x and y, the other variable y will be entirely determined by the equation, so that the determinate values of the variables x and y cannot be taken independently of one another; so that x having obtained a certain value, y can-

not take any arbitrary value. Thus, in the equation $ay + bx = xy$, making x equal to c, we have $y = \frac{bc}{c - a}$.

In this example, after determining the arbitrary value of x, we have found the value of y. But we might otherwise begin by determining the arbitrary value of y, and then finding x in the equation.

In an equation of two variables, that which may be determined at pleasure is called the *independent* or *primitive* variable, and the other, whose value must afterwards be found in the given equation, is called the *dependent* variable.

We shall for the future represent the independent variable by the letter x, and the dependent variable by the letter y, except where a particular limitation is made.

In an explicit equation of two variables, the dependent variable y as a function is sometimes expressed by $f(x)$ or $\phi(x)$

The form, or general expression of explicit equations of two variables, is $y = f(x)$, when $f(x)$ contains only the single independent variable x, besides the constant quantities.

5. Let us now place, in an explicit equation of two variables, $x + \Delta x$ instead of the independent x; the function $y = f(x)$ is then changed into

$$y + \Delta y = f(x + \Delta x).$$

According to this notation, Δx indicates the change or difference of the independent x, Δy the change or difference of the dependent or function y, which results from it; then $x + \Delta x$ will indicate the modified independent variable, and $y + \Delta y$ the modified dependent variable. We express by $f(x)$ the proposed or primitive function, by $y = f(x)$ the proposed or primitive equation.

We now propose to develope $f(x + \Delta x)$ in a series arranged according to the integral and positive powers of Δx. Let

$$y + \Delta y = f(x + \Delta x) = \mathrm{A} + \mathrm{B} \quad \Delta x + \mathrm{C} \ . \ \Delta x^2 + \ldots .$$

A, B, C, &c. contain only x, and not Δx; we must now determine them.

The coefficient A may be immediately determined; for when we make in the above series $\Delta x = 0$, Δy will also be equal to zero, and the series is reduced to its first term; we then have $y = \mathrm{A}$, that is to say, A is the primitive function. Removing $y = \mathrm{A}$ from the equation, we have

$$\Delta y = \mathrm{B} \ . \ \Delta x + \mathrm{C} \ . \ \Delta x^2 + \ldots,$$

or the change of a function expressed by a series arranged according to the integral powers of Δx.

This series may also be written

$$\Delta y = y'\Delta x + y''\Delta x^2 + \ldots\ldots,$$

$$\text{or, } \Delta y = f'(x)\,\Delta x + f''(x) \ . \ \Delta x^2 + \ldots\ldots,$$

$$\text{or abridged, } \Delta y = y'\Delta x + \psi\Delta x^2,$$

$$\text{where } \psi = y'' + y'''\Delta x + \ldots\ldots.$$

Example. Let the primitive function be $y = ax^2$; we then have

$$y + \Delta y = a \ . \ (x + \Delta x)^2 = ax^2 + 2ax\Delta x + a\Delta x^2;$$

taking away $y = ax^2$, there remains

$$\Delta y = 2ax\Delta x + a\Delta x^2.$$

So that the coefficients A, B, C, D . . ., are here determined by the values ax^2, $2ax$, a, 0 . . .

6. Dividing by Δx the two members of the series,

$$\Delta y = y'\Delta x + y''\Delta x^2 + \ldots\ldots,$$

$$\text{we have } \frac{\Delta y}{\Delta x} = y' + y''\Delta x + \ldots.,$$

where $y' + y''\Delta x + \ldots$ is the exponent of the ratio of the two changes, Δx and Δy, or the ratio of *the differences*. The first term of the ratio of the differences is therefore y'. The notation most in use for this term, besides $f'(x)$, of which we have treated above, is $\frac{dy}{dx}$. The first notation of the first term of $\frac{\Delta y}{\Delta x}$ indicates that it is a function of x, and more particularly a function derived from $y = f(x)$ (art. 5); the last announces that it has its origin in $\frac{\Delta y}{\Delta x}$. $\frac{dy}{dx}$ is commonly called the *differential coefficient*, which it is the object of the *differential calculus* to find. The expression $y' = \frac{dy}{dx}$ is usually denoted by $dy = y'dx$, or $dy = \frac{dy}{dx} \,.\, dx$, and dy is called the *differential* of y, dx that of x; and the differential of y is said to be equal to the quantity y' or $\frac{dy}{dx}$, multiplied by the differential of x. But dy and dx must not be regarded as true quantities in themselves, nor $y'dx$ and $\frac{dy}{dx} \,.\, dx$ as real products, nor $\frac{dy}{dx}$ as a real quotient.

When henceforward we seek the differential of a function, we shall only require to find the differential coefficient of that function, and then express it under the form of the differential.

The differential coefficient $\frac{dy}{dx}$ is also expressed, often advantageously, by p, and consequently the differential of a function y by $dy = p \,.\, dx$.

We have found in the above example

$$y = ax^2, \ \Delta y = 2ax\Delta x + a\Delta x^2,$$

whence results the ratio of the differences

$$\frac{\Delta y}{\Delta x} = 2ax + a\Delta x;$$

We have here, therefore, the differential coefficient

$$y' = \frac{dy}{dx} = p = 2ax,$$

and the differential is

$$dy = 2ax \,.\, dx.$$

We have just shown that if y is the function of x, we have

$$\Delta y = y'\Delta x + y''\,\Delta x^2 + \ldots,$$

$$\text{and } dy = y'dx, \text{ or } dy = \frac{dy}{dx} \,.\, dx.$$

Consequently, we have in the primitive equation $x = f(y)$, considering x reciprocally as the function of y,

$$\Delta x = \frac{dx}{dy} \,.\, \Delta y + x'' \,.\, \Delta y^2 + \ldots.,$$

$$\text{and } dx = \frac{dx}{dy} \,.\, dy.$$

Moreover, if for example z is the function of u, we have

$$\Delta z = \frac{dz}{du} \,.\, \Delta u + z'' \,.\, \Delta u^2 + \ldots,$$

$$\text{and } \Delta z = \frac{dz}{du} \,.\, du.$$

Reciprocally, the expression $\frac{du}{dy} \cdot dy$ being given, it may be otherwise represented by du, which may be substituted for it.

Finally, we may derive the difference from a given differential; suppose

$$dy = \frac{dy}{dx} \cdot dx,$$

$$\text{we have } \Delta y = \frac{dy}{dx} \cdot \Delta x + y'' \cdot \Delta x^2 + \ldots.$$

7. In considering the expression

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + y'' \cdot \Delta x + \ldots,$$

it will be at once evident that the value of $\frac{\Delta y}{\Delta x}$ (that is to say, the series $\frac{dy}{dx} + y''\Delta x + \ldots$), supposing $\Delta x = 0$, reduces itself to $\frac{dy}{dx}$, that is to say, to the differential coefficient. This value will differ very slightly indeed from $\frac{dy}{dx}$, if Δx is supposed very small; and this difference will continue to diminish as Δx becomes smaller. We may therefore regard $\frac{dy}{dx}$ as the limit of the value of $\frac{\Delta y}{\Delta x}$, since $\frac{\Delta y}{\Delta x}$ approaches this limit in proportion as Δx becomes smaller, but that it only reaches the limit when Δx is equal to zero. $\frac{dy}{dx}$ is therefore the limit of the ratio of the differences of y and x; which is also called the *last* or *ultimate* ratio of the differences of x and y themselves. According to this, it will be seen (art 6.) how the notations $\frac{dy}{dx}$ and $\frac{\Delta y}{\Delta x}$ express the intimate connection which subsists between the values which they indicate.

8. It will also be easily seen, after what has been said, how to proceed with regard to functions which contain many terms. Let

$$y = u + v + z + \ldots,$$

in which u, v, z, ... are functions of x.

Putting $x + \Delta x$ instead of x, we have (according to art. 5 and 6), instead of u,

$$u + \Delta u = u + \frac{du}{dx}\Delta x + u''\Delta x^2 + \ldots,$$

instead of v, $v + \Delta v = v + \frac{dv}{dx}\Delta x + v''\Delta x^2 + \ldots,$

instead of z, $z + \Delta z = z + \frac{dz}{dx}\Delta x + z''\Delta x^2 + \ldots$, &c.

We have then

$$y + \Delta y = u + \Delta u + v + \Delta v + z + \Delta z + \ldots = u + \frac{du}{dx}\Delta x + u''\Delta x^2 + \ldots$$
$$+ v + \frac{dv}{dx}\Delta x + v''\Delta x^2 + \ldots + z + \frac{dz}{dx}\Delta x + z''\Delta x^2 + \ldots$$

Taking away the primitive function, we have

$$\Delta y = \Delta u + \Delta v + \Delta z + \ldots = \left(\frac{du}{dx} + \frac{dv}{dx} + \frac{dz}{dx} + \ldots\right)\Delta x$$
$$+ (u'' + v'' + z'' + \ldots)\Delta x^2 + \ldots;$$

whence we obtain (according to art. 6),

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dz}{dx} + \quad . \quad . \quad . \quad . \quad . \quad ,$$

that is to say, that a function composed of many functions of the same independent variable, joined by means of addition or subtraction, has for its difference the sum of the differences of those functions, and for its differential co-efficient the sum of the differential co-efficients of each of those functions.

Example.—Let $y = ax^2 + bx$; we have

$$u = ax^2\,,\ v = bx\,;$$

$$\text{Whence } \frac{du}{dx} = 2ax\,,\ \frac{dv}{dx} = b\,;$$

$$\text{we then have } \frac{dy}{dx} = 2ax + b.$$

$$\text{From the expression } \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dz}{dx} + \quad . \quad . \quad .$$

we may also determine, according to art. 6,

$$dy = \left(\frac{du}{dx} + \frac{dv}{dx} + \frac{dz}{dx} + \ . \ . \ . \ \right) dx = \frac{du}{dx}\,dx + \frac{dv}{dx}\,dx + \frac{dz}{dx}\,.\,dx + \quad . \quad .$$

We have also (art. 6.)

$$dy = du + dv + dz + \quad . \quad . \quad . \quad ,$$

a proposition which is thus expressed; the differential of a function which is composed of several functions of the same independent variable, is equal to the sum of the differentials of each of those functions.

Thus, for example, we have in the function $y = ax^2 + bx$, ...

$$dy = 2axdx + bdx.$$

When one of the terms of which y is composed is constant, it is manifest that this portion has no influence in the differential of the function. Thus the functions

$$y = ax^2 + bx \text{ and } y = ax^2 + bx + c,$$

have the same differential.

9. In the preceding article we have considered x as the independent in the equation $y = f(x)$. Now let y be the independent. According to the first supposition, we have (art. 6,) $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + y''\,\Delta\,x + \ . \ . \ . \ . \ . \ ,$

$$\text{whence } \frac{\Delta x}{\Delta y} = 1 \div \left(\frac{dy}{dx} + y''\,\Delta\,x + \ . \ . \ .\right)$$

Placing in the second member of this equation, y being now the independent (according to art. 6,)

$$\Delta x = \frac{dx}{dy}\,\Delta y + x''\,\Delta y^2 + \quad . \quad . \quad . \quad ,$$

$$\text{we obtain } \frac{\Delta x}{\Delta y} = 1 \div \left[\frac{dy}{dx} + y''\left(\frac{dx}{dy}\,\Delta y + \quad . \ . \ . \right) + \ . \ . \ . \right];$$

$$\text{Now put } 1 \div \left[\frac{dy}{dx} + y''\left(\frac{dx}{dy}\,\Delta y + \ . \ . \ .\right) + \ . \ . \ .\right] = a + \beta\Delta y + \ . \ . \ .$$

In determining a by the method of indeterminate coefficients, vol. i. p. 267, we obtain

$$a = 1 \div \frac{dy}{dx}.$$

It follows (art. 6,) that

$$\frac{dx}{dy} = 1 \div \frac{dy}{dx}.$$

Example. In the equation $y = ax^2$, we find the differential coefficient.

$$\frac{dy}{dx} = 2ax \text{ (art. 6)};$$

we have therefore, as we have just proved,

$$\frac{dx}{dy} = \frac{1}{2ax}\left[x \text{ being } = \left(\frac{y}{a}\right)^{\frac{1}{2}}\right] = \frac{1}{2 \cdot (ay)^{\frac{1}{2}}}.$$

an expression of the differential coefficient of the function $x = \left(\frac{y}{a}\right)^{\frac{1}{2}}$, in which y is considered as the independent.

Hence it is evident that we may work the expression $\frac{dy}{dx}$ precisely as the fraction of which dy is the numerator, and dx the denominator; notwithstanding which, the assertion made in art. 6, that it is not a true fraction, remains in force.

10. Let y be expressed by z, and z by x, so that y shall be a mediate function of the independent x, as in the equation $y = z^2$, in which z is supposed $= ax$.

According to this supposition, we have

$$\Delta y = \frac{dy}{dz}\Delta z + y'' \Delta z^2 + \ldots,$$

$$\text{and } \Delta z = \frac{dz}{dx}\Delta x + z'' \Delta x^2 + \ldots;$$

by substituting, we have

$$\Delta y = \frac{dy}{dz} \cdot \frac{dz}{dx}\Delta x + \frac{dy}{dz} \cdot z'' \Delta x^2 + \ldots;$$

$$\text{whence } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \text{ and } dy = \frac{dy}{dz} \cdot \frac{dz}{dx} \cdot dx;$$

that is to say, that in order to obtain the differential coefficient of a function which is mediate of x and immediate of z, the two differential coefficients of the mediate function (y) and the immediate (z) must be multiplied.

We may also observe, that the same result may be obtained by substituting in the expressions

$$dy = \frac{dy}{dz} \cdot dz, \quad dz = \frac{dz}{dx} \cdot dx,$$

the value of dz, obtained from the last of these expressions, in the first; we then have, as above,

$$dy = \frac{dy}{dz} \cdot \frac{dz}{dx} \cdot dx.$$

Example. Required, to differentiate $y = z^2$, when $z = ax$.

We have (art. 6.) $\frac{dy}{dz} = 2z$ and $\frac{dz}{dx} = a$;

therefore . . . $\frac{dy}{dx} = 2z \cdot a = 2a^2 x$,

a result which would have been immediately obtained by differentiating the equation $y = a^2 x^2$.

11. Suppose $y = u \cdot v$; u, v being functions of x. Substituting $x + \Delta x$ for x, we obtain

$$y + \Delta y = (u + \Delta u) \cdot (v + \Delta v) = uv + v\Delta u + u\Delta v + \Delta u \cdot \Delta v.$$

Taking away the primitive equation, we have

$$\Delta y = v\Delta u + u\Delta v + \Delta u \cdot \Delta v.$$

In which we have $\Delta u = \frac{du}{dx}\Delta x + u'' \Delta x^2 + \ldots,$

$$\text{and } \Delta v = \frac{dv}{dx}\Delta x + v'' \Delta x^2 + \ldots \text{ (art. 6.)}$$

By substituting, we have

$$\Delta y = v \cdot \frac{du}{dx}\Delta x + u \cdot \frac{dv}{dx}\Delta x + v \cdot u'' \Delta x^2 + u \cdot v'' \Delta x^2 + \ldots$$

$$+ \left(\frac{du}{dx}\Delta x + \ldots\right) \cdot \left(\frac{dv}{dx}\Delta x + v'' \Delta x^2 + \ldots\right)$$

$$\text{whence } dy = v \cdot \frac{du}{dx} \cdot dx + u \cdot \frac{dv}{dx} \cdot dx.$$

Having also (art. 6.) $\frac{du}{dx} \cdot dx = du$, and $\frac{dv}{dx} \cdot dx = dv$.

we obtain at length

$$dy = vdu + udv;$$

that is to say, that in order to obtain the differential of the product of two functions of x, each must be multiplied by the differential of the other, and the two results added together.

If we have $y = a \cdot u$, a being constant, and u a function of x, we have, according to the proposition proved above [since $da = 0$, (art. 8.)], $dy = a \cdot du$.

Next, taking three functions of x, let $y = u \cdot v \cdot z$; supposing $u \cdot v = n$, we have

$$dy = ndz + zdn,\ dn, = udv + vdu;$$

substituting the values of n and of dn in the value of dy, we have (art 10.)

$$dy = uzdv + vzdu + uvdz.$$

Hence results the mode of finding the differential of a product of the functions of the same independent variable, whatever their number may be; thus: multiply the differential of each of these functions by the product of all the other functions, and add together the results.

12. We are now able to differentiate the equation $y = x^n$. We have $x^n = x \cdot x \cdot x \ldots$, in which the functions of x generally denoted in the preceding number by $u, v, z \ldots$ are consequently all equal to each other and at the same time equal to x. Each of the terms forming the differential will then be equal to $x^{n-1}\,dx$. Now these terms are in number n; we have therefore

$$dy = d \cdot x^n = nx^{n-1}\,dx.$$

According to art. 9, we shall easily find the differential of $y = x^n$, y being the independent variable, or of $x = y^{\frac{1}{n}}$ Thus, we have (art. 9.)

$$dx = \frac{1}{nx^{n-1}}dy \text{ [because } x = y^{\frac{1}{n}}] = \frac{dy}{ny^{\frac{1}{n}(n-1)}} = \frac{dy \cdot y^{\frac{1}{n}-1}}{n}.$$

Interchanging the letters x and y we have

$$dy = d \cdot x^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}\,dx.$$

Hence may be easily found, with the help of art. 10, the differential of $y = x^{\frac{m}{n}}$ putting $x^{\frac{1}{n}} = z$. We then have

$$y = z^m,\ z = x^{\frac{1}{n}};$$

therefore $dy = mz^{m-1}dz$, $dz = \frac{1}{n}x^{\frac{1}{n}-1}dx$;

consequently (art. 10.) $d . y = d . x^{\frac{m}{n}} = \frac{m}{n}x^{\frac{m}{n}-1}dx$.

Thus, in the function $y = x^m$, whether m be integral or fractional, but a positive number, we have always

$$dy = d . x^m = mx^{m-1}dx.$$

The rule is: to find the differential of x^m, m being an integral or fractional positive number, multiply dx by the exponent of the independent variable, and by the proposed power, its exponent being diminished by unity.

13. Let $y = \frac{u}{v}$, u and v being fractions of x; we have (art. 6.)

$$\Delta u = \frac{du}{dx}\Delta x + \ldots, \quad \Delta v = \frac{dv}{dx}\Delta x + \ldots .$$

We derive from the proposed function

$$v . y = u.$$

Whence we have, substituting $x + \Delta x$ for x,

$$(v + \Delta v)(y + \Delta y) = u + \Delta u,$$

$$\text{or, } vy + v . \Delta y + y . \Delta v + \Delta v . \Delta y = u + \Delta u,$$

taking away $vy = u$, we have

$$v . \Delta y + y . \Delta v + \Delta v . \Delta y = \Delta u,$$

whence

$$\Delta y = \frac{\Delta u - y . \Delta v}{v + \Delta v} = \frac{\frac{du}{dx}\Delta x + u''\Delta x^2 + \ldots - y\left[\frac{dv}{dx}\Delta x + \ldots\right]}{v + \frac{dv}{dx}\Delta x + v''\Delta x^2 + \ldots}.$$

Making the last fraction equal to the following series,

$$\alpha \Delta x + \beta\Delta x^2 + \ldots,$$

in which α, β, . . . are indeterminate co-efficients, we obtain

$$\frac{du}{dx} - y . \frac{dv}{dx} = v . \alpha,$$

$$\text{whence, } \alpha = \frac{\frac{du}{dx} - y . \frac{dv}{dx}}{v} = \frac{v . \frac{du}{dx} - u . \frac{dv}{dx}}{v^2};$$

$$\text{therefore (art. 6) } dy = \frac{v . \frac{du}{dx} . dx - u . \frac{dv}{dx} . dx}{v^2};$$

But, we have (art. 6) $\frac{du}{dx} . dx = du$, and $\frac{dv}{dx} . dx = dv$;

we shall therefore have, finally,

$$dy = d . \frac{u}{v} = \frac{vdu - udv}{v^2}.$$

Hence it results that the differential of a fraction whose numerator and denominator are functions of the same variable, is equal to the differential of the numerator multiplied by the denominator, minus the differential of the denominator multiplied by the numerator, and the result divided by the square of the denominator.

Let there be given the equation

$$y = x^{-m} = \frac{1}{x^m}, \text{ in which } u = 1,\ v = x^m,$$

then $du = 0$ (art. 8), $dv = mx^{m-1}dx$ (art. 12);

we have therefore

$$d \cdot \frac{1}{x^m} = \frac{-mx^{m-1}dx}{x^{2m}} = -mx^{-m-1}dx.$$

Let also

$$y = x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}}, \text{ in which } u = 1,\ v = x^{\frac{m}{n}},\ dv = \frac{m}{n}x^{\frac{m}{n}-1}dx \text{ (art. 12)};$$

then,

$$d \cdot x^{-\frac{m}{n}} = \frac{-\frac{m}{n}x^{\frac{m}{n}-1}dx}{x^{\frac{2m}{n}}} = -\frac{m}{n}x^{-\frac{m}{n}-1}dx,$$

whence we may infer that the differential of x^m, in which m is a negative number, integral or fractional, will be

$$mx^{m-1}dx.$$

Comparing this with what was said in the preceding article on the differential of x^m (in which m was regarded only as a positive number) we obtain the following proposition: dx^m is equal to $mx^{m-1}dx$, whether m designates an integral or fractional, positive or negative number; that is to say, the differential of x^m, whether m be a positive, negative, integral or fractional number, is equal to dx multiplied by the exponent of the variable, and by the power in which the exponent m is diminished a unit.

14. The rules enunciated in the preceding pages will be sufficient for the differentiation of all the algebraic functions. Some examples are added, of the application of these rules.

Ex. 1. Let $y = (a + bx^m)^n$.

Putting $a + bx^m = z$, we have

$$y = z^n,\ dy = nz^{n-1}dz \text{ (art. 12)};$$

$$dz = mbx^{m-1}dx \text{ (art. 8, 11, 12)};$$

therefore (art. 10), $dy = nmb\,(a + bx^m)^{n-1} \cdot x^{m-1}dx$.

Ex. 2. $y = \dfrac{ax}{x + (a + x^2)^{\frac{1}{2}}}$.

Comparing this function with u and v in art. 13, we have

$$u = ax,\ du = a \cdot dx \text{ (art. 11)},$$

$$v = x + (a + x^2)^{\frac{1}{2}}; \text{ put } a + x^2 = z, \text{ there results}$$

$$d \cdot (a + x^2)^{\frac{1}{2}} = \tfrac{1}{2}z^{-\frac{1}{2}}dz \text{ (art. 12)},$$

$$\text{and } dz = 2xdx \text{ (art. 8, 12)};$$

$$\text{therefore } d \cdot (a + x^2)^{\frac{1}{2}} = x(a + x^2)^{-\frac{1}{2}}dx \text{ (art. 10)};$$

$$\text{consequently } dv = dx + \frac{xdx}{(a + x^2)^{\frac{1}{2}}} \text{ (art. 8)},$$

whence $dy = d \cdot \dfrac{ax}{x + (a + x^2)^{\frac{1}{2}}}$

$$= \text{(art. 31.)} \frac{a\mathrm{d}x\,[x + (a + x^2)^{\frac{1}{2}}] - ax\mathrm{d}x \left[1 + \frac{x}{(a+x^2)^{\frac{1}{2}}}\right]}{[x + (a + x^2)^{\frac{1}{2}}]^2},$$

$$\text{or, by reduction,} = \frac{a^2\mathrm{d}x}{(a + x^2)^{\frac{1}{2}} \cdot [x + (a + x^2)^{\frac{1}{2}}]^2}.$$

The reader will do well to differentiate the following functions, carefully applying the enumerated rules, as has been done in the two preceding examples.

Ex. 3. $y = \frac{a}{x^{\frac{1}{2}}} + \left(a + \frac{b}{x}\right)^m$. Required, the differential coefficient.

$$\text{Answer.} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = p = \frac{-a}{2x^{\frac{3}{2}}} - \frac{bm}{x^2}\left(a + \frac{b}{x}\right)^{m-1}.$$

Ex. 4. $y = \frac{1 + x}{(1 - x)^{\frac{1}{2}} - (1 + x)^{\frac{1}{2}}}$. Required the differential coefficient.

$$\text{Answer.} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = p = \frac{3 - x - (1 - x^2)^{\frac{1}{2}}}{2(1 - x)^{\frac{1}{2}} \cdot [(1 - x)^{\frac{1}{2}} - (1 + x)^{\frac{1}{2}}]^2}.$$

OF DIFFERENTIAL COEFFICIENTS AND DIFFERENTIALS OF SUPERIOR ORDERS.

15. The differential coefficient $\frac{\mathrm{d}y}{\mathrm{d}x} = p$ being a function of x (art. 6), we may obtain from it, as from a primitive function, the difference, the ratio of the differences, and the differential coefficient. Now, the notation of the differential coefficient is $\frac{\mathrm{d}y}{\mathrm{d}x}$, if y is the primitive function; if then, we take $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a primitive function, we should, from analogy, designate the differential coefficient of the differential coefficient by $\frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x}$; but it is usually designated by $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, or by $f''(x)$; and frequently by the letter q. It is properly called *the second differential coefficient*, or *the differential coefficient of the second order* of the function y.

It is also written, as has been explained, for the first differential coefficient (art. 6), under this form,—

$$\mathrm{d}^2y = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \cdot \mathrm{d}x^2, \text{ or } \mathrm{d}^2y = q \cdot \mathrm{d}x^2,$$

considering in this expression d^2y as the second differential of y, and $\mathrm{d}x^2$ the square of the differential of x.

Now $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ being considered as the primitive function, we obtain the third differential coefficient of y, that is to say, $\frac{\mathrm{d}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)}{\mathrm{d}x}$. This notation is most frequently superseded by $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$, by r, or by $f'''(x)$. The third differential of the function y will consequently be

$$d^3y = \frac{d^3y}{dx^3} \cdot dx^3, \text{ or } d^3y = r \cdot dx^3.$$

From this the notation of differential coefficients, and of the ulterior differentials, may easily be settled.

Differential coefficients, and differentials above the first order, are called *differential coefficients of the superior orders*, or *superior differential coefficients*, and *superior differentials* of the primitive function y.

It is easily seen that the second differential

$$d^2y = \frac{d^2y}{dx^2} \cdot dx^2 = q \cdot dx^2$$

may be obtained by differentiating the first differential

$$dy = \frac{dy}{dx} \cdot dx = pdx,$$

in such manner that dy and dx are considered as true quantities, and pdx as a product; dy and p as variables, but dx as constant. In fact, we then obtain from $dy = \frac{dy}{dx} \cdot dx$,

$$d \cdot dy = d \cdot \frac{dy}{dx} dx,$$

$$\text{or, } d^2y = \frac{d^2y}{dx} dx \text{ (art. 11)};$$

multiplying by dx and dividing

$$= \frac{d^2y}{dx^2} \cdot dx^2, \text{ or } d^2y = q \cdot dx^2.$$

In the same way the third differential may be obtained from the second differential $d^2y = q \cdot dx^2$, differentiating so as to regard d^2y and q as variables, and dx^2 as constant; and so on for successive differentials.

Having in the preceding articles explained the method of differentiating any algebraic function, we are now prepared to find also the superior differentials, each of the successive differential coefficients of an algebraic function being also an algebraic function.

We may develop, by way of example, the differentials of the superior orders of the function $y = ax^m$. We have (art. 12)

$$\frac{dy}{dx} = max^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1)\,ax^{m-2},$$

$$\frac{d^3y}{dx^3} = m(m-1)\,(m-2)\,ax^{m-3}, \text{ \&c.}$$

Moreover, $dy = max^{m-1}dx$, $d^2y = m(m-1)ax^{m-2}dx^2$,
$d^3y = m(m-1)\,(m-2)ax^{m-3}dx^3$, &c.

Ex. 2. Find the successive differential coefficients of

$$y = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + gx + k.$$

ON THE TRANSFORMATION OF A FUNCTION IN A SERIES ARRANGED ACCORDING TO THE INTEGRAL POWERS OF THE VARIABLE.

16. We shall now, by means of the superior differentials, express a function of a single variable by a series arranged according to the integral powers of this variable.

Let the function be represented by y, and the variable included in it by x. Supposing at pleasure the series

$$y = A + Bx + Cx^2 + \ldots,$$

it is proposed to determine the co-efficients A, B, C . . ., that is to say, the series itself.

First, to determine A, x must be made equal to zero in the two members of the equation; the value to which y is reduced when x is made $= 0$, is represented by

$$y_{x=0};$$

we then have $A = y_{x=0}$.

The other coefficients may be determined, by making $x = 0$ in the successive differential coefficients of the equation

$$y = A + Bx + Cx^2 + Dx^3 + \ldots.$$

We thus have

$$p = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots$$
$$q = 2C + 6Dx + 12Ex^2 + \ldots$$
$$r = 6D + 24Ex + \ldots, \text{\&c.}$$

Therefore $p_{x=0} = B,\ q_{x=0} = 2C,\ r_{x=0} = 6D$, &c.;

in which the notation $x = 0$ joined to the differential coefficients (which might be expressed by the phrase, if x is made equal to zero) indicates that x ought then to be made $= 0$.

We shall thus obtain

$$B = p_{x=0},\ C = q_{x=0} \cdot \frac{1}{1\,.\,2},\ D = r_{x=0} \cdot \frac{1}{1\,.\,2\,.\,3}, \text{\&c.}$$

And the final result

$$y = y_{x=0} + p_{x=0} \cdot x + q_{x=0} \cdot \frac{x^2}{1.\,2} + r_{x=0} \cdot \frac{x^3}{1.\,2.\,3} + \ldots.$$

Example 1. The function $y = \frac{a}{b+x}$ is to be transformed into a series regulated according to the integral powers of x. We have here, according to the preceding rules for differentiating,

$$p = \frac{-a}{(b+x)^2},\ q = \frac{2a}{(b+x)^3},\ r = \frac{-6a}{(b+x)^4}, \ldots.$$

We have, moreover,

$$y_{x=0} = \frac{a}{b},\ p_{x=0} = \frac{-a}{b^2},\ q_{x=0} = \frac{2a}{b^3},\ r_{x=0} = \frac{-6a}{b^4}, \text{\&c.}$$

These values found, and substituted in the series developed above,

$$y = y_{x=0} + p_{x=0} \cdot x + q_{x=0} \cdot \frac{x^2}{1\,.\,2} + \ldots,$$

we obtain $\frac{a}{b+x} = \frac{a}{b} - \frac{a}{b^2} \cdot x + \frac{a}{b^3} \cdot x^2 - \frac{a}{b^4} \cdot x^3 + \ldots.$

We may obtain the same result by determining the indeterminate coefficients in the expression $\frac{a}{b+x} = A + Bx + Cx^2 + \ldots$ by the usual method; we may also obtain the same result by dividing a by $b + x$. The reader may easily prove this for himself by actual calculation.

Example 2. Suppose the function $y = (a + x)^m$, to be developed according to the integral powers of x. We have

$$p = m(a+x)^{m-1},\ q = m(m-1)(a+x)^{m-2};$$
$$r = m(m-1)(m-2)(a+x)^{m-3}, \ldots \&c.$$
$$y_{x=0} = a^m,\ p_{x=0} = ma^{m-1},\ q_{x=0} = m(m-1)a^{m-2}, \&c$$

These values substituted in the general series above, we have

$$y = (a+x)^m = a^m + \frac{m}{1} a^{m-1} x + \frac{m(m-1)}{1 \cdot 2} a^{m-2} x^2 + \ldots.$$

It is evident that the expression of this series retains its value, whatever number may be designated by m, whether a positive, negative, integral, or fractional number, because the differential coefficients already developed are the same for all cases (art. 12, 13). Thus the binomial theorem is demonstrated both generally and according to the affection of any exponent.

As, in an expression of many quantities, each of them may be regarded either as arbitrary or variable, unless there is something which determines it otherwise, the proposition enunciated in this article shows in fact how to develop an expression according to either of the quantities included in it. It is often differently expressed; but the same in effect: and is called, from its inventor, *Maclaurin's theorem*. Thus, in the first example, we may develop the function $y = \frac{a}{b+x}$ according to the integral powers of b, taking the values of

$$\frac{dy}{db} = \frac{-a}{(b+x)^2},\ \frac{d^2y}{db^2} = \frac{2a}{(b+x)^3}, \&c.,$$
$$p_{b=0} = \frac{-a}{x^2} \text{ (in which } p = \frac{dy}{db}\text{) } \&c.,$$

and the series in this case.

$$y = A + Bb + Cb^2 + \ldots$$

$$\text{becomes } \frac{a}{b+x} = \frac{a}{x} - \frac{a}{x^2} b + \frac{a}{x^3} b^2 - \ldots.$$

In the second example, if the function $y = (a+x)^m$ is developed with respect to a, we obtain

$$(a+x)^m = x^m + \frac{m}{1} x^{m-1} a + \frac{m(m-1)}{1 \cdot 2} x^{m-2} a^2 + \ldots.$$

17. Given a function developed according to the integral powers of x, then

$$y = A + Bx + Cx^2 + \ldots,$$

in which (art. 16) A, B, C . . are equal to

$$y_{x=0},\ p_{x=0},\ q_{x=0} \cdot \frac{1}{1 \cdot 2}, \ldots.$$

If instead of x we put $x + \Delta x = x + h$, y becomes $y + \Delta y = y + k$ (so that $\Delta x = h$ here designates the difference of the independent, $\Delta y = k$ that of the dependent or of the function). From this substitution we have (A, B, C . . not containing x),

$$y + k = A + B(x+h) + C(x+h)^2 + D(x+h)^3 + \ldots$$
$$= A + Bx + Cx^2 + Dx^3 + \ldots + h(B + 2Cx + 3Dx^2 + \ldots)$$
$$+ h^2(C + 3Dx + \ldots)\ h^3(D + \ldots$$

But the function proposed being $y = A + Bx + Cx^2 + \ldots$, we deduce from it

$$p = B + 2Cx + \ldots,\ q = 2C + 6Dx + \ldots = 2(C + 3Dx + \ldots), \&c.$$

as we have already obtained in the preceding article. Thus we have, finally,

$$y + k = y + \Delta y = y + p \cdot h + q \cdot \frac{h^2}{1 \cdot 2} + r \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots$$

Taking away y from the two members of the equation, we obtain

$$k = p \cdot h + q \cdot \frac{h^2}{1 \cdot 2} + r \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots,$$

$$\text{or } \Delta y = \frac{\mathrm{d}y}{\mathrm{d}x} \Delta x + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \cdot \frac{\Delta x^2}{1 \cdot 2} + \ldots.$$

Thus, not only is the function changed, but the change of the function is expressed in a series according to the integral powers of the difference of the independent. This series is called, from its author, *Taylor's theorem*.

This series is also expressed by

$$\Delta y = p\Delta x + \psi \Delta x^2,$$

$$\text{in which } \psi = \frac{q}{1 \cdot 2} + r \cdot \frac{\Delta x}{1 \cdot 2 \cdot 3} + \ldots \text{ (See art. 5.)}$$

$$\text{Or thus, } f(x+h) = fx + \frac{\mathrm{d}fx}{\mathrm{d}x} \cdot \frac{h}{1} + \frac{\mathrm{d}^2 fx}{\mathrm{d}x^2} \cdot \frac{h^2}{1 \cdot 2} + \frac{\mathrm{d}^3 fx}{\mathrm{d}x^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

We can, therefore, by means of this article, and the preceding, develop a function according to the integral powers of the independent, and likewise the change of the function according to the powers of the variation of the independent. The coefficients of these series are the differential coefficients of the proposed function, which are determined in the first series by a particular supposition of the independent, namely, by $x = 0$.

These two methods of development may be used for every function whose successive differential coefficients can be found. This having been shown hitherto only for all the algebraic functions, these two methods of development can thus far be applied only to algebraic functions.

For an example, develop the functions

$$y = \frac{1+x}{1-x} \text{ and } y = (1-x^2)^{\frac{1}{2}}.$$

ON THE DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

18. Let y be $= a^x$. Substituting $x + \Delta x$ for x, we have

$$y + \Delta y = a^{x+\Delta x}, \text{ whence } \Delta y = a^{x+\Delta x} - a^x = a^x. \ (a^{\Delta x} - 1).$$

Putting $a = 1 + b$, we obtain

$$a^{\Delta x} - 1 = (1+b)^{\Delta x} - 1 = \text{(art. 16) } \frac{\Delta \cdot x}{1} b + \frac{\Delta x (\Delta x - 1)}{1 \cdot 2} b^2$$
$$+ \frac{\Delta x (\Delta x - 1)(\Delta x - 2)}{1 \cdot 2 \cdot 3} b^3 + \ldots$$

This expression being arranged according to the powers of Δx, we have

$$a^{\Delta x} - 1 = \Delta x \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \ldots\right) + \frac{\Delta x^2}{1 . 2}(b^2 - b^3 + \ldots) + \ldots.$$

Expressing the coefficient of Δx by k, (that is to say

$$b - \frac{b^2}{2} + \frac{b^3}{3} - \ldots = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \ldots,)$$

$$\text{we have } \Delta y = a^x \cdot \Delta x \cdot k + a^x \cdot \frac{\Delta x^2}{1 . 2}\left[(a-1)^2 - (a-1)^3 + \ldots\right] + \ldots$$

$$\text{Whence (art. 6) } \mathrm{d}y = \mathrm{d} \cdot a^x = k \cdot a^x \cdot \mathrm{d}x.$$

From thence, according to art. 15, we may deduce the differentials of the superior orders; we have, in fact,

$$\mathrm{d}^2 y = k^2 \cdot a^x \mathrm{d}x^2, \ \mathrm{d}^3 y = k^3 \cdot a^x \cdot \mathrm{d}x^3, \ \&c.$$

By means of these differentials, we may develop (according to art. 16) the function $y = a^x$ in a series arranged according to the integral powers of x.

Thus we have

$$y_{x=0} = a^x_{x=0} = a^0 = 1, p_{x=0} = k,$$
$$q_{x=0} = k^2, r_{x=0} = k^3, \&c.$$

The series of Maclaurin (art. 16) thus gives

$$y = a^x = 1 + \frac{k}{1} \cdot x + \frac{k^2 x^2}{1 \cdot 2} + \frac{k^3 x^3}{1 \cdot 2 \cdot 3} + \ldots.$$

In this series, making $x = 1$, we have

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \ldots$$

Taking here $k = 1$, and making the value of a, changed by this supposition, equal to x, we have, by developing a few terms,

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots = 2{\cdot}7182818 \ldots.$$

We have thence

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots.;$$

replacing in this series $x = k$, we have

$$e^k = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \ldots.$$

We have therefore $$e^k = a,$$

whence it follows that $k \cdot \log. e = \log. a$, or $k = \frac{\log. a}{\log. e}$.

It is to be understood that in this expression we must take both logarithms in the same system.

19. We have just seen that from $y = a^x$ is deduced

$$dy = k \cdot a^x \cdot dx,$$

and thence, if y is considered as the independent (according to art. 9)

$$dx = \frac{dy}{k \cdot a^x} = \frac{dy}{k \cdot y} = (\text{art. } 18)\ \frac{dy \log. e}{y \cdot \log. a}.$$

At the same time we find from $y = a^x$,

$$x \cdot \log. a = \log. y;$$

$$\text{therefore } x = \frac{\log. y}{\log. a} \text{ and } dx = \frac{1}{\log. ad} \cdot \log. y;$$

$$\text{consequently } \frac{1}{\log. a} \cdot d \cdot \log. y = \frac{\log. e \cdot dy}{y \cdot \log. a},$$

$$\text{or } d \cdot \log. y = \log. e \cdot \frac{dy}{y}.$$

($\log. y$ and $\log. e$ must here be considered as in the same system.)

If we take the system whose base is equal to $e = 2{\cdot}71828 \ldots.$ (art. 18), we have $\log. e = 1$, and designating the logarithm of y in this system by $\log.' y$, we have

$$d \cdot \log.' y = \frac{dy}{y}.$$

This system, whose base is e, is called *the system of natural logarithms, or Neperian or hyperbolic.*

The proposition contained in this article, on the differential of the logarithm of a number, is then

$$d \cdot \log. y = \log. e \cdot \frac{dy}{y}, \text{ and } d \cdot \log.' y = \frac{dy}{y},$$

that is to say, that the differential of the logarithm of a number taken in any system is equal to the differential of the number, divided by the number itself, and multiplied by the logarithm of $e = 2{\cdot}71828 \ldots$ taken in the system adopted. For the Neperian system, the differential of the logarithm of a number is equal to the differential of this number divided by the number.

If we take the logarithms in the Neperian system, (according to art. 18) we have also the following series:

$$a^x = 1 + \log.' a, x + \frac{(\log.' a)^2 . x^2}{1 . 2} + \frac{(\log.' a)^3 . x^3}{1 . 2 . 3} + \dots .$$

20. We shall now develop (by the method in art. 16) the function $y = \log (1 + x)$ in a series according to the integral powers of x. We have

$$p = \frac{\log. e}{1 + x} \; (19), \; q = \frac{-\log. e}{(1 + x)^2} \; (13), \; r = \frac{2 . \log. e}{(1 + x)^3}, \text{ \&c.,}$$

Whence $y_{x=0} = \log. 1 = 0$, $p_{x=0} = \log. e$, $q_{x=0} = -\log. e$, $r_{x=0} = 2 . \log. e$, &c.

Substituting these values in Maclaurin's series in art. 16, we have

$$y = \log. (1 + x) = \log. e \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right).$$

It is observable that the terms of this series become smaller in proportion as they are distant from the commencement, supposing x to be a true fraction (whose numerator is less than the denominator); while they become greater when x is a fraction greater than unity, because the value of the successive powers of a proper fraction diminishes as they proceed, while that of the successive powers of a fraction greater than unity, increases. The first terms then are not sufficient for finding the logarithm by approximation, unless x is a true fraction. This, as is well known, has been thus expressed: "The above series is convergent only when x is a proper fraction."

Another inconvenience of this series consists in the change of the signs by which the respective terms are affected.

The following is a mode of finding a series, more convenient for calculation. In the above series, putting $-x$ instead of $+x$, we have

$$\log. (1 - x) = \log. e \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right).$$

Taking away this series from the first, we obtain

$$\log. (1 + x) - \log. (1 - x) = \frac{\log. (1 + x)}{\log. (1 - x)} = 2 \log. e \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

Now making $\frac{1 + x}{1 - x} = 1 + \frac{z}{n}$, whence $x = \frac{z}{2n + z}$,

we have $$\log. (n + z) - \log. n = 2 \log. e \left[\frac{z}{2n + z} + \frac{1}{3}\left(\frac{z}{2n + z}\right)^3 + \frac{1}{5}\left(\frac{z}{2n + z}\right)^5 + \dots\right],$$

or, $$\log. (n + z) = \log. n + 2 \log. e \left[\frac{z}{2n + z} + \frac{1}{3}\left(\frac{z}{2n + z}\right)^3 + \dots\right].$$

By means of this series, the logarithm of a number may, with tolerable exactness, be found by a few of the first terms, because the series converges sufficiently, and the successive terms diminish sufficiently quickly. If n is greater than unity, the convergence of the series will increase still more. To find the logarithm of the number $n + z$, we must evidently first know that of the number n.

For the Neperian system, whose base is e, the above series is changed, according to the notation adopted in art. 19, into

$$\log.'(n + z) = \log.'n + 2 \left[\frac{z}{2n + z} + \frac{1}{3}\left(\frac{z}{2n + z}\right)^3 + \dots\right].$$

The first terms of this series will be sufficient to find the Neperian logarithm

of a number with tolerable exactness. If, for example, we make $n=1$ and $z=1$, we have for the neperian logarithm of 2,

$$\log.'2 = 2\left(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \ldots\right);$$

next making $n=2$, substituting the value we have just found in place of $\log.'n$, and making $z=1$, we obtain the neperian logarithm of 3. To find the logarithm of 4, we take $4=2^2$, whence $\log.'4 = 2\log.'2$. We may easily deduce log. 5 from log. 4, by making $n=4$, and $z=1$. To find the logarithm of 6, we take $6 = 3 \cdot 2$, whence we find $\log.'6 = \log.'3 + \log.'2$. These two last logarithms being already found, we know also that of 6. Hence it is plain that in general we need only find the neperian logarithms of the first numbers with the assistance of the series found above: those of all the compound numbers may be easily deduced from them.

The expression found in the preceding article,

$$\log.(1+x) = \log. e\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right),$$

becomes, for the neperian logarithm,

$$\log.'(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

It follows from thence, that if we wish to pass from the neperian logarithm to that of the same number in any system, we have only to multiply the neperian logarithm by $\log. e$, that is to say, by the logarithm of the number $2{\cdot}7182818\ldots$ taken in the system adopted. This factor $\log. e$, which is different for each system, is called the *modulus* of the system. For *Briggs's* system, whose base, it is well known, is equal to zero, the modulus, which is frequently designated by com. log. e, or log. tab. e, or log. Brig. e, is $= 0{\cdot}434294\ldots$ We may find it easily by the series given at the end of the preceding article, making $n=1$, $z=9$. In fact, we thus obtain

$$\text{log. tab. } 10 = \text{log. tab. } 1 + 2\,\text{log. tab. } e\left[\frac{9}{11} + \frac{1}{3}\left(\frac{9}{11}\right)^3 + \ldots\right],$$

that is to say, $1 = 2\,\text{log. tab. } e\left[\frac{9}{11} + \frac{1}{3}\left(\frac{9}{11}\right)^3 + \ldots\right]$,

whence we find the above value for log. tab. e, or for the modulus of Briggs's system. It may be more convenient to seek $\log.'10 = \log.'5 + \log.'2$, by the series found at the commencement of this article, according to the method exhibited above; we then have

$$\text{com. log. } 10 = \text{com. log. } e \log.'10,$$

$$\text{or, com. log. } e = \frac{1}{\log.'10}.$$

Reciprocally, to pass from the logarithm of a number in any system to that of Napier, it is plain, from what has been said, that we must multiply the known logarithm by $\frac{1}{\log. e}$, taking $\log e$ in the given system. Passing, for example, from Briggs's system to the Neperian,

$$\frac{1}{\log. e} \text{ is equal to } \frac{1}{0{\cdot}43429448\ldots} = 2{\cdot}302585\ldots$$

Consequently it is by this fraction that we must multiply the logarithm of a number in Briggs's system, to obtain the Neperian logarithm of the same number.

Remark.—The sign *log.* for the future, in this Appendix, will always designate the Neperian logarithm. Until lately it has been usually denoted by H. L. or Hyp. log.

22. The following are some examples of the differentiation of exponential and logarithmic functions.

(1.) Let $y = x^z$, z and y being functions of x. Taking the logarithms of the two members of the equation, we have, $\log. y = z \,.\, \log. x$, whence

$$(11)\ \mathrm{d}\,.\,\log. y = z\mathrm{d}\,.\,\log. x + \log. x\,.\,\mathrm{d}z,$$

$$\text{or, } (19)\ \frac{\mathrm{d}y}{y} = z\,.\,\frac{\mathrm{d}x}{x} + \log. x\,.\,\mathrm{d}z,$$

whence finally

$$\mathrm{d}y = \mathrm{d}\,.\,x^z = \frac{x^z\,.\,z\,.\,\mathrm{d}x}{x} + \log. x\,.\,x^z\,.\,\mathrm{d}z.$$

(2.) Let $y = \log.\left[\dfrac{(1+x)^{\frac{1}{2}} + (1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}}\right].$

First multiplying the numerator and the denominator of the fraction under the sign of the logarithm by $(1+x)^{\frac{1}{2}} + (1-x)^{\frac{1}{2}}$, we have

$$y = \log.\left[\frac{1 + (1-x^2)^{\frac{1}{2}}}{x}\right];$$

$$\text{now let } \frac{1 + (1-x^2)^{\frac{1}{2}}}{x} = z, \text{ and } 1 + (1-x^2)^{\frac{1}{2}} = u;$$

$$\text{then } \mathrm{d}u = \frac{-x\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}},\ \mathrm{d}z = \frac{-\mathrm{d}x\,[1 + (1-x^2)^{\frac{1}{2}}]}{x^2\,.\,(1-x^2)^{\frac{1}{2}}};$$

$$\text{therefore } \mathrm{d}y = \frac{-\mathrm{d}x}{x\,(1-x^2)^{\frac{1}{2}}}.$$

The reader may, for practice, resolve the following problems, of which only the results are given.

(3.) Let $y = \log.\left[\dfrac{(1+x^2)^{\frac{1}{2}} + x}{(1+x^2)^{\frac{1}{2}} - x}\right]$

First multiplying the fraction of the radical in the numerator and denominator by $(1+x^2)^{\frac{1}{2}} + x$, we have

$$y = \log.\left[x + (1+x^2)^{\frac{1}{2}}\right].$$

$$\text{we have } \mathrm{d}y = \frac{\mathrm{d}x}{(1+x^2)^{\frac{1}{2}}}.$$

(4.) Let $y = \log.\left[\dfrac{x}{(1+x^2)^{\frac{1}{2}}}\right]$

$$\text{We find } \mathrm{d}y = \frac{\mathrm{d}x}{x\,(1+x^2)}.$$

ON THE DIFFERENTIAL OF THE ARC.

23. We will now again consider the series in art. 7.

$$\frac{\Delta y}{\Delta x} = y' + y'' \Delta x + \ldots,$$

which is also designated (5) by

$$\frac{\Delta y}{\Delta x} = y' + \psi \Delta x;$$

y' or p (6) is a function of x only; but $\psi\Delta x$ includes Δx as well as x. If the assertion made in the article quoted were generally true, that is to say, that $\psi\Delta x$ vanishes on the supposition of $\Delta x = 0$, or that the series is reduced to p, it would apply to every value of x. To examine this by means of Taylor's theorem, let us develop the series $\frac{\Delta y}{\Delta x}$ for the function $y = (x^2 - a^2)^{\frac{3}{2}}$.

We have $$\frac{\Delta y}{\Delta x} = 3x\,(x^2 - a^2)^{\frac{1}{2}} + \frac{(6x^2 - 3a^2)\,\Delta x}{(x^2 - a^2)^{\frac{1}{2}}\,1\,.\,2} + \ldots.$$

On the supposition of $\Delta x = 0$ and at the same time of $x = a$, each term of the series $\psi\Delta x$ becomes $\frac{0}{0}$, the expression of an indeterminate quantity, which we shall presently consider, and which does not always vanish Hence we may conclude that, for $\Delta x = 0$, $\psi\Delta x$ only *generally* becomes equal to zero, that is to say, not for *every* value of x.

In like manner the second proposition in art. 7, namely that the series for $\frac{\Delta y}{\Delta x}$ approaches nearer and nearer to p in proportion as Δx diminishes, is not true for every value of x; which will easily appear from the preceding example, when, for $x = a$, p becomes nothing, and each term of $\psi\Delta x$ infinite; at least it is not clear how a series (in this case, $\psi\Delta x$) of which every term is infinite, can approach nearer to zero, after being multiplied by a continually decreasing quantity.

We proceed then to consider more particularly this decrease of $\psi\Delta x$, which augments with the decrease of Δx, and which we have shown to be only generally true. No limits can be put to the decrease of Δx, nor to that of $\psi\Delta x$ resulting from it: for, if by the decrease of Δx, $\psi\Delta x$ were to become an extremely small quantity, we might still diminish Δx, so that $\psi\Delta x$ would become still less than the very small quantity mentioned before.

This may be thus expressed: "$\psi\Delta x$ may, by a decrease of Δx, become smaller than any assignable quantity, however small it may be."

Consequently, it would be contradictory to endeavour to determine the greatness or smallness of the expression $\psi\Delta x$ in its least value, since, according to the above, there exists no value which can be considered the least, the value $\Delta x = 0$ being excluded from it.

To make this still more clear by an example from elementary Geometry, let us imagine a circle in which is inscribed a regular polygon of an indefinite number of sides. Thus, comparing the difference of the two areas of the polygon and of the circle, with our Δx, we shall see that it, as well as $\psi\Delta x$, has a certain assignable value, which however, if we continue to divide the sides of the polygon into equal parts, may become less than any area, however small.

24. After what has been said, we are ready to appreciate the truth of the following proposition. If these two expressions

$$\frac{\Delta\phi}{\Delta x} > a + \psi\Delta x \text{ and } \frac{\Delta\phi}{\Delta x} < a + \psi'\Delta x,$$

are true with respect to any value either of x or of Δx, we have

$$\frac{\mathrm{d}\phi}{x} = a$$

ϕ and a being here functions of x ; ψ and ψ' designate the series ordered according to the integral powers of Δx, whose coefficients are functions of x. Consequently, $\psi\Delta x$ and $\psi'\Delta x$, though differing as to value, are conformable to each other in this, that they can in general, by the decrease of Δx, become for any value of x, smaller than any assignable quantity, however minute it may be.

To demonstrate this $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ not being equal to a may be either greater or less. Let us first suppose

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} > a, \text{ or } \frac{\mathrm{d}\phi}{\mathrm{d}x} = a + \beta,$$

in which β being of the same nature as a, that is to say, a function of x, and independent of Δx, may, for any value of x, take an assignable value as small as we please. We thus obtain (6)

$$\frac{\Delta\phi}{\Delta x} = a + \beta + \pi \,.\, \Delta x,$$

where $\pi \,.\, \Delta x$ is of the same nature as $\psi\Delta x$ and $\psi'\Delta x$. Compared with the second supposition.

$$\frac{\Delta\phi}{\Delta x} < a + \psi'\Delta x,$$

$$\text{we have } a + \beta + \pi \,.\, \Delta x < a + \psi' \,.\, \Delta x,$$
$$\text{or } \beta < \psi' \,.\, \Delta x - \pi \,.\, \Delta x,$$

which may take place for any value of x, because ψ', $\Delta x - \pi \,.\, \Delta x$ (23), may for a sufficiently small value of Δx, become less than any assignable quantity, however small it may be, therefore also less than β.

In the second place, if we make

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = a - \beta,$$

$$\text{we obtain } \frac{\Delta\phi}{\Delta x} = a - \beta + \pi \,.\, \Delta x,$$

whence we see, compared with the first supposition, that

$$a - \beta + \pi \,.\, \Delta x > a + \psi \,.\, \Delta x,$$
$$\text{or } \pi \,.\, \Delta x - \psi\Delta x > \beta,$$

An absurd result like the preceding. Therefore since $\frac{\Delta\phi}{\Delta x}$ can be neither greater nor less than a, it follows of necessity that $\frac{\mathrm{d}\phi}{\mathrm{d}x} = a$.

The reader may easily prove that the preceding demonstration and the proposition established by it are correct, when in the two suppositions one of the two quantities ψ and ψ' should vanish, or that ψ or ψ', or both of them, should be negative.

25. Let us now exhibit geometrically, in a curve, the expressions Δx and $\Delta y = p \,.\, \Delta x + \psi \,.\, \Delta x^2$ (5).

In the annexed figures, let EMN represent any curve, AP the axis of the abscisses, A the origin of the rectangular coordinates. The abscisses are taken in a positive sense from A to P, and the ordinates are positive above the axis of the abscisses.

In figs. 2 and 4, the curve has its concavity turned towards the axis of the abscisses: in figs. 1 and 3, its convexity.

The tangent MM and the secant MN are drawn through the point M, called the *point of contact*, whose coordinates are AP $= x$, and PM $= y$. The first of these right lines cuts the axis of the abscisses at the point B, the last at the point C. These two points will be found on the negative side in figs. 1 and 2, and on the positive side of the ordinate in figs. 3 and 4, PP′ being $= \Delta x$; the ordinate raised at P′ cuts the curve as well as the secant at the point N, and the tangent at the point M′; we have therefore P′N $= y + \Delta y$: and if through M a parallel is drawn to the axis of the abscisses MQ, we have NQ $= \Delta y$.

In the four figures, we obtain first from the two similar triangles M′MQ and MBP,

$$\mathrm{M'Q} : \mathrm{MQ} = \mathrm{MP} : \mathrm{BP},$$

$$\text{or } \mathrm{M'Q} : \Delta x = y : \mathrm{BP}.$$

$$\text{whence PB} = \frac{y \,.\, \Delta x}{\mathrm{M'Q}};$$

and the similar triangles MNQ and MPC give

$$\mathrm{NQ} : \mathrm{MQ} = \mathrm{MP} : \mathrm{PC},$$

$$\text{or } \Delta y : \Delta x = y : \mathrm{PC};$$

$$\text{whence PC} = \frac{y \,.\, \Delta x}{\Delta y},$$

Or [y being a function of x, (art. 5, 17), $\Delta y = p \,.\, \Delta x + \psi \,.\, \Delta x^2$],

$$\mathrm{PC} = \frac{y \,.\, \Delta x}{p \,.\, \Delta x + \psi \,.\, \Delta x^2} = \frac{y}{p + \psi \,.\, \Delta x}.$$

Now, it is evident, from the figures, that while Δx decreases the secant tends towards the tangent, and the point C towards the point B; so that, when $\Delta x = 0$, the secant is confounded with the tangent, and we have PC $=$ PB. Consequently, if we make $\Delta x = 0$ in the above expression of PC, and at the same time PB instead of PC, we obtain PB $= \frac{y}{p}$; comparing this value with that which we have already obtained for PB, we have

$$\frac{y \,.\, \Delta x}{\mathrm{M'Q}} = \frac{y}{p}.$$

and we thence obtain M′Q $= p \,.\, \Delta x$.

This is therefore the absolute value of M′Q; let us now examine the value of this line with respect to its *position* in the four different figures, or its value relative to its *sign*. In the first place, it is evident that the position of M′Q in figs. 3 and 4, is opposite to that in figs. 1 and 2, with respect to MQ parallel to the axis of the abscisses; M′Q is therefore positive in figs. 1 and 2, negative in figs. 3 and 4. We see also that NQ or Δy is positive in figs. 1 and 2, but negative in figs. 3 and 4. We have consequently

Fig. 1. Fig. 2. Fig. 3. Fig. 4.

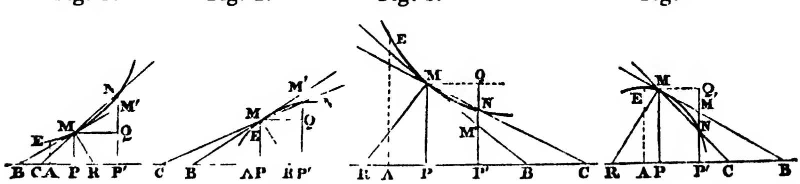


Fig. 1. M′N $= \Delta y - p \,.\, \Delta x = + \psi \,.\, \Delta x^2$,
Fig. 2. M′N $= p \,.\, \Delta x - \Delta y = - \psi \,.\, \Delta x^2$,
Fig. 3. M′N $= -p \,.\, \Delta x - (-\Delta y) = -p \,.\, \Delta x + \Delta y = + \psi \,.\, \Delta x^2$,
Fig. 4. M′N $= -\Delta y - (-p \,.\, \Delta x) = - \psi \,.\, \Delta x^2$.

Hence we see that the relative value of M′N is $+ \psi \,.\, \Delta x^2$ in figs. 1 and 3, and that it is $- \psi \,.\, \Delta x^2$ in figs. 2 and 4; the absolute value of M′N is therefore $\psi \,.\, \Delta x^2$.

For the respective positions of the tangent and of the curve, with respect to the axis of the abscisses, we have the following relative values of Δy, as it is represented in the four figures.

Fig. 1. $\Delta y = +p \,.\, \Delta x + \psi \,.\, \Delta x^2$,
Fig. 2. $\Delta y = +p \,.\, \Delta x - \psi \,.\, \Delta x^2$,
Fig. 3. $\Delta y = -p \,.\, \Delta x + \psi \,.\, \Delta x^2$,
Fig. 4. $\Delta y = -p \,.\, \Delta x - \psi \,.\, \Delta x^2$.

From what has been said, it results, therefore, that in the formula

$$\Delta y = p \,.\, \Delta x + \psi \,.\, \Delta x^2,$$

in which p and ψ are taken in their absolute value, the first part, $p \,.\, \Delta x$ represents the line M'Q; the last, $\psi \,.\, \Delta x^2$, the line M'N, considering the two lines in an absolute sense.

Remark—The supposition made at the beginning of this article, that positive abscisses go towards the right, and positive ordinates above the axis of the abscisses, will, as is usual, be adopted in all the figures, unless the contrary is specified.

26. We shall easily, then, with the assistance of the two preceding articles, find the differential of the arc of a curve.

Suppose (fig. 1, 2, 3, 4) the arc EM $= \phi$; ϕ being a function of x, MN will be the difference of ϕ; for the difference of x, that is to say PP' $= \Delta x$, and consequently MN $= \Delta\phi$.

The arc MN is greater than the chord MN, and less than the sum of the two lines MM' + NM. Now, we have the chord

$$\text{MN} = (\text{MQ}^2 + \text{NQ}^2)^{\frac{1}{2}} = (25)\ [\Delta x^2 + \Delta y^2]^{\frac{1}{2}} = (5)$$

$$[\Delta x^2 + (p \,.\, \Delta x + \psi \,.\, \Delta x^2]^{\frac{1}{2}} = \Delta x\,(1 + p^2 + 2p\psi \,.\, \Delta x + \psi^2 \,.\, \Delta x^2)^{\frac{1}{2}}.$$

Developing the radical as a binomial, considering $1 + p^2$ as the first, and $2p\psi \,.\, \Delta x + \psi^2 \,.\, \Delta x^2$ as the second part, and using π to designate all the terms which include Δx, we find

$$\text{MN} = \Delta x\,(1 + p^2)^{\frac{1}{2}} + \pi \,.\, \Delta x^2.$$

We have also

$$\text{MM}' = (\text{MQ}^2 + \text{M'Q}^2)^{\frac{1}{2}} = (25)\ (\Delta x^2 + p^2 \,.\, \Delta x^2)^{\frac{1}{2}} = \Delta x\,(1 + p^2)^{\frac{1}{2}},$$

$$\text{therefore MM}' + \text{NM}' = (25)\ \Delta x\,(1 + p^2)^{\frac{1}{2}} + \psi \,.\, \Delta x^2.$$

We have, according to the preceding, $\Delta\phi > \Delta x\,(1 + p^2)^{\frac{1}{2}} + \pi \,.\, \Delta x^2$,

$$\text{and } \Delta\phi < \Delta x\,(1 + p^2)^{\frac{1}{2}} + \psi \,.\, \Delta x^2,$$

$$\text{or } \frac{\Delta\phi}{\Delta x} > (1 + p^2)^{\frac{1}{2}} + \pi \,.\, \Delta x,$$

$$\text{and } \frac{\Delta\phi}{\Delta x} < (1 + p^2)^{\frac{1}{2}} + \psi \,.\, \Delta x,$$

$$\text{whence (by art. 24), } \frac{d\phi}{dx} = (1 + p^2)^{\frac{1}{2}}, \text{ or } = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}},$$

$$\text{or, finally, } d\phi = dx\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = (dx^2 + dy^2)^{\frac{1}{2}},$$

that is to say, the differential of the arc is equal to the square root of the sum of the squares of the differentials of the two variables.

ON THE DIFFERENTIATION OF CIRCULAR FUNCTIONS.

27. Let us propose, for example, to find the differential of the arc of the circle.

The equation of the circle referred to the centre is $y^2 + x^2 = r^2$, whence $\frac{dy}{dx} = \frac{-x}{y}$. Whence results

$$d\phi = dx\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = \frac{dx}{y}\,(x^2 + y^2)^{\frac{1}{2}} = \frac{r dx}{y}.$$

Making the radius $r = 1$, we have $d\phi = \frac{dx}{y}$.

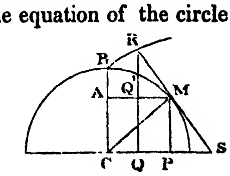
The differential of the arc of the circle enables us to find the differentials of all the trigonometrical lines by the following process.

Supposing the radius of the circle $= 1$, we have for the equation of the circle referred to the centre,

$$y^2 + x^2 = 1.$$

We then have, in the marginal figure,

$$\text{CP} = \text{AM} = x, \text{ and MP} = \text{AC} = y.$$



Let the arc BM $= \phi$, the angle BCM will also be $= \phi$, as the radius is equal to unity. It is again evident that AM $= x =$ sin. ϕ and AC $= y =$ cos. ϕ. Substituting these values of x and y in the differential of the arc of the circle $d\phi = \frac{dx}{y}$, we obtain $d\phi = \frac{d \sin. \phi}{\cos. \phi}$,

$$\text{or } d \,.\, \sin. \phi = \cos. \phi \,.\, d\phi:$$

hence, the differential of the sine of an arc is equal to the cosine of the arc, multiplied by the differential of the arc.

Putting sin. $\phi = (1 - \cos. {}^2\phi)^{\frac{1}{2}}$, or $(1 - \cos. \phi^2)^{\frac{1}{2}}$, we have (12)

$$d \,.\, \sin. \phi = \frac{-\cos. \phi \,.\, d \,.\, \cos. \phi}{(1 - \cos. \phi^2)^{\frac{1}{2}}} = \frac{-\cos. \phi \,.\, d \,.\, \cos. \phi}{\sin. \phi};$$

Now, we have already found d . sin. ϕ = cos. ϕ. d ϕ; we have therefore

$$\frac{-\cos. \phi \,.\, d \,.\, \cos. \phi}{\sin. \phi} = \cos. \phi \,.\, d\phi,$$

$$\text{or, } d \,.\, \cos. \phi = -\sin. \phi \,.\, d\phi,$$

that is, the differential of the cosine of an arc is equal to the sine of an arc, multiplied by the differential of the arc, and the result taken with the negative sign.

Moreover, we find from tan. $\phi = \frac{\sin. \phi}{\cos. \phi}$

$$d \,.\, \tan. \phi = d\left(\frac{\sin. \phi}{\cos. \phi}\right) = (\text{art. } 13)\ \frac{\cos. \phi \,.\, d \,.\, \sin. \phi - \sin. \phi \,.\, d \,.\, \cos. \phi}{\cos. \phi^2}.$$

Substituting the values previously found of d . sin. ϕ and of d . cos. ϕ, we have

$$d \,.\, \tan. \phi = \frac{\cos. \phi^2 \,.\, d\phi + \sin. \phi^2 d\phi}{\cos. \phi^2} = \frac{d\phi}{\cos. \phi^2};$$

that is, the differential of the tangent of an arc of a circle is equal to the differential of the arc, divided by the square of the cosine of the arc.

Because cot. $\phi = \frac{1}{\tan. \phi}$, we have

$$d \,.\, \text{cotan. } \phi = d \,.\, \frac{1}{\tan. \phi} = (\text{art. } 13)\ \frac{-d \,.\, \tan. \phi}{\tan. \phi^2}$$

$$= \frac{-d\phi}{\cos. \phi^2 \tan. \phi^2} = \frac{-d\phi}{\sin. \phi^2},$$

that is, the differential of the cotangent of the arc of a circle is equal to the differential of the arc, divided by the square of the sine, the result being taken with the negative sign.

By a similar process we find the differentials of other trigonometrical lines.

Thus, we find from sec. $\phi = \frac{1}{\cos. \phi}$,

$$d \,.\, \sec. \phi = \frac{-d \,.\, \cos. \phi}{\cos. \phi^2} = \frac{\sin. \phi \,.\, d\phi}{\cos. \phi^2};$$

and from cosec. $\phi = \frac{1}{\sin. \phi}$,

we get $d \,.\, \text{cosec. } \phi = \frac{-d \,.\, \sin. \phi}{\sin. \phi^2} = \frac{-\cos. \phi \,.\, d\phi}{\sin. \phi^2};$

from versin. $\phi = 1 - \cos. \phi$,

we deduce d . versin. $\phi = -$ d . cos. ϕ = sin. ϕ . dϕ;

and finally, from co-versin. $\phi = 1 - \sin. \phi$,

$$d \,.\, \text{co-versin. } \phi = -d \,.\, \sin. \phi = -\cos. \phi \,.\, d\phi.\text{*}$$

* By the aid of the above formulæ, we may find the differentials of expressions involving sines, cosines, tangents, &c. We here present an example.

[Let

Remark.—According to this, we have also, if u represents a constant number, since $\mathrm{d}\,.\,(nx) = n\,.\,\mathrm{d}x$ (art. 11), $\mathrm{d}\,.\,\sin.\,nx = n\,.\,\mathrm{d}x\,.\,\cos.\,nx$, $\mathrm{d}\,.\,\cos.\,nx = -\,n\,.\,\mathrm{d}x\,.\,\sin.\,nx$,

$$\mathrm{d}\,.\,\tan.\,nx = \frac{n\mathrm{d}x}{(\cos.\,nx)^2}, \text{ \&c.}$$

28. We may easily find from the differentials of the trigonometrical lines just obtained, the differentials of the arc in terms of the respective trigonometrical lines.

Thus, we obtain from $\mathrm{d}\,.\,\sin.\,\phi = \cos.\,\phi\,.\,\mathrm{d}\phi$,

$$\mathrm{d}\phi = \frac{\mathrm{d}\,.\,\sin.\,\phi}{\cos.\,\phi} = \frac{\mathrm{d}\,.\,\sin.\,\phi}{(1-\sin\,\phi^2)^{\frac{1}{2}}},$$

that is to say, the differential of the arc is equal to the differential of the sine of the arc, divided by the cosine of the arc.

We find again from $\mathrm{d}\,.\,\cos.\,\phi = -\sin.\,\phi\,.\,\mathrm{d}\phi$,

$$\mathrm{d}\phi = \frac{-\,\mathrm{d}\,.\,\cos.\,\phi}{\sin.\,\phi} = \frac{-\,\mathrm{d}\,.\,\cos.\,\phi}{(1-\cos.\,\phi^2)^{\frac{1}{2}}},$$

that is, the differential of the arc is equal to the differential of the cosine, divided by the sine, the result taken with the negative sign.

By a similar process we find from $\mathrm{d}\,.\,\tan.\,\phi = \dfrac{\mathrm{d}\phi}{\cos.\,\phi^2}$,

$$\mathrm{d}\phi = \cos.\,\phi^2\,.\,\mathrm{d}\tan.\,\phi = \frac{\mathrm{d}\,.\,\tan.\,\phi}{\sec.\,\phi^2} = \frac{\mathrm{d}\,.\,\tan\,\phi}{1+\tan.\,\phi^2},$$

that is, the differential of the arc is equal to the differential of the tangent, divided by the square of the secant.

From $\mathrm{d}\,.\,\mathrm{cotan.}\,\phi = \dfrac{-\,\mathrm{d}\phi}{\sin.\,\phi^2}$, we obtain

$$\mathrm{d}\phi = -\sin.\,\phi^2\,.\,\mathrm{d}\,.\,\mathrm{cotan.}\,\phi = \frac{-\,\mathrm{d}\,.\,\mathrm{cotan.}\,\phi}{\mathrm{cosec.}\,\phi^2} = \frac{-\,\mathrm{d}\,.\,\mathrm{cotan.}\,\phi}{1+\mathrm{cotan.}\,\phi^2},$$

that is, the differential of the arc is equal to the differential of the cotangent, divided by the square of the cosecant, the result taken with the negative sign.

By means of the formulæ in the preceding article the reader may easily express the differential of the arc by other trigonometrical lines.

29. We here subjoin for more convenient reference, the formulæ demonstrated above. Representing by x successively each of the trigonometrical lines of the arc ϕ, we have

for $x = \sin.\,\phi$, $\quad \mathrm{d}x = -\,\mathrm{d}\phi\,(1-x^2)^{\frac{1}{2}}, \quad \mathrm{d}\phi = \dfrac{\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}}$;

for $x = \cos.\,\phi$, $\quad \mathrm{d}x = -\,\mathrm{d}\phi\,(1-x^2)^{\frac{1}{2}}, \quad \mathrm{d}\phi = \dfrac{-\,\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}}$,

for $x = \tan.\,\phi$, $\quad \mathrm{d}x = \mathrm{d}\phi\,(1+x^2), \quad \mathrm{d}\phi = \dfrac{\mathrm{d}x}{1+x^2}$,

for $x = \mathrm{cotan}\,\phi$, $\quad \mathrm{d}x = -\,\mathrm{d}\phi\,(1+x^2), \quad \mathrm{d}\phi = \dfrac{-\,\mathrm{d}x}{1+x^2}$,

Let $y = (\cos.\,x)^{\sin x}$. Make $\cos.\,x = z$, $\sin.\,x = u$, then $y = z^u$, and

$$\mathrm{d}y = \mathrm{d}z^u = z^u\left(\mathrm{d}u\,log.\,z + \frac{u\mathrm{d}z}{z}\right)\text{ [art. 32]}$$

$$= \mathrm{d}x\,.\,(\cos.\,x)^{\sin x}\left(\cos.\,x\,log.\,\cos.\,x - \frac{\sin.^2 x}{\cos\,x}\right).$$

$$\text{for } x = \text{sec. } \phi \quad dx = d\phi \, . \, x\,(x^2 - 1)^{\frac{1}{2}}, \; d\phi = \frac{dx}{x\,(x^2-1)^{\frac{1}{2}}},$$

$$\text{for } x = \text{cosec. } \phi, \; dx = -\,d\phi \, . \, x\,(x^2 - 1)^{\frac{1}{2}}, \; d\phi = \frac{-\,dx}{x\,(x^2-1)^{\frac{1}{2}}},$$

$$\text{for } x = \text{versin. } \phi, \; dx = d\phi \, . \, [x\,(2 - x)]^{\frac{1}{2}}, \; d\phi = \frac{dx}{[x\,(2-x)]^{\frac{1}{2}}},$$

$$\text{for } x = \text{co-vers. } \phi, \; dx = -\,d\phi \, . \, [x(2 - x)]^{\frac{1}{2}}, \; d\phi = \frac{-\,dx}{[x\,(2-x)]^{\frac{1}{2}}}.$$

In these formulæ the radius of the circle is (as in art. 27) supposed equal to unity. But it will be easy to refer them to radius $= r$, remembering that in the elements of Geometry, arcs as well as trigonometrical lines, relative to the same angles in circles of different radii, are as their respective radii.

To exemplify this, let us designate a certain arc whose radius $= 1$ by ϕ, and that whose radius $= r$ by ϕr, and also the corresponding trigonometrical line by x and x_r, we shall have $\phi : \phi_r = 1 : r$. and $x : x_r = 1 : r$,

$$\text{whence } \phi = \frac{\phi_r}{r}, \text{ and } x = \frac{x_r}{r};$$

$$\text{therefore } d\phi = \frac{d\phi_r}{r}, \text{ and } dx = \frac{dx_r}{r}.$$

Substituting in place of x, dx, ϕ and $d\phi$ these values, in the above formulæ, we obtain the differentials of the arc and of the trigonometrical lines for the radius $= r$. Thus, for example, the formulæ for the sine

$$dx = d\phi\,(1 - x^2)^{\frac{1}{2}} \text{ and } d\phi = \frac{dx}{(1 - x^2)^{\frac{1}{2}}}$$

are changed into these

$$\frac{dx_r}{r} = \frac{d\phi_r}{r}\left(1 - \frac{x_r^2}{r^2}\right)^{\frac{1}{2}} = \frac{d\phi_r}{r^2}\,(r^2 - x_r^2)^{\frac{1}{2}},$$

$$\text{and } \frac{d\phi_r}{r} = \frac{\frac{dx_r}{r}}{\left(1 - \frac{x_r^2}{r^2}\right)^{\frac{1}{2}}} = \frac{dx^r}{(r^2 - x_r^2)^{\frac{1}{2}}}$$

or in $dx_r = \frac{d\phi_r\,(r^2 - x^2_r)^{\frac{1}{2}}}{r}$, and $d\phi_r = \frac{r\,.\,dx_r}{(r^2 - x_r^2)^{\frac{1}{2}}}$, that is, we have, when x represents the sine of the arc ϕ, whose radius $= r$,

$$dx = \frac{d\phi\,(r^2 - x^2)^{\frac{1}{2}}}{r} \text{ and } d\phi = \frac{rdx}{(r^2 - x^2)^{\frac{1}{2}}}.$$

Proceeding in this manner on all the above formulæ, we obtain for radius $= r$,

$$\text{for } x = \text{sin. } \phi, \quad dx = \frac{d\phi\,(r^2 - x^2)^{\frac{1}{2}}}{r}, \; d\phi = \frac{rdx}{(r^2 - x^2)^{\frac{1}{2}}},$$

$$\text{for } x = \text{cosin. } \phi, \quad dx = \frac{-d\phi\,(r^2 - x^2)^{\frac{1}{2}}}{r}, \; d\phi = \frac{-\,rdx}{(r^2 - x^2)^{\frac{1}{2}}},$$

$$\text{for } x = \text{tan. } \phi, \quad dx = \frac{d\phi\,(r^2 + x^2)}{r^2}, \; d\phi = \frac{r^2dx}{r^2 + x^2}$$

$$\text{for } x = \text{cotan. } \phi, \quad dx = \frac{-\,d\phi\,(r^2 + x^2)}{r^2}, \; d\phi = \frac{-\,r^2dx}{r^2 + x^2},$$

for $x = \sec.\ \phi$, $\quad dx = \frac{d\phi\ .\ x\,(x^2-r^2)^{\frac{1}{2}}}{r^2}$, $d\phi = \frac{r^2 dx}{x\,(x^2-r^2)^{\frac{1}{2}}}$

for $x = \text{cosec.}\ \phi$, $\quad dx = \frac{-d\phi\ .\ x\,(x^2-r^2)}{r^2}$, $d\phi = \frac{-r^2 dx}{x\,(x^2-r^2)^{\frac{1}{2}}}$.

Henceforward, however, in treating of trigonometrical functions, the radius will always be supposed $= 1$, unless the contrary is expressly stated.

30. We are now prepared to find the differential coefficient, and by it also the differentials of the superior orders of trigonometrical functions which contain arcs of the circle and trigonometrical lines: we can also, by art. 16, develope any trigonometrical function in a series, according to the integral powers of the independant. We have first in the function $y = \sin.\ x$ (art. 27).

$$\frac{dy}{dx} = p = \cos.\ x,\ q = -\sin.\ x,\ r = -\cos.\ x,\ s = +\sin.\ x,\ \&c.;$$

therefore (art. 16) $y_{x=0} = 0,\ p_{x=0} = 1,\ q_{x=0} = 0,\ r_{x=0} = -1,\ s_{x=0} = 0$, &c.

whence, $y = \sin. x = x - \frac{x^3}{1\ .\ 2\ .\ 3} + \frac{x^5}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5} - \frac{x^7}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6\ .\ 7} + \ldots$,*

a series which expresses the sine of an arc by the arc itself. For $y = \cos.\ x$; we obtain (art. 27.)

$$p = -\sin.\ x,\ q = -\cos.\ x,\ r = \sin.\ x,\ s = \cos.\ x,\ \&c.$$

whence $y_{x=0} = 1,\ p_{x=0} = 0,\ q_{x=0} = -1,\ r_{x=0} = 0,\ s_{x=0} = 1$, &c.;

therefore art. 16.

$$y = \cos.\ x = 1 - \frac{x^2}{1\ .\ 2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} - \ldots,$$

a series which expresses the cosine of an arc by the arc itself.

31. By the same process by which we have expressed in a series the sine and cosine by the corresponding arc, we may also (art. 16) express every other trigonometrical line by its respective arc. But not dwelling on this, we propose to seek reciprocally a series for the arc expressed by its trigonometrical line.

Thus, let y represent the arc whose sine $= x$, or according to the clearest notation, $y = \text{arc}\ (\sin. = x)$ †; we have by art. 28 $\frac{dy}{dx} = p = (1 - x^2)^{-\frac{1}{2}}$,

whence (art. 13)

* There are many other methods of arriving at this valuable theorem, for one of which see the commencement of the *Trigonometry*, in vol. i. Dr. James Thomson deduces it from the following simple process:

$$\text{Assume, } \sin.\ x = x + Ax^3 + Bx^5 + Cx^7 + \ldots\ (a)$$

differentiate this and divide by dx; then

$$\cos.\ x = 1 + 3Ax^2 + 5Bx^4 + 7Cx^6 + \ldots\ldots$$

differentiate again, and divide by $-dx$; then

$$\sin.\ x = -2\cdot 3Ax - 4\cdot 5Bx^3 - 6\cdot 7Cx^5 - \ldots\ldots$$

equating the coefficients of this with the corresponding ones of (a)

we get $-2\cdot 3A = 1$; $4\cdot 5B = A$; $6\cdot 7C = B$, &c.

Hence, $A = -\frac{1}{2\cdot 3}$; $B = -\frac{A}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4\cdot 5}$; &c. and the substitution of these respectively in the assumed series (a), gives the series for sin. x in the text.

In a similar manner, the series for cos. x would be found by assuming $\cos.\ x = 1 + Ax^2 + Bx^4 +$ &c.

The student should recollect that the series for the sine and cosine possess the curious property of reproducing one another continually by repeated differentiations and divisions by dx or $-dx$.

† See, however, the notes at pp. 41, 221, of this volume, as to a notation now brought much into use.

$$q = x(1-x^2)^{-\frac{3}{2}},\ r = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}, \text{ \&c.},$$

moreover, $y_{x=0} = 0$ (regarding the arc and the diameter to commence at one extremity of the latter)

$$p_{x=0} = 1,\ q_{x=0} = 0,\ r_{x=0} = 1, \text{ \&c.}$$

We have consequently (art. 16),

$$y = \text{arc}(\sin. = x) = x + \frac{x^3}{1.2.3} + \frac{3.3 \,.\, x^5}{1.2.3.4.5} + \frac{3.3.5.5 \,.\, x^7}{1.2.3.4.5.6.7} + \ldots$$

For $y = \text{arc}(\cos. = x)$ or $= \cos.^{-1} x$, we obtain by proceeding in like manner, the following series:

$$y = \text{arc}(\cos. = x) = \frac{\pi}{2} - \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{3.3\, x^5}{1.2.3.4.5} - \frac{3.3.5.5. x7}{1.2.3.4.5.6.7} - \ldots;$$

this series may also be found immediately by the preceding, found for the sine, observing that

$$\text{arc}(\cos. = x) = \frac{\pi}{2} - \text{arc}(\sin. = x).$$

Moreover, if we make $y = \text{arc}(\tan. = x)$, we have (art. 28),

$$\frac{dy}{dx} = p = \frac{1}{1+x^2},\ q = -2x(1+x^2)^{-2}$$

$$r = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}, \text{ \&c.},$$

whence $y_{x=0} = 0,\ p_{x=0} = 1,\ q_{x=0} = 0,\ r_{x=0} = -2$, &c.

therefore (art. 16) $y = \text{arc}(\tan. = x) = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.$

Students, in order to become well versed in differentiating, should themselves develop several series, both for trigonometrical lines, for example, for the tangent expressed by the arc, and for arcs expressed by a trigonometrical line. They will make use of the same process as that employed in arts. 30 and 31.

32. The series given above for arc ($\sin. = x$) is sufficiently convergent, when x is taken a small fraction, and some of the first terms will be sufficient to estimate with tolerable exactness the value of the arc. If, for example, we take $x = \frac{1}{2}$, we have $y = \frac{\pi}{6}$, or the sixth part of the half of the circumference of the circle. We obtain also

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1.1}{1.2\; 3.2^3} + \frac{3.3}{1.2.3.4.5} \cdot \frac{1}{2^5} + \ldots;$$

taking the six first terms of this series, and multiplying the result by 6, we obtain a tolerably exact value for the half of the circumference, namely, $\pi = 3\;1415926$.

Forserieses still more rapidly approximative, see p. 245 of this volume.

ON VANISHING FRACTIONS, FRACTIONAL FUNCTIONS WHICH, FOR A CERTAIN VALUE OF x, BECOME $\frac{0}{0}$.

33. In Mathematical researches we sometimes meet with fractional functions which vanish for a certain value of x, both in the numerator and in the denominator, which, however, if numerator and denominator are divided by a common factor, are equal to a determinate quantity called the *true value of the function.*

Thus, the fraction $\frac{ax - x^2}{a^2 - x^2}$ becomes $\frac{0}{0}$ in the supposition of $x = a$; but dividing both numerator and denominator by $a - x$, the fraction becomes

$\frac{x}{a+x}$, and when $x = a$, it becomes equal to $\frac{1}{2}$; which is consequently the true value of the proposed fractional function.

To find the true value of any fractional function which becomes $\frac{0}{0}$ when $x = a$, we make use of the process previously given for the development of any function in a series according to the integral powers of x.

Denoting in general the proposed fractional function by $\frac{u}{v}$, putting instead of x, $a + h$ or $a - h$, we may develop u as well as v according to the powers of h. Thus $\frac{u}{v}$ will be transformed to

$$\frac{mh^{\alpha} + nh^{\beta} + \ldots}{m'h^{\alpha'} + n'h^{\beta'} + \ldots},$$

where m, n . . ., and m', n' . . ., include only a, and the exponents α, β . . ., as well as α', β' . . ., form an ascending series.

It is not necessary that these exponents be integral numbers, though Maclaurin's series there employed develops the function in a series according to integral powers (art. 17). Let, for example, $u = (x^2 - a^2)^{\frac{1}{2}}$, a function which becomes $= 0$ when $x = a$; supposing $x = a + h$, the function becomes

$$(2ah + h^2)^{\frac{1}{2}} = h^{\frac{1}{2}} (2a + h)^{\frac{1}{2}}$$

or, expanding the radical according to art. 16,

$$= (2a)^{\frac{1}{2}} \cdot h^{\frac{1}{2}} + \frac{1}{2} (2a)^{-\frac{1}{2}} \cdot h^{\frac{3}{2}} - \frac{1}{8} (2a)^{-\frac{3}{2}} \cdot h^{\frac{5}{2}} + \ldots,$$

where the exponents of h are evidently fractional. If we put in the fraction obtained above,

$$\frac{mh^{\alpha} + nh^{\beta} + \ldots}{m'h^{\alpha'} + n'h^{\beta'} + \ldots},$$

$h = 0$, we necessarily obtain the same value that $\frac{u}{v}$ takes, when $x = a$. However, not to obtain $\frac{0}{0}$, but the true value, we make use of the following process.

We will first distinguish the three cases, $\alpha > \alpha'$, $\alpha < \alpha'$, and $\alpha = \alpha'$. In the first case, the numerator and the denominator being divided by $h^{\alpha'}$, we obtain

$$\frac{mh^{\alpha-\alpha'} + nh^{\beta-\alpha'} + \ldots}{m' + n'h^{\beta'-\alpha'} + \ldots\ldots}$$

In the second case, dividing by h^{α}, we obtain

$$\frac{m + nh^{\beta-\alpha} + \ldots}{m'h^{\alpha'-\alpha} + n'h^{\beta'-\alpha} + \ldots\ldots}$$

In the third case, we divide by $h^{\alpha} = h^{\alpha'}$, and obtain

$$\frac{m + nh^{\beta-\alpha} + \ldots\ldots}{m' + n'h^{\beta'-\alpha} + \ldots\ldots}.$$

We must then put $h = 0$; we obtain for the true value in the first case, 0; in the second case ∞; in the third, $\frac{m}{m'}$.

34. The process just explained will be applied in the following examples, in which we shall shew at the same time the common factor of the numerator and denominator.

Ex. 1. Let it be required to find the true value of the function

$$\frac{m \,.\, \text{arc}\left(\sin. = \frac{x}{a}\right)}{x},$$

which becomes $\frac{0}{0}$, when $x = a$.

We here put (art. 34) $x = 0 + h = h$; whence we obtain

$$\frac{m \text{ arc}\left(\sin. = \frac{h}{a}\right)}{h} = (\text{by art. } 31) \frac{m\left(\frac{h}{a} - \frac{h^3}{a^3 . 1 . 2 . 3} + \ldots\right)}{h};$$

Now dividing both numerator and denominator by h, and then making $h = 0$, we obtain for the true value, $\frac{m}{a}$. To find the common factor, we must develop the numerator of the proposed fraction in a series according to x, by the method of art. 31, we thus obtain

$$\frac{m\left(\frac{x}{a} + \frac{x^3}{a^3 . 1 . 2 . 3} + \ldots\right)}{x},$$

whence it is evident that the required factor is $= x$, because, dividing both numerator and denominator by x, and making $x = 0$, we obtain the true value already found.

In some cases, the operation is facilitated by a convenient substitution, as will be seen in the following example.

Ex. 2. Let the proposed function be $\frac{1 - \sin. x}{\cos. x}$, which becomes $\frac{0}{0}$, when $x = \frac{\pi}{2}$, or when $\sin. x = 1$.

According to the process in art. 34, we must make $x = \frac{\pi}{2} + h$, and then develop both the numerator and the denominator, (art. 30) in a series according to h. But still more to facilitate the development of the powers of h, it will be well in the proposed function, to substitute z for $\sin x$; whence we find

$$\frac{1 - z}{(1 - z^2)^{\frac{1}{2}}}.$$

This expression being $\frac{0}{0}$, when $z = 1$, we must make (34) $z = 1 - h$: we thus obtain

$$\frac{h}{(2h - h^2)^{\frac{1}{2}}} = (\text{art. } 34) \; \frac{h}{h^{\frac{1}{2}}(2 - h)^{\frac{1}{2}}},$$

or, expanded according to art. 16,

$$= \frac{h}{2^{\frac{1}{2}} . h^{\frac{1}{2}} = \frac{1}{3^{\frac{3}{2}}} . h^{\frac{3}{2}} - \ldots .}$$

dividing the numerator and denominator, according to art. 34, by $h^{\frac{1}{2}}$, then making $h = 0$, we obtain the true value $= 0$.

To obtain the common factor, it will be seen that we have

$$\frac{1 - z}{(1 - z^2)^{\frac{1}{2}}} = \frac{(1 - z)^{\frac{1}{2}} . (1 - z)^{\frac{1}{2}}}{(1 - z)^{\frac{1}{2}} . (1 + z \tfrac{1}{2}};$$

therefore dividing the numerator and the denominator by $(1 - z)^{\frac{1}{2}}$, we obtain

$$\frac{(1 - z)^{\frac{1}{2}}}{(1 + z)^{\frac{1}{2}}},$$

a fraction which does not become $\frac{0}{0}$, in the supposition of $z = 1$, but which gives the true value $= 0$, as we have found above. The required factor will therefore be $(1 - z)^{\frac{1}{2}}$, or $(1 - \sin. x)^{\frac{1}{2}}$.

Ex. 3. Let the proposed function be

$$\frac{(x^2 - a^2)^{\frac{3}{2}}}{(x - a)^{\frac{3}{2}}},$$

which becomes $\frac{0}{0}$, when $x = a$. Making $x = a + h$, we have

$$\frac{(2ah + h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}} = \frac{h^{\frac{3}{2}}(2a + h)^{\frac{3}{2}}}{h^{\frac{3}{2}}}$$

(developing according to art. 16)

$$= \frac{(2a)^{\frac{3}{2}} h^{\frac{3}{2}} + \frac{3}{2}(2a)^{\frac{1}{2}} . h^{\frac{5}{2}} + \ldots}{h^{\frac{3}{2}}}$$

dividing both numerator and denominator by $h^{\frac{3}{2}}$, and then making $h = 0$, we obtain the true value $= (2a)^{\frac{3}{2}}$.

Putting the proposed function under the form

$$\frac{(x + a)^{\frac{3}{2}} . (x - a)^{\frac{3}{2}}}{(x - a)^{\frac{3}{2}}}$$

it is manifest that the common factor is $(x - a)^{\frac{3}{2}}$; for dividing both numerator and denominator by this quantity, we obtain $(x + a)^{\frac{3}{2}}$, an expression which, when $x = a$, gives the true value found above $= (2a)^{\frac{3}{2}}$.

From the whole, this practical rule is readily reduced for finding the value of a vanishing fraction :

Differentiate both terms of the fraction the same number of times, until one or other ceases to become 0 on the supposition that $x = a$, or $x = c$, &c. Then substitute a, &c. for x in both terms of the fraction, and the result will be the value required.

Ex. 4. Required the true value of the fraction

$$\frac{ax^2 + ac^2 - 2acx}{bx^2 - 2bcx + bc^2}; \text{ which becomes } \frac{0}{0} \text{ when } x = c.$$

Ex. 5. Required the true value of the fraction

$$\frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2}; \text{ which becomes } \frac{0}{0} \text{ when } x = a.$$

Ex. 6. Find the value of $y = \frac{\tan. x}{\tan. 3x}$, when $x = \frac{1}{2}\pi$.

Ex. 7. Let the function be $\frac{x(x - a)^2}{x^3 + a^3 - ax(a + x)}$, which becomes $\frac{0}{0}$, when

Ex. 8. Let the function be $\frac{1 - \sin. x + \cos. x}{\sin. x + \cos. x - 1}$, which becomes $\frac{0}{0}$ when $x = \frac{1}{2}\pi$, or $\sin. x = 1$.

Ex. 9. Let the function be $\frac{2\pi a^3}{x} \log. \frac{x + (a^2 + x^2)^{\frac{1}{2}}}{a}$; which becomes $\frac{0}{0}$ when $x = 0$.

ON THE MAXIMA AND MINIMA OF A FUNCTION.

35. If in a function of x, represented by y, we put, instead of x, all the possible consecutive values, y itself receives a different value for each of the respective values of x. The values of y which are greater than those which immediately precede or follow them, are called *maxima* of the function y, and the values of y, which are less than those which immediately precede and follow them, are called *minima* of the function y. which is thus expressed "the function y is for a certain value of the independent variable x a *maximum* or a *minimum*."

For example, let the function $y = a^2 + (x - m)^2$. Putting m instead of x, we have $y = a^2$; but if we take $m + c$ or $m - c$ instead of x, in which c designates any positive or negative quantity, as small as we please, we have $y = a^2 + c^2$.

Thus, since the values of x which immediately precede and follow that of $x = m$, make the function greater, the function y is a minimum when $x = m$, that is to say $= a^2$. Moreover, the function for a greater value of x, taken either in a positive or negative sense, becomes still greater and consequently there is no value of x for which y is greatest. The function y has therefore no maximum.

36. We shall hereafter shew how the differential calculus is used for determining the maximum and minimum of a proposed function; but first present some considerations.

Let there be given the series

$$ax + bx^2 + \ldots + ux^{m2};$$

it will, evidently, be convergent, that is to say, each consecutive term will be less than the preceding, when $x < 1$. But it is also observable that if we assign a sufficiently small value to x, the first term of the series, namely ax, may become greater than the sum of all the consecutive terms, however large their number may be. We may prove the truth of this proposition by determining the value of x for which it takes place. We must therefore have

$$ax > bx^2 + cx^3 + \ldots + ux^m,$$

$$\text{or } a > x(b + cx + \ldots + ux^{m-2}).$$

$$\text{In the series, } b + cx + \ldots + ux^{m-2},$$

let s be the greatest of the coefficients; the quantity x will then have the required value, when it is so determined that we have

$$a > x(s + sx + \ldots sx^{m-2}),$$

$$\text{or } a > sx(1 + x + \ldots + x^{m-2}),$$

$$\text{or again } a > sx\left(\frac{x^{m-1} - 1}{x - 1}\right).$$

x being evidently < 1, from what has been said, we shall designate its value by $\frac{1}{p}$, where $p > 1$; we must therefore determine p so as to have

$$\frac{\left(\frac{1}{p^{m-1}}-1\right)}{\frac{1}{p}-1} \text{ or } a > \frac{s}{p^{m-1}}\left(\frac{1-p^{m-1}}{1-p}\right),$$

$$\text{or again } ap^{m-1} > s\left(\frac{p^{m-1}-1}{p-1}\right)$$

$$\text{that is to say } ap^{m-1} > \frac{sp^{m-1}}{p-1} - \frac{s}{p-1},$$

for this we need only make

$$ap^{m-1} = \frac{sp^{m-1}}{p-1}, \text{ that is to say } a = \frac{s}{p-1},$$

whence $p = \frac{s+a}{a}$; the desired value of x is therefore $= \frac{a}{s+a}$. Substituting this value instead of x in the series

$$ax + bx^2 + \ldots + ux^m,$$

the term ax will be greater than the sum of all the consecutive terms.

Hence it will also be easy to determine the value of x in such manner that *any* term of a proposed series becomes greater than the sum of all the consecutive terms. If, for example, it were proposed to make the terms px^n greater than the sum of all those which follow it, we need only, according to the preceding make $x = \frac{p}{p+s}$, s designating the greatest coefficient of all those which follow p.

37. In a function of x, represented by y, instead of x, putting successively $x+h$ and $x-h$, and designating the corresponding values of y by $y_{,}$ and ${}_{,}y$, we obtain by Taylor's theorem (17),

$$y_{,} = y + p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1.2} + r \cdot \frac{h^3}{1.2.3} + \ldots$$

$${}_{,}y = y - p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1.2} - r \cdot \frac{h^3}{1.2.3} + \ldots$$

Now instead of x, putting in these two expressions, a value represented by a, for which p becomes $= 0$, and designating the values of $y_{,}$, ${}_{,}y$, p, q, r . . . changed by the supposition of $x = a$, by

$$y_{,x=a},\ {}_{,}y_{x=a},\ y_{x=a},\ p_{x=a},\ q_{x=a},\ r_{x=a}\ \ldots \text{ (see art. 16)},$$

or, in an abridged form by $y_{,}'$, ${}_{,}y'$, y', p', q', r' . . ., we obtain

$$y'_{,} = y' + q' \cdot \frac{h^2}{1.2} + r' \cdot \frac{h^3}{1.2.3} + \ldots,$$

$${}_{,}y' = y' + q' \cdot \frac{h^2}{1.2} - r' \cdot \frac{h^3}{1.2.3} + \ldots,$$

If h is made so small that $q' \cdot \frac{h^2}{1.2}$ is greater than the sum of all the consecutive terms (36), and if we suppose q' positive, the expression

$$q' \cdot \frac{h^2}{1.2} + r' \cdot \frac{h^3}{1.2.3} + \ldots,$$

will be positive as well as

$$q' \cdot \frac{h^2}{1.2} - r' \cdot \frac{h^3}{1.2.3} + \ldots;$$

$y_{,}'$, as well as ${}_{,}y'$ will therefore be greater than y', that is to say $y' = y_{x=a}$ will be less than the value which immediately precedes and follows it, or (35) y is a *minimum* when $x = a$. If, on the contrary, on the same supposition for the value of h, we make q' negative, $y_{,}'$, as well as ${}_{,}y'$ will be less than y', or (35) y is

a *maximum* when $x = a$. The condition under which y becomes a *minimum*, is therefore that instead of x, we suppose a value $= a$ which makes p vanish and renders q positive; and the condition under which y becomes a *maximum*, is that instead of x, we suppose a value $= a$ which makes p vanish and renders q negative. Therefore, in order in a proposed function, to examine whether y is a *maximum* or a *minimum*, and for what value of x it obtains, we must make $p = 0$, and determine x by this equation. If q becomes positive for the value found of x, which we designate by a, y is a *minimum* for $x = a$; if, on the contrary, q becomes negative by this supposition, y is a *maximum* for $x = a$.

38. It is manifest that we must examine for each real value of x which we obtain from the equation $p = 0$, the sign by which q is affected, and that only the imaginary values of x may be neglected in the equation $p = 0$; it also hence results that a function may have several maxima and minima.

If q does not contain x, which will be the case when the equation $p = 0$ includes x only in the first power, q indicates immediately by its sign, whether the function is a maximum or a minimum.

A few examples will elucidate what has been said.

$$\text{Let (1) } y = \frac{x^3}{3} + ax^2 - 3a^2x,$$

whence $p = x^2 + 2ax - 3a^2$, and $q = 2x + 2a$.
we have therefore $p = 0 = x^2 + 2ax - 3a^2$,
whence $x = -a \pm 2a$,
that is to say, x has the two values $+ a$ and $- 3a$; whence we find the two values of q

$$+ 4a \text{ and } - 4a.$$

Therefore the proposed function is for $x = + a$ a *minimum*, and for $x = - 3a$ a *maximum*. The value of the minimum (as may easily be found by the substitution of the value of x in the proposed function) is $= - \frac{5}{3} a^3$, and that of the maximum is $= + 9a^3$.

$$\text{(2.) Let } y = 2x^4 + a^3x,$$
$$\text{then } p = 8x^3 + a^3,\ q = 24x^2.$$

From $p = 0$, we obtain the values $x = - \frac{a}{2}$, and $x = \frac{a}{4}(1 \pm \sqrt{-3})$. The two last values being imaginary, we need only consider the first, by which q becomes $= 6a^2$. The proposed function is therefore a minimum for $x = - \frac{a}{2}$. The value of the minimum is $= \frac{-3a^4}{8}$.

(3.) Let the function proposed be that already considered in art. 35.

$$y = a^2 + (x - m)^2,$$
$$\text{whence } p = 2(x - m),\ q = 2.$$

The function is a *minimum* for the value of $x = m$, which we deduce from $p = 0$, which results immediately from the positive sign by which q is affected.

39. If q as well as p, vanishes for $x = a$, the two series of No. 37 change into the following:

$$y'_{,} = y' + r' \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots,$$
$$\text{and } {}_{,}y' = y' - r' \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots$$

If we here make the expression $r' \cdot \frac{h^3}{1 \cdot 2 \cdot 3}$ by a sufficient diminution of h

greater than the sum of all the consecutive terms (36), it will, because of the opposite signs by which it is affected in the two series, make $y'_{,}$ greater and $_{,}y'$ less than y'. The function consequently will be neither maximum nor minimum in the case of p and q vanishing for $x = a$. But if, on the contrary, on the supposition of $x = a$, r as well as p and q becomes equal to zero, a circumstance in which we have

$$y'_{,} = y' + s' \cdot \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots,$$

$$\text{and } {}_{,}y' = y' + s' \cdot \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} - \ldots.$$

y' will be a maximum or a minimum, according as s is negative or positive for $x = a$. Designating generally by $\frac{d^m y}{dx^m}$ the first of the differential coefficients of the function y which, under the supposition of $x = a$, does not vanish; the function y will be for $x = a$, neither a maximum nor a minimum when m is an odd number; but it will be a maximum or a minimum when m is an even number; that is to say, a maximum, if $\frac{d^m y}{dx^m}$ becomes negative for $x = a$, and a minimum if $\frac{d^m y}{dx^m}$ becomes positive for $x = a$.

Let us, for example, consider the function

$$y = (x - a)^n,$$

where n is a positive whole number. We have

$$p = n\,(x - a)^{n-1},\ q = n\,(n - 1)\,(x - a)^{n-2}, \text{ \&c.},$$

$$\text{to } \frac{d^n y}{dx^n} = n\,(n - 1) \ldots (x - a)^{n-n} = n\,(n - 1)\,(n - 2) \ldots 1.$$

From $p = 0$, we obtain $x = a$; for this value of x all the successive differential coefficients vanish, as far as that of the order n: the proposed function may therefore, according to what has just been said, be a *maximum* or a *minimum*, if n is an even number: but it is in fact only a *minimum*, because the first of the differential coefficients which do not vanish, is $= n\,(n - 1) \ldots 1$, therefore positive.

40. We now present some problems on this subject.

Ex. 1. A given number m is to be divided into two parts, such that their product shall be a maximum. In this, as in the following problems, we must in the first place determine the independent x and the function y. Let x be one of the two parts, and y the product of the two parts susceptible of a maximum, and which depend on x. Now, if one of the two parts is $= x$, the other is $= m - x$; therefore, according to the problem,

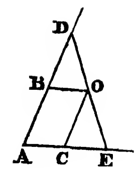
$$y = x\,(m - x).$$

Whence we obtain $p = m - 2x$, $q = -2$;

the value of $x = \frac{m}{2}$, found from the equation $p = 0$, gives consequently a *maximum*, because of the negative sign of q (art. 38,)

The solution of the proposed problem consists therefore in dividing the given number m into two equal parts.

It is at once obvious that this problem includes also the following: "Let it be proposed to divide a given right line into two parts, in such manner that we may with it describe a rectangle whose area shall be a maximum." The solution gives a square whose area will in fact be the greatest among all rectangles of the same perimeter.



Ex. 2. Within an angle BAC, whose sine is $= a$, is given a point O; it is proposed to draw through O a right line to the two lines bounding the angle, in such manner that the triangle so formed shall be a minimum.

The position of O being given, we may consider the lines OB, OC, drawn through the point O, parallel to the two sides of the angle, as given; denote them by a and b. Suppose the line DE drawn through O to answer the required condition, namely, to be, among all the lines drawn through O, that which includes the smallest triangle DAE.

$$\text{Put DB} = x, \text{ then } x : a = x + b : \text{AE, or AE} = \frac{a(b+x)}{x};$$

the perpendicular drawn from D to AE is $= a(b + x)$; therefore the triangle

$$\text{DAE} = \frac{aa(b+x)^2}{2x}.$$

Now, the triangle being the function which is susceptible of a minimum, let it be denoted by y. We have thus

$$y = \frac{aa(b+x)^2}{2x}, \text{ whence } p = \frac{aa(a^2 - b^2)}{2x^2}, q = \frac{aab^2}{x^3}.$$

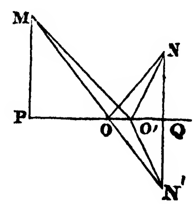
From $p = 0$ we obtain $x = \pm b$.

The first value $x = + b$ renders q positive, and consequently indicates a *minimum* whose value is $= 2aab$, that is to say double the parallelogram ABOC. For the construction of this said triangle, make BD $=$ AB $= b$, and then from the point D, which will be thus determined, draw a right line through O.

The other value $x = - b$ renders q negative, and thus indicates a maximum. But, it is manifest, that the triangle cannot be a maximum; for if x diminishes, the triangle augments more and more, and increases continually, as may be plainly seen from the figure (compare art 35): therefore $x = - b$ cannot indicate the greatest triangle. It is nevertheless certain (according to art. 37) that the value of the function $y = \frac{aa(b+x)^2}{2x}$ answers to a maximum if considered in itself, or algebraically, that is, without reference to the proposed problem. To make this still plainer, it will be well to construct the function geometrically, as is often done, that is to say in a curve (in which x and y are considered as coordinates). We thus obtain a hyperbola in which the points $x = \pm b$ evidently indicate the least positive ordinate and the least negative ordinate, that is to say, the ordinates of which one is a *maximum*, the other a *minimum*.

It will be evident that $x = - b$ gives in fact a maximum of the function regarded in itself, but not a solution for the proposed problem. We have therefore the same case here as in the problem, which lead to certain equations, in which some values answer to the equations, but not to the solution of the proposed problem.

The particular determination which the solution takes may be easily found if the given angle is a right angle, where $a = 1$; and also when the point O is so taken in the midst of the angular space that $a = b$.



Ex. 3. It is proposed to find in the line PQ the point O, such that the sum of its distances from the two given points M and N, taken out of the line shall be a minimum.

Let the perpendiculars MP and NQ be drawn from M and N, and let them be denoted by a and b; let PQ $= c$; let also OP $= x$, we then have

$$y = \mathrm{MO} + \mathrm{NO} = (a^2 + x^2)^{\frac{1}{2}} + [b^2 + (c - x)^2]^{\frac{1}{2}}$$

From $p = 0$, we obtain two values for x, namely, $x = \frac{ac}{b + a}$, and $x = \frac{-ac}{b - a}$, both of which, making q positive, indicate minima. But it will be at once obvious, from the geometrical construction, that there will be only one minimum. To this end let NQ be prolonged until QN′ = QN, and draw MN′. The point of intersection O will be the point sought, since the sum

$$\mathrm{MO} + \mathrm{N'O} = \mathrm{MO} + \mathrm{NO}$$

is manifestly less than the sum of any other two lines, for example MO′ + N′O′ = MO′ + NO′. Finding the value of PO, by means of the similar triangles MPO and N′OQ, we obtain for it the first value of x, found above. That first value of x passes evidently to the other $\frac{-ac}{b - a}$, if a is taken in a negative sense, that is, so that the two points M and N are found on opposite sides of the line PQ.

What will be the value of the *minimum?* and what particular determinations does the solution undergo when $a = b$?

Ex. 4. Out of all the right cones, whose surface $= m^2$, to find that whose volume is a maximum.

Ex. 5. Required to find, among all right cylinders whose whole surface $= m^2$, that whose volume is a maximum: take also the case when the curve surface + one end $= m^2$.

Ex. 6. Is the function $y = \sqrt{2px}$ susceptible of either a maximum or a minimum?

Ex. 7. Let $y = \frac{x^2 - 3x + 2}{x^2 + 3x + 2}$, or $y = \frac{(x - 1)(x - 2)}{(x + 1)(x + 2)}$.

Ex. 8. Let $y = 2x^3 - \frac{1}{2}ax^2 - a^2x + \frac{1}{2}a^3$, to find such a value of x as shall make y a maximum or a minimum.

Ex. 9. Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$.

Ex. 10. Let $y = \frac{(x + a)^3}{(x + b)^2} = \frac{1}{v}$.

Ex. 11. Find an arc such that the rectangle under its tangent and the cosine of its double may be a maximum or a minimum.

Ex. 12. In a spherical triangle, right angled at B, given the angle A, to find the sides so that their difference $b - c$ shall be a maximum.

Ex. 13. Find the least parabola which shall circumscribe a given circle.

Ex. 14. Given the base of a plane triangle, and its altitude greater than the base, to determine it so that the vertical angle shall be a maximum.

Ex. 15. Having given a line, AB, in length, and another, CPD, in position, to find the point P at which the line AB will appear under the greatest angle possible.

Ex. 16. To find on the line which joins two lights of given intensities, the point which will be least illuminated by both together*.

* For other problems on maxima and minima, see pages 237—239 of this volume.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO GEOMETRY.

Fig. 1.

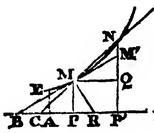


Fig. 2.

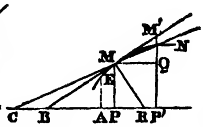


Fig. 3.

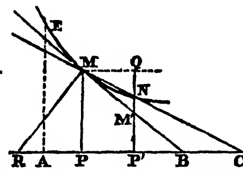
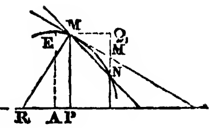


Fig. 4.



41. We have seen in art. 25 that, in figs. 1, 2, 3, 4, M'Q is $= p \,.\, \Delta x$, that is to say $= + p \,.\, \Delta x$ in figs. 1 and 2, and $= - p \,.\, \Delta x$ in figs. 3 and 4.

MQ being $= \Delta x$ (25) it results that

$$\tan.\ M'MQ = \tan.\ MBP = p,$$

that is to say that the trigonometrical tangent of the angle MBP, included between the tangent MB of a curve and the axis of the abscisses, is equal to $p = \frac{dy}{dx}$, that is to say, $= + p$, in figs. 1 and 2, when of the two adjacent angles about B, the acute angle MBP, is turned towards the positive side of the abscisses; and $= - p$, in figs. 3 and 4, when the acute angle MBP is turned towards the negative side of the abscisses.

Now, this angle varies with the point of contact; if we designate the co-ordinates of certain points of contact by x' and y', we must in $\frac{dy}{dx}$ put x' and y' instead of x and y, to obtain the trigonometrical tangent for this determinate point of contact, which will be expressed by $\frac{dy'}{dx'}$, or p'.

But it is to be observed that of these two quantities x' and y', x' only can be taken at pleasure, and the corresponding value of y' must be sought by the primitive equation of the curve.

To elucidate the preceding, let it be proposed to seek the trigonometrical tangent of the circle for the co-ordinates of the point of contact x' and y.

The equation of the circle referred to the centre being $y = (r^2 - x^2)^{\frac{1}{2}}$, it follows that $y' = (r^2 - x'^2)^{\frac{1}{2}}$; moreover p being $= \frac{-x}{y} = \frac{-x}{(r^2 - x^2)^{\frac{1}{2}}}$, the trigonometrical tangent required will be $= \frac{-x'}{y'} = \frac{-x'}{(r^2 - x'^2)^{\frac{1}{2}}}$. Taking $y' = (r^2 = x'^2)^{\frac{1}{2}}$ positively, that is, considering the part of the circle above the diameter, in which the ordinates are positive (art. 25), we see by the sign of the preceding expression, on which side of the origin the co-ordinates of the tangent meet the axis of the abscisses; thus, x' being *positive*, the tangent meets the axis of the abscisses on the positive side of the abscisses, as in figs. 3 and 4; but x' being *negative*, as in figs. 1 and 2, it meets it on the negative side.

Reciprocally, when in a given curve is required the point of contact where the tangent with the axis of the abscisses includes an angle whose trigonometrical tangent is $= a$, the co-ordinates of the point of contact x' and y' must be determined by the equation $\frac{dy'}{dx'} = a$, conjointly with the given equation of the curve.

Suppose, for example, that the tangent of the circle makes with the axis of the abscisses an angle $= 60°$, whose trigonometrical tangent is $= \pm 3^{\frac{1}{2}}$, we have the two equations

$$y' = (r^2 - x'^2)^{\frac{1}{2}}, \text{ and } \frac{dy'}{dx'} = \frac{-x'}{y'} = \pm 3^{\frac{1}{2}}.$$

It hence results that

$$x' = \frac{r}{2} \cdot \pm 3^{\frac{1}{2}}, \text{ and } y' = +\frac{r}{2} \text{ (if } y' \text{ is taken positively).}$$

If we make $\frac{dy'}{dx'} = 0$, we shall obtain the co-ordinates of the point of contact, where the tangent with the axis of the abscisses, makes an angle $= 0$, or where the tangent is parallel to the axis of the abscisses.

If, on the contrary, we make $\frac{dy'}{dx'} = \infty$ (a condition which is satisfied by making the denominator in the value of $\frac{dy'}{dx'}$ equal to zero) the corresponding angle is $= 90°$, that is, the tangent is perpendicular to the axis of the abscisses. For the circle, whose equation $y = (r^2 - x^2)^{\frac{1}{2}}$, we have for $\frac{dy'}{dx'} = \infty$.

$$x' = \pm r \text{ and } y' = 0.$$

Now, $\frac{dy'}{dx'} = 0$ indicates a maximum or a minimum (art. 37); therefore, since from $\frac{dy'}{dx'} = \infty = \frac{1}{0}$, it results that $\frac{dx'}{dy'} = 0$ (art. 9), the value of $\frac{dy'}{dx'} = \infty$ indicates a *maximum* or a *minimum*, but in reference to the absciss x'. The correctness of these two propositions will appear clearly from the inspection of a figure, for example, of a circle.

42. We may easily find the sub-tangent BP (fig. 1, 2, 3, 4) by the preceding method. In the triangle MPB, we have

$$\text{BP} = \text{MP} \div \text{tan. MBP} = y' \div \frac{dy'}{dx'} = y' \cdot \frac{dx'}{dy'}$$

and, according to art. 41, $\frac{dy'}{dx'}$ being positive in figs. 1 and 2, and negative in figs. 3 and 4, the expression for the subtangent BP will also be positive in figs. 1 and 2, and negative in figs. 3 and 4.

In the same manner the position of the subtangent may at once be determined by the *sign* by which its expression is affected.

We find readily by the right-angled triangles BMR and PMR (figs. 1, 2, 3, 4) that the sub-normal PR is equal to

$$y'^2 \div \text{PB} = y'^2 \div y' \cdot \frac{dx'}{dy'} = y' \cdot \frac{dy'}{dx'}.$$

Its position, as well as that of the subtangent, may be ascertained by the *sign* of its expression. The *length* of the tangent MB and that of the normal MR may at once be deduced from the value of the sub-tangent and that of the sub-normal.

For the former we obtain $y' \cdot \left(1 + \frac{dx'^2}{dy'^2}\right)^{\frac{1}{2}}$,

and for the latter $y' \cdot \left(1 + \frac{dy'^2}{dx'^2}\right)^{\frac{1}{2}}$.

43. It may be well to recal to the reader, in a few words, some propositions of the elementary Analytical Geometry, concerning the equation of the right line.

The most general equation of the right line is

$$y = ax + b,$$

in which x and y denote the co-ordinates, and a and b constant (though indetermined) quantities, so that b represents the ordinate at the origin of the co-ordinates, and a the trigonometrical tangent of the angle comprehended between the right line and the axis of the abscisses, supposing that, as usual, rectangular co-ordinates have been adopted. The sign a is positive when, of the two angles found by the line and the axis of the abscisses, that which is acute is towards the side of the positive abscisses; which takes place with the tangent BM, which here represents the right line in general, in figs. 1 and 2, when the angle MBP $< 90°$.

On the contrary, a is negative when of the two angles, that which is obtuse is towards the side of the positive abscisses, which takes place with the tangent BM in figs. 3 and 4. Moreover, for the tangent BM, the point of intersection B, in figs. 1 and 2 is on the *negative* side, and in figs. 3 and 4 on the *positive* side of the origin A; therefore the ordinate is positive in all the four figures, because the ordinates are taken in a *positive* sense *above* the axis of the abscisses (art. 25). In the contrary case, the ordinate, or b in the general equation of the right line is negative. In this manner the reader may, without difficulty, trace and project the four following forms.

(1) $y = + ax + b$, (2) $y = + ax - b$,
(3) $y = - ax + b$, (4) $y = - ax - b$.

We will now consider some of these determinations.

Let it be proposed to find the equation of a right line, which, including with the axis of the abscisses an angle whose trigonometrical tangent $= a'$, passes by a point whose co-ordinates are α and β.

Besides the general equation $y = ax + b$, we then have also, for the given point, $\beta = a\alpha + b$;

eliminating b from the two equations, we obtain $y - \beta = a (x - \alpha)$.

Moreover, because of the given angle, we have for the equation sought,

$$y - \beta = a' (x - \alpha).$$

This equation may also be expressed by

$$y = a'x + \beta - a'\alpha,$$

and it is obvious that the letter b of the general equation is represented by the expression $\beta - a'\alpha$, the *sign* of which remains still to be found, in order to determine the corresponding figure of the line according to the four particular forms given above.

In a similar manner we obtain for the right line which passes through two points whose co-ordinates are α, β, and α', β', the following equation:

$$y - \beta = \left(\frac{\beta' - \beta}{\alpha' - \alpha}\right) \cdot (x - \alpha).$$

Finally, let it be proposed to find the equation of the right line perpendicular to another whose equation is $y = a'x + b'$.

The required line, in consequence of being perpendicular to the axis of the abscisses, forms an angle whose trigonometrical tangent $= \frac{1}{a'}$. Moreover, the position of the required line belonging to the two latter of the four forms above, we have for the required equation

$$y = - \frac{1}{a'} \cdot x + b.$$

If moreover, the line sought passes through a point whose co-ordinates are α and β, we have for its equation

$$y - \beta = -\frac{1}{a'}(x - \alpha).$$

44. Hence it is easy to find the equations of the tangent and of the normal for any curve.

The tangent to a point whose co-ordinates are x', y', and including with the axis of the abscisses an angle whose trigonometrical tangent $= \frac{dy'}{dx'}$ (41) has for equation (43)

$$y - y' = \frac{dy'}{dx'} \cdot (x - x').$$

Moreover, the normal being perpendicular to the tangent and passing through a point whose co-ordinates are x', y', has for equation (43)

$$y - y' = -\frac{dx'}{dy'} \cdot (x - x').$$

Example. For the circle whose equation $y^2 + x^2 = r^2$, we have, as has been already shown,

$$\frac{dy'}{dx'} = \frac{-x'}{y'},$$

whence the equation of the tangent

$$y - y' = \frac{-x'(x - x')}{y'},$$

or (the equation of the circle giving $y'^2 + x'^2 = r^2$)

$$yy' + xx' = r^2.$$

We have also for the circle $\frac{-dx'}{dy'} = \frac{y'}{x'}$; therefore the equation of the normal is

$$y - y' = \frac{y'}{x'}(x - x') \text{ or } yx' = xy'.$$

The absence of the constant term in this equation shews that the ordinate is $= 0$ (43) at the origin of the co-ordinates, which is here the centre, and that consequently the normal passes through the centre, that is, it is one of the radii.

For *another example*, let there be given the equation

$$y^2 = 2mx + nx^2,$$

a general equation of all the sections of the cone referred to the summit and to the axis. It is proposed to find $\frac{dy}{dx}$, and then the equation of the tangent and of the normal. This is left as an exercise for the student.

45. We will now return to the expression considered in No. 25,

$$\Delta y = p \cdot \Delta x + \psi \cdot \Delta x^2.$$

We have seen, in the above article, that ψ is positive in figs. 1 and 3, where the curve turns its convexity towards the axis of the abscisses; negative in figs. 2 and 4, where it turns its concavity towards the axis of the abscisses. Now, we have (17)

$$\psi = \frac{q}{1 \cdot 2} + \frac{r}{1 \cdot 2 \cdot 3} \cdot \Delta x + \frac{s}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \Delta x^2 + \ldots,$$

and, if Δx is taken sufficiently small, ψ will be positive or negative, according as q is positive or negative. Hence we see that the curve will in any point present its convexity or its concavity towards the axis of the abscisses, according as the differential coefficient of the second order q, obtained from the equation of the

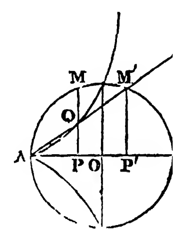
7

curve, for the corresponding value of the absciss x, is positive or negative. It follows at the same time, that for a *maximum* in which $p = 0$ and q is negative (36) the curve necessarily turns its concavity towards the axis of the abscisses, and that for a *minimum* in which $p = 0$ and q is positive, the curve necessarily turns its convexity towards the axis of the abscisses: which will at once appear from the inspection of a figure *.

It is evident, both from the preceding and from art. 41, that in order to ascertain the position of a curve with respect to the axis of the abscisses in a given point, the *signs* taken at that point by p and q must be examined.

If q change its sign immediately before and after a given point, it may be concluded that on one of the sides of this point the curve is convex, and on the other concave towards the axis of the abscisses; such a point is called a *point of inflexion*. It is evident that in this point, in which its sign passes to the opposite, q either vanishes or is infinite, the latter circumstance taking place if q is a fraction whose denominator vanishes for this point.

In order more clearly to understand how a quantity passes from positive to negative, through the infinite, it will be well to call to mind that in Trigonometry, the quantity $\frac{r \sin. x}{\cos. x}$ which expresses the length of the tangent of an angle x, becomes infinite when $\cos. x = 0$, or when the tangent ceasing to be positive, begins to become negative.



But even when q becomes, for a point of inflexion, either $= 0$, or $= \infty$, the reciprocal of the proposition does not necessarily obtain, and there does not always exist a point of inflexion for $q = 0$, or $= \infty$; for the equation of the curve represented in the margin, which at A, has a point of regression, must necessarily contain a radical of an even degree. Thus only can ordinates become imaginary, and the existence of the curve on one of the sides of this point, impossible. But in equations of this kind, the first differential coefficients become $= 0$ for that value of x which makes the radical vanish, and the consecutive coefficients become $= \infty$, as has been already shewn, art. 23 and 26. In what follows, it will therefore be observed: if for a certain value of x, q is $= 0$ or $= \infty$, the primitive equation must be examined, whether it indicates the existence of the curve on one side only of this point, or on both sides at the same time; it is only in the latter case that the point examined can be a point of inflexion, while in the former case it may be a point of regression.

It often happens that for some values of x, for which $p = 0$ or $= \infty$, q also becomes $= 0$ or $= \infty$, and yet neither of these two points exist, which will be examined further on. For this reason, and because some values of x, in which both p and q become $= 0$ or $= \infty$, do not correspond with any ordinate (as the primitive equation then shows) that it is absolutely necessary to consider the expressions of y, p and q, conjointly. But all this requires a more lengthened analysis than can here be given to it †.

46. Let us now consider two curves whose equations are

$$y = f(x) \text{ and } y = \phi(x).$$

* For the principles employed in the determination of asymptotes, see page 241.

† It is very elegantly treated, with various kindred inquiries, in *Hind's Differential Calculus*.

Suppose that these two curves meet in a point whose co-ordinates are x' and y', in such manner that we have for that point

$$y' = f(x') \text{ and } y' = \phi(x').$$

The differential coefficients, after substituting x' for x, are represented for the first curve by $p', q', r' \ldots$ (41) and for the other curve by $P', Q', R', \ldots$; we moreover designate the *difference* of the ordinates, or Δy, by k' for the first curve, and K' for the second. We have thus, by Taylor's theorem, putting $x' + h$ instead of x',

$$k' = p' . h + q' . \frac{h^2}{1 . 2} + r' . \frac{h^3}{1 . 2 . 3} + \ldots,$$

$$K' = P' \; h + Q' . \frac{h^2}{1 . 2} + R' . \frac{h^3}{1 . 2 . 3} + \ldots,$$

in which $p'. q', r', \ldots, P', Q', R', \ldots$ include only x' and constant quantities. If it should now happen that we had $p' = P'$, the two curves having a common point, would have the same tangent at this point (according to art. 41), and at the same time

$$k' - K' = (q' - Q') . \frac{h^2}{1 . 2} + (r' - R') . \frac{h^3}{1 . 2 . 3} + \ldots,$$

an expression which gives the *difference* of the ordinates of the two curves or the distance of these curves extremely near the point of contact, supposing h to be made sufficiently small. This distance will become still less, that is to say equal to

$$(r' - R') \frac{h^3}{1 . 2 . 3} + \ldots,$$

if we have at the same time $p' = P'$ and $q' = Q'$.

It is farther manifest that this distance will become *still* smaller, that is to say that the two curves are proportionally nearer to each other, immediately approximating to the point of contact (or as h becomes smaller) as the quantities $p' - P', q' - Q', r' - R' \ldots$, successively vanish: which may be thus expressed: "The two curves which pass through a point whose co-ordinates are x', y', have a contact of the *first* order if $p' - P' = 0$; a contact of the *second* order if $p' - P' = 0$ and $q' - Q' = 0$; a contact of the *third* order if $p' - P' = 0$, $q' - Q' = 0$ and $r' - R' = 0$; and so on for a contact of any order whatever."

From the preceding may easily be deduced the process by which we can determine for a curve, whose equation is $y = \phi(x)$, a contact of any order n, in a point whose co-ordinates are x', y', with a determinate curve whose equation is $y = f(x)$, that is with a curve in which all the constant quantities are supposed as entirely determinate. Putting for this purpose in the n first differential coefficients x' instead of x, we determine by the n equations

$$p' - P' = 0, q' - Q' = 0 \ldots, \&c \;,$$

combined with the primitive equation

$$y' = f(x') = \phi(x'),$$

$n + 1$ constant quantities which enter into the equation $y = \phi(x)$.

The equation of a curve therefore, as is evident, that it may be susceptible of a contact of the *first* order, must contain *two* constant quantities, for a contact of the *second* order *three* constants; in general, for a contact of the nth order, $n + 1$ constants.

A curve of which all the constant quantities which enter into its equation have been determined in the manner just explained, is called an *osculatory* curve *.

* It may here be remarked that in contact of an *even* order there are both contact and intersection, while in contact of an *odd* order, there is contact without intersection. Also, that with

47. Let us now, for an example, give to the *right line*, whose equation is $y = ax + b$, a contact of the *first* order in a point whose co-ordinates are x', y', with the determinate curve (46) $y = f(x)$. We here have

$$\phi(x') = ax' + b, \text{ and } P' = \frac{dy'}{dx'} = a.$$

We thus obtain, according to the preceding article, the two equations

$$f(x') = y' = ax' + b \text{ and } a = p' = \frac{dy'}{dx'}.$$

It hence results that

$$b = y' - x' \cdot \frac{dy'}{dx'}.$$

The equation of the right line which has a contact of the first order (46) with the proposed curve, in a point whose co-ordinates are x' and y', or the equation of the right which is tangent to this curve at the point named, will be therefore

$$y = \frac{dy'}{dx'} \cdot x + y' - x' \cdot \frac{dy'}{dx'}, \text{ or } y - y' = \frac{dy'}{dx'}(x - x'),$$

as we found, No. 44.

For a second example, let us consider the circle. Its most general equation, as is well known, is

$$(y - \beta)^2 + (x - \alpha)^2 = r^2,$$

where r denotes the radius, and α, β, the co-ordinates of the centre. The circle is, consequently, susceptible of a contact of the second order, its equation including three constant quantities; it therefore can only become *an osculatory circle*, by a contact of this order (46). Differentiating the equation of the circle twice successively, we obtain

$$\frac{dy}{dx} = P = \frac{-(x - \alpha)}{[r^2 - (x - \alpha)^2]^{\frac{1}{2}}}, \quad \frac{d^2y}{dx^2} = Q = \frac{-r^2}{[r^2 - (x - \alpha)^2]^{\frac{3}{2}}}.$$

And therefore from the preceding article we have,

$$y - \beta = [r^2 - (x' - \alpha)^2]^{\frac{1}{2}} \ldots \text{ (1)},$$

$$p' = \frac{-(x' - \alpha)}{[r^2 - (x' - \alpha)^2]^{\frac{1}{2}}} \ldots \text{ (2)},$$

$$q' = \frac{-r^2}{[r^2 - (x' - \alpha)^2]^{\frac{3}{2}}} \ldots \text{ (3)}.$$

Squaring both members of the equations (2) and (3), and eliminating $x' - \alpha$, we obtain

$$r = \frac{\pm(1 + p'^2)^{\frac{3}{2}}}{q'}.$$

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respect to the circle of curvature, there are but three constants, the radius and the co-ordinates of the centre; and hence the equality of y and of the first and second differential coefficients in the equations of the curve and circle, will give three equations by which the magnitude and position of the circle may be determined. With respect to any osculatory curve, the degree or order of contact will be *one* less than the number of constants in its equation. Yet at particular points the degree of contact may be higher than this, as the values of the variables at those points may be such as to render other differential coefficients equal. Thus, the circle of curvature will have with any curve, a contact of the third order at the point of greatest or of least curvature. This will be seen, by finding the point at which the third differential coefficient of the circle is equal to $\frac{d^3y}{dx^3}$, the like coefficient of the curve; and also by finding the point at which r is a maximum or minimum; as those points will be found to agree. It may, farther, be remarked that if *all* the differential coefficients be *equal* the curves entirely coincide.

The expression $r = \frac{\pm (1 + p^2)^{\frac{3}{2}}}{q}$ belongs therefore to the radius of the osculatory circle for every point of the proposed curve whose co-ordinates are x and y.

The double sign, which relates to the opposite position of two radii together forming a diameter, shows that the centre of the osculatory circle and the axis of the abscisses are found either on the same side, or on different sides of the curve. In the former case, the curve is concave towards the axis of the abscisses, and q *negative* (45). In the second case, the curve is convex towards the axis of the abscisses, and q *positive*. Therefore, if $(1 + p^2)^{\frac{3}{2}}$ is always taken positive, r and q will have the same sign. From this it is easy to determine the position of the radius r.

Example. For the parabola, we have $y^2 = mx$:

$$\text{whence } p = \frac{m^{\frac{1}{2}}}{2x^{\frac{1}{2}}} \text{ and } q = \frac{-m^{\frac{1}{2}}}{4x^{\frac{3}{2}}};$$

$$\text{therefore } r = \frac{-(4x + m)^{\frac{3}{2}}}{2m^{\frac{1}{2}}},$$

negative because q is negative.

Problem. Let it be proposed to find the value of r for a conic section generally, whose equation is $y^2 = 2mx + nx^2$, and then to express this value by the normal.

What value will r have, when $x = 0$?

From the equation (2) we obtain

$$\frac{dy}{dx'} = \frac{-(x' - a)}{[r^2 - (x' - a)^2]^{\frac{1}{2}}} = \frac{-(x' - a)}{y' - \beta};$$

$$\text{whence we deduce } y' - \beta = -\frac{dx'}{dy'} \cdot (x' - a),$$

which, according to art. 44, is the equation of the normal belonging to the point x', y', and which passes through the point a, β. It will be understood therefore that the centre of the osculatory circle falls in the normal drawn through the point of contact. It is hence easy to construct the osculatory radius for a determinate point of a curve, since we know not only its length, but also that it is found in the normal of the given point and on the concave side of the curve.

The curvature of the circle is uniform in all its points, and is inversely as its radius, while the curvature of every other curve is variable. But as the curve has in each point very nearly the same curvature as its osculatory circle there (46), its curvature in each point may be judged of by that of the osculatory circle belonging to it; hence the osculatory circle is called *the circle of curvature*, and the osculatory radius, the *radius of curvature.*

48. Since each x' and y' corresponds to another a and β, we perceive that the centres must necessarily form a continuous curve as well as the proposed curve. The equation of this curve, which we call the *locus* or the *curve of the centres*, or the *evolute*, is obtained by the equations (1), (2), (3) in the preceding article. In fact, we obtain

$$x' - a = \frac{p'(1 + p'^2)}{q'}, \text{ and } y' - \beta = \frac{-(1 + p'^2)}{q'}.$$

Substituting in these two expressions for y', p', and q' their values in x', and then eliminating x', we obtain the equation sought between a and β.

Example. Let it be proposed to find the *evolute* or *locus* of the centres for the ordinary parabola, which has for equation $y'^2 = mx'$. We have

$$p' = \frac{m^{\frac{1}{2}}}{2x'^{\frac{1}{2}}},\ q' = \frac{-m^{\frac{1}{2}}}{4x'^{\frac{3}{2}}};$$

$$\text{therefore } 1 + p'^2 = \frac{m + 4x'}{4x'}.$$

$$\text{whence we have } x' - a = \frac{-(m + 4x')}{2},\ \text{or } x' = \frac{2a - m}{6} = \frac{a - \frac{m}{2}}{3};$$

$$\text{we have moreover } y' - \beta = m^{\frac{1}{2}} . x'^{\frac{1}{2}} - \beta = \frac{x'^{\frac{1}{2}}(m + 4x')}{m^{\frac{1}{2}}},$$

$$\text{whence } mx'^{\frac{1}{2}} - m^{\frac{1}{2}} . \beta = mx'^{\frac{1}{2}} + 4x'^{\frac{3}{2}},$$

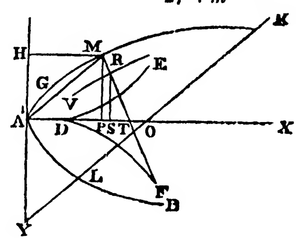
$$\text{or } 4x'^{\frac{3}{2}} = -m^{\frac{1}{2}}\beta,\ \text{or finally } 16x'^3 = m\beta^2;$$

substituting the value of x' previously found, we obtain, for the equation sought,

$$\frac{16\left(a - \frac{m}{2}\right)^3}{27m} = \beta^2;$$

putting $a - \frac{m}{2} = a'$; that is to say, transporting the origin of the abscisses from A to D, so that AD $= \frac{m}{2}$, we obtain the equation $\beta^2 = \frac{16}{27 . m} . a'^3$

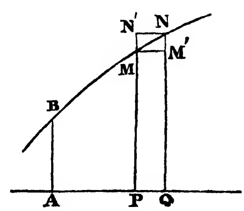
In the annexed curve the branch DF is the locus of the centres of the osculatory circles of the branch AK of the parabola BAK.



ON THE DIFFERENTIAL OF THE AREA.

49. The proposition in art. 24, of which we have made use in finding the differential of the arc of a curve, may now lead us to the differential of the area of a curve.

Let A the origin of the co-ordinates, AP $= x$, PQ $= \Delta x$, PM $= y$, QN $= y + \Delta y$; let two parallels be drawn through M and N to the axis of the abscisses MM′ and NN′. Designating by s the area ABMP, which will necessarily be a function of x, Δs must designate the portion of the area PQNM. Now, the area Δs is greater than the parallelogram PQMM′, whose measure is $= y . \Delta x$, and it is less than the parallelogram PQN′N, whose measure is $= (y + \Delta y) \cdot \Delta x$, or, acccording to Taylor's theorem (17),



$$= \Delta x\left(y + p . \Delta x + \frac{q}{1 . 2}\Delta x^2 + \ldots\right).$$

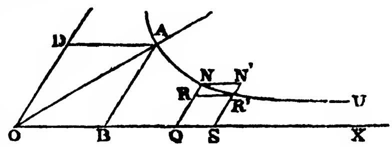
We have therefore $\Delta s > y\Delta x$, and $\Delta s < \Delta x . (y + p . \Delta x + \ldots)$,

$$\text{or } \frac{\Delta s}{\Delta x} > y,\ \text{and } \frac{\Delta s}{\Delta x} < y + p . \Delta x + q . \frac{\Delta x^2}{1 . 2} + \ldots;$$

whence, from art. 24, $\frac{ds}{dx} = y$, or $ds = y dx$;

that is, the differential of the area of a curve is equal to the ordinate of the curve, into the differential of the absciss.

We may also by a slight variation in the process find the differential of the area of a curve referred to oblique co-ordinates.



Let the angle ABX, formed by the two axes of the co-ordinates BA and BX $= \omega$; let BQ $= x$, QN $= y$, and QS $= \Delta x$. Drawing SR′ parallel to QN, and through N and R′ the lines NN′ and RR′ parallel to the axis of the abscisses, we have N′R′ $= \Delta y$.

Designating by S the area ABQN, and by ΔS the area QNR′S, we have

$$\Delta S < \text{parallelogram QSNN}',$$
$$\text{and } \Delta S > \text{parallelogram QSR}'\text{R};$$
$$\text{that is, } \Delta S < (y + \Delta y) \,.\, \sin.\, \omega \,.\, \Delta x,$$
$$\text{and } \Delta S > y \,.\, \sin.\, \omega \,.\, \Delta x.$$

If we develope the first of these two expressions by art. 17, and reason as above, we obtain $dS = y \,.\, \sin.\, \omega \,.\, dx$.

We might next proceed to shew the application of the Differential Calculus to investigations regarding arcs and areas in relation to *polar* co-ordinates, the differentials of surfaces and solids of various kinds, &c.; but as the volume has already been extended beyond its proposed magnitude, we are obliged to relinquish that part of our design, as well as any exhibition of the elements of the Integral Calculus. What is here exhibited, will enable the student to comprehend the principles of the Differential Calculus, and to compare them with those of Fluxions: for a more enlarged view of the science he is earnestly referred to some of the authors already named at page 203. He may also read with advantage Mr. Woolhouse's Essay on the Fundamental Principles of the Differential and Integral Calculus, in the Appendix to the Gentleman's Diary for 1835 and 1836.

THE END.

ADDITIONAL ERRATA

IN

VOLUME I.

PAGE

61. line 14. 15, from bottom, *for* foot *read* root.

65. note, l. 9, annex the factor b to the bracketted expression.

69. note, l. 15, *for* powers of n *read* powers of x.

71. line 6 from bottom, *for* tenth *read* truth.

79. — 5 from bottom, *for* $2 - = 6$ *read* $2-8 = -6$.

80. — 20 from bottom, *for* 2, 4, 6, 10, &c. *read* 2, 4, 6, 8, 10, &c.

104. — 11 from bottom, *dele* should.

108. — 15, *for* dn *read* dx.

110. — 5, *for* n *read* m; and in note † *for* $m + r = m - r + r = m$ *read* $m + r = n - r + r = n$; and in note ‡ for $a^{\frac{2}{3}} a^{\frac{7}{3}}$ *read* $a^{\frac{2}{3}} b^{\frac{7}{3}}$, and for $c^{\frac{1}{3}}$ *read* $b^{\frac{1}{3}}$.

111. — 16, *for* $\sqrt[4]{(a + z)}$ *read* $\sqrt[4]{(a + x)}$.

114. V. *for* $4x^1$ *read* $4x^{-1}$.

115. line 1. *for* $(2ac + c^{2\frac{1}{2}}$ *read* $(2ac\ c^2)^{\frac{1}{2}}$; line 5, *for* $\sqrt{2ac + c}$ *read* $\sqrt{2ac + c^2}$; line 14, *for* $acd - d + 23$ *read* $acd - d = 23$.

117. Ex. 10, *for* xxy *read* xy; and in Case II. Ex. 7. l, 1, *for* a *read* $\sqrt{a}$; and in l. 4, x *read* $a^{\frac{1}{2}}$.

118. Ex. 9. line 5, *for* $(x + y)^{\frac{1}{3}}$ *read* $(a + y)^{\frac{1}{3}}$; and in Ex. 10. line 2, *for* $\sqrt{\ }$ *read* $\sqrt[4]{\ }$.

119. — 1. line 2, *for* cdy *read* cdx; and in Ex. 2. Answer, for $(3d - 2b)$ *read* $(3d - b)$.

124. line 1, *for* x *read* a; l. 2 from bottom, *for* $(a - b)$ *read* $(a + b)$.

125. — 23, *for* $x + c$ *read* $a + e$.

128. — 1, *for* $+ x^3$ *read* $- x^3$; Question 12, *for* $\frac{3}{2} x^2$ *read* $\frac{3}{2} x^3$; and *for* $-\frac{7}{2} x$ *read* $+ \frac{7}{2}$.

129. — 24, *for* division *read* divisor.

130. — 9, same erratum; and l. 28, *for* x^{1-4} *read* x^{-4}.

131. Ex. 9. *for* $(u + x + y)$ *read* $(u + x + y)^2$.

133. — 11, *for* $1 - n$ *read* $1 - x$, and *for* a^7 *read* a^3. Ex. 3. Ans. *for* $- 3a^2$ *read* $- \frac{3a^2}{a}$.

140. — 5, *for* $- \frac{3a^3 + 3a^3}{3a^2}$ *read* $- \frac{3a^3 + 3b^3}{3a^2}$.

142. note, line 5, *for* a^{-2} *read* a^{3-2}; and in line 1 from bottom, *for* y *read* z.

144. line 9 from bottom, *for* $2x$ *read* $2z$; and in 7, *for* $(- 1^4)$ *read* $(- 1)^4$.

148. last line, *for* 3 *read* $3\frac{5}{10}$.

149. Ans. 2nd *for* $(5^{-\cdot 2})$ *read* (5^{-2}); and 3rd *for* (5^2) *read* (5^{-2}).

150. Ex. 3. Prob. III. *for* 75 *read* 45. Note 11. line 3, *for* $\sqrt[3]{\ }$ *read* $\sqrt[3]{a}$, in the denom.

152. Prob. VII. Ex. 2. write $\sqrt[3]{\ }$ before 280, 56, 8; and *for* $\sqrt{7}$ *read* $\sqrt[3]{7}$.

154. Ex. 10, *for* $\sqrt[3]{bx}$ *read* $\sqrt{bx}$; and Ex. 12, *for* $2x \times x^2$ *read* $2x + x^2$.

156. line 3, *for* 109 *read* 199; line 15, *for* x *read* x^2. In Ex. 8, interchange the signs of the answers; in Ex. 9, *for* n^2 *read* $\frac{n(n + 1)}{2}$; in Ex. 10, *for* 10 *read* 0; and omit the second part of the question.

162. — 23, *for* or the ratio *read* r the ratio.

164. Ex. 1, *for* $1 + 2^{11}$ *read* 1×2^{11}.

168. quotient for 2^3 *read* $2b^3$, and in the work *for* 1^2 *read* a^2.

170. line 2 from the bottom, *for* reference to p. 129 *read* 143.

PAGE

171. line 7, *for* reference to p. 130 *read* 144; line 25, *for* $\frac{b^2}{a}$ *read* $\frac{b^2}{a^2}$; and line 30, *for* $\frac{b^2}{2}$ *read* $\frac{b^2}{2a}$.

172. — 1, *for* $\frac{x}{a}$ *read* $-\frac{x}{a}$; and *for* $x = -2$ *read* $n = -2$; line 6, *for* $\frac{1}{a^3}$ *read* $\frac{1}{a^2}$.

173. — 2, *for* $\frac{x^3}{a}$ *read* $\frac{x^2}{a}$; line 3, *for* $\sqrt{\frac{a^2}{a^2+x^2}}$ *read* $\sqrt{\frac{1}{a^2+x^2}}$; line 6, *for* $\frac{4b^3}{a^2}$ *read* $\frac{4b^3}{a^3}$; line 14, *for* $-\frac{b^2}{2a^4}$ *read* $-\frac{b^3}{2a^3}$; line 16, *for* 6^3 *read* b^3.

174. — 1, *for* conveyancy *read* convergency; Table, line 2, *for* $-\frac{7}{8}$ *read* $\frac{7}{16}$; line 7, *for* $\frac{39}{35}$, &c. *read* $-\frac{29}{35}$, $-\frac{35}{42}$, $-\frac{41}{49}$, $-\frac{47}{56}$, $-\frac{53}{63}$, $-\frac{59}{70}$; next line, *for* 2040 *read* 2048; line 3 from bottom, *for* $-\frac{21}{23}$ *read* $-\frac{21}{33}$; and last line insert — between $\frac{1}{86}$ and $\frac{1}{11}$.

175. — 6, prefix another 0; and dele 0 from line 8; line 15, *for* 1081424 *read* 1103424, and correct the resulting work, giving for 672, in the answer, line 6 from the bottom, 708.

182. Qu. 11. Ans. $\frac{3}{8}a$. Qu. 14, *for* a^4 *read* x^4. Qu. 15. Ans. $\frac{ac}{b}$.

183. — 19, *for* $-\frac{13x-2}{17x-32}$ *read* $-\frac{13x-22}{17x-32}$. Qu. 22, *for* z *read* x. Qu. 25. Ans. ·333 .... Qu. 26, *for* $\frac{c}{d}x$ *read* $\frac{c}{b}x$. Qu. 31, *for* $v = 4$ *read* $v = 2$.

185. — 8. Ans. *for* $\frac{c}{2}\sqrt{\left(\frac{a}{b}+\sqrt{\frac{b}{a}}\right)}$ *read* $\frac{2}{c}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right.$.

187. — 7, *for* $x^3 = y^3$ *read* $x^3 - y^3$. Qu. 8, *for* $(a - a)$ *read* $a\,(x - y)$. Qu. 12. Ans. *for* 26^2 *read* $2b^2$.

188. — 6. Ans. $x = 5\frac{17}{140}$, $y = 6\frac{29}{70}$.

189. — 8, *for* $a = (bc - xy) =$ *read* $x\,(bc - xy) =$.

191. — 10, last denominator in the answer, *for* $bcln$ *read* $beln$. Qu. 13, *for* $az, bz, cz,$ *read* a^2z, b^2z, c^2z.

194. — 9. Ans., *for* 6*s.* 3*d.* *read* 4*s.* 3*d.*

195. — 26, *for* $8\frac{4}{7}$ sec. *read* $24\frac{18}{23}$. Qu 30, 31, belong to Quadratics.

196. note, line 13, *for* $\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}$ *read* $\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$.

[illegible]. Qu. 10. Ans. *read* or $(-\frac{1}{2} \pm \frac{1}{2}\sqrt{-59})^2$. Qu. 11. Ans. *read* $x = \pm\sqrt{\frac{1\pm\sqrt{5}}{2}}$.

[illegible]. — 12. Ans. add, and $x = \frac{a}{2}$. Qu. 13. second equation, *for* $x^{\frac{6}{3}}$ *read* $x^{\frac{16}{3}}$. Qu. 15. Ans. *for* 2·5 *read* — 2·25. Qu. 16, second equation, *for* $2x^2$ *read* $2x^3$.

201. line 16, 17 from bottom, *for* $-\frac{1}{3}$ and -12 *read* $-\frac{1}{6}$ and -6.

202. *for* b *read* b^2 throughout the wrought question. Qu. 3, second equation *for* $(x^2+y^2)^2$ *read* $(x^2+y)^2$.

203. Qu. 9, *for* $\frac{18u}{y}$ *read* $\frac{18x}{y}$; Qu. 13, *for* $y = \sqrt{xa}$ *read* $y = \sqrt{xz}$. Qu. 15, for uw *read* uv.

206. note, *for* $\mp\frac{11\mp\sqrt{233}}{2}$ *read* $y = \frac{11\mp\sqrt{233}}{2}$.

208. Qu. 26, after "parcel" insert *valued* 18 *shillings*; Qu. 27, *for* 100 *read* 120.

209. note, *for* a^2 *read* $x^{\cdot 2}$; *for* y^{10} *read* y^6; *for* — 10 *read* 4; and *for* $y^{10} - 4y^6 + \cdot 1y^3 = \cdot 5$, *read* $y^9 - 2\cdot 5y^6 + \cdot 25y^3 = 1\cdot 25$.

210. line 4, *for* $6x^2$ *read* $6z^2$; in l. 13, *for* $34\cdot 5x^2$ *read* $34\cdot 5z^2$; and in l. 19, *for* $15z^2$ *read* $15z^{\cdot 2}$; in note, l. 4, *for* $\frac{a^2}{x^2}$ *read* $\frac{a^2}{n^2}$

211. — 7, *for* — 2 and 4 *read* $-\frac{8}{3}$ and 2.
— 9, *for* $\sqrt[2]{2+\frac{10}{9}\sqrt{3}}$ *read* $\sqrt[3]{2-\frac{10}{9}\sqrt{3}}$.
note, line 2, *for* ax *read* az; l. 7, *for* $\left(\frac{a}{3}\right)$ *read* $\left(\frac{a}{3}\right)^3$; and l. 9, *for* x^3 and y^3 *read* $2x^3$ and $2y^3$.

212. — 18, *for* x *read* y.

213. — 3, *read* $\pm\left\{\sqrt{\overline{(p^2+2n+q)}\,x}-\sqrt{n^2+s}\right\}$, and enclose the preceding line in a vinculum, in the same manner.

PAGE

214. Ex. 6. line 2, *for* $-400x$ *read* $400x^2$.

216. note, *for* $\frac{k_, - k_,}{k - x_,} = \frac{x_{,,} - x_,}{x_, - x_,}$ *read* $\frac{k_{,,} - k_,}{k - k_,} = \frac{x_{,,} - x_,}{x - x_,}$

221. line 24, dele is; and, l. 30, *for* ceases *read* cease.

223. — 19, *for* -6 *read* $+6$, and make the corresponding alteration in the subsequent line. Question 3, *for* $\pm \frac{1}{2} \sqrt{2} \pm \sqrt{\mp 3 \sqrt{2} - \frac{1}{2}}$ *read* $\pm \left\{\frac{1}{2} \sqrt{2} \pm \sqrt{\mp 3 \sqrt{2} - \frac{1}{2}}\right\}$

224. — 5, note, *for* $\times$ *read* $+$.

225. in the last example, *for* 3·438 *read* 2·292, and correct the result. Also, in the last line, *for* x *read* y.

227. lines 25, 26, *for* $25x^2$ *read* $25y^2$.

228. — 21, *for* $(-5$ *read* $(5$.

229. Ex. 1, *for* $9x^3 + 6x^2$ *read* $6x^3 + 9x^2$; and *for* $(x+1)^3(x-1)$ *read* $(x+1)^3(x-1)^2$; and make the next line agree. *For* VI. *read* VII. Note, line 3, *for* B *read* B′; line 6, *for* $x - i$ *read* $x + i$; line 8, *for* $x - 1$, $x - 2$, *read* $n - 1$, $n - 2$; line 9, *for* x^{n-3}, x^{n-4} *read* x^{n-2}, x^{n-3}; in line 10 *read* x^{n-3} and x^{n-4}; line 14, *for* $m - n$ *read* m; line 15, annex i^2 to the third term; and in the second vinculum, *for* $Y'i \times Z'i^2$ *read* $Y'i + Z'i^2$.

230. line 9, *for* $x = 3$ *read* $x - 3$; line 14, dele to $2m + 1$; and note, line 15, *for* $-72x^2$ *read* $+72x^2$.

232. — 13, 17, 18, *for* 4. 5, 6, *read* 6, 7, 8, respectively. In Ex. 2, interchange the signs of the addends to x throughout, and the consequent phraseology. Also, line 30, *read* their *for* three; and, line 32, read $(\cdot 5) + 1 - 1\cdot 5 + 1\cdot 5 - 3\cdot 25 + 4\cdot 5625 - 2\cdot 59375$; and correct the consequent numbers in the text.

234. bottom line, *for* increased division *read* increase.

235. line 16, *read* $30 + 5$ or 35 is, &c.; and *for* $x + 602x^2$ *read* $x^3 + 602x^2$.

236. Ex. 2, *read* $x^5 + 4x^4 - 2x^3 + 10x^2 - 2x - 962 = 0$; and in the second col. headed -2, insert between the 6th and 7th lines 997·1611
169·4514

239. line 16, *for* cx^2, on second side of equation, *read* $c_, x^2$; and lines 17 and 18, *for* $x + 1$ *read* $n + 1$.

240. — 30, *for* x *read* n.

241. in II. *for* m *read* n.

243. line 20, *for* $C = \frac{A^3}{3}$, $D = \frac{A^4}{4}$ *read* $C = \frac{A^3}{2\cdot 3}$, $D = \frac{A^4}{2\cdot 3\, 4}$; line 31, *for* $-\frac{1}{2}(a-1)$ *read* $-\frac{1}{2}(a-1)^2$.

247. — 6, *for* x *read* n, and line 20, *for* $\frac{n}{1}$ *read* $\frac{x}{1}$; line 30, 31, 32, *for* $\frac{1}{4}$, *read* $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{16}$

249. — 28, *for* $-1, 2$, *read* $-1, -2$.

254. — 14, *for* $(109684)^7$ *read* $(1\cdot 09684)^7$.

270. — 18, *for* $ax =$ *read* $^n x =$.

271. — 12, *for* Δa *read* Δx.

277. — 5 from bottom, *for* invention *read* insertion; and *for* expressions *read* equations

292. — 3 from bottom, *for* minus *read* minor.

315. — 14 from bottom, interchange the words triangles and rectangles.

338. — 13, *for* HK, *read* to HK; and line 22, *for* is the *read* is in the.

352. — 3 from bottom, *for* base a *read* base of a.

353. — 6, *for* to the sum *read* to double the sum; and line 1 from bottom, *for* therefore *read* therefrom.

396. note, *for* $\frac{c}{2}$ *read* $\frac{C}{2}$.

397. In the value of cos. A dele a from the denominator; and in the value of cos. B, *for* $-b$ *read* $-b^2$.

414. Ex. 30, *for* 800 and 750 *read* 80 and 75; and *for* 58° 15′ *read* 68° 15′.

467. Question 51, answer, *for* 8 *read* 2 in the numerator.

468. last line, *for* same *read* curve.

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